This paper deals with the existence and uniqueness of solutions for a class of infinite-horizon systems derived from optimal control. An existence and uniqueness theorem is proved for such Hamiltonian systems under some natural assumptions.

1. Introduction

We begin with a simple example to introduce the background of the considered problem. Let $U$ be a bounded closed subset of $\mathbb{R}^m$ and let functions $f : \mathbb{R}^n \times \mathbb{R}^m \times [a, \infty) \to \mathbb{R}^n$, $L : \mathbb{R}^n \times \mathbb{R}^m \times [a, \infty) \to \mathbb{R}$ be differentiable with respect to the first variable. Consider an optimal control system of the form

$$\text{Minimize } J[u(\cdot)] = \int_a^\infty L(x(t), u(t), t) \, dt$$

over all admissible controls $u(\cdot) \in L^2([a, \infty); U)$, where the trajectories $x : [a, \infty) \to \mathbb{R}^n$ are differentiable on $[a, \infty)$ and satisfy the dynamic system

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(a) = x_0.$$  

From control theory, the well-known Pontryagin maximum principle, an important necessary optimality condition, is usually applied to get optimal controls for this system. By doing this, the following infinite-horizon Hamiltonian system is derived:

$$\dot{x}(t) = \frac{\partial H(x(t), p(t), t)}{\partial p}, \quad x(a) = x_0,$$

$$\dot{p}(t) = -\frac{\partial H(x(t), p(t), t)}{\partial x},$$

$$x(\cdot) \in L^2([a, \infty); \mathbb{R}^n), \quad p(\cdot) \in L^2([a, \infty); \mathbb{R}^n).$$
Here, \( H(x, p, t) = \lambda L(x, \bar{u}, t) + \langle p, f(x, \bar{u}, t) \rangle \) is the Hamiltonian function for (1.1)-(1.2), \( \langle \cdot, \cdot \rangle \) stands for inner product in \( \mathbb{R}^n \), \( \bar{u} \) is an optimal control, and \( x(t) \) is the optimal trajectory corresponding to the optimal control \( \bar{u} \).

The existence and uniqueness of solutions for system (1.3) is a very interesting question; if solutions to (1.3) are unique, then the optimal control for system (1.1)-(1.2) can be solved analytically or numerically through (1.3). When we consider the generalization of (1.3) in infinite-dimensional spaces, the following Hamiltonian system is obtained:

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + F(x(t), p(t), t), \\
x(a) &= x_0, \\
\dot{p}(t) &= -A^*(t)p(t) + G(x(t), p(t), t), \\
x(\cdot) &\in L^2([a, \infty); X), \\
p(\cdot) &\in L^2([a, \infty); X),
\end{align*}
\]

(1.4)

where both \( x(t) \) and \( p(t) \) take values in a Hilbert space \( X \) for \( a \leq t < \infty \). It is always assumed that \( F, G : X \times X \times [a, \infty) \to X \) are nonlinear operators, that \( A(t) \) is a closed operator for each \( t \in [a, \infty) \), and that \( A^*(t) \) is the adjoint operator of \( A(t) \).

The following system is called a linear Hamiltonian system, which is a special case of (1.4),

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)p(t) + \varphi(t), \\
x(a) &= x_0, \\
\dot{p}(t) &= -A^*(t)p(t) + C(t)x(t) + \psi(t), \\
x(\cdot) &\in L^2([a, \infty); X), \\
p(\cdot) &\in L^2([a, \infty); X),
\end{align*}
\]

(1.5)

where \( \varphi(\cdot), \psi(\cdot) \in L^2([a, \infty); X) \), and \( B(t), C(t) \) are selfadjoint linear operators from \( X \) to \( X \) for all \( t \in [a, \infty) \).

In [2], Lions has discussed the existence and uniqueness of solutions for system (1.5) and gave an existence and uniqueness result. In [1], Hu and Peng considered the existence and uniqueness of solutions for a class of nonlinear forward-backward stochastic differential equations similar to (1.3) but on finite horizon, they provided an existence and uniqueness theorem for (1.3). Peng and Shi in [3] dealt with the existence and uniqueness of solutions for (1.3) using the techniques developed in [1]. In this paper, we consider the existence and uniqueness of solutions for infinite-dimensional system (1.4).

Throughout the paper, the following basic assumptions hold.

(I) There exists a real number \( L > 0 \) such that

\[
\begin{align*}
||F(x_1, p_1, t) - F(x_2, p_2, t)|| &\leq L(||x_1 - x_2|| + ||p_1 - p_2||), \\
||G(x_1, p_1, t) - G(x_2, p_2, t)|| &\leq L(||x_1 - x_2|| + ||p_1 - p_2||)
\end{align*}
\]

(1.6)

for all \( x_1, p_1, x_2, p_2 \in X \) and \( t \in [a, \infty) \).
(II) There exists a real number $\alpha > 0$ such that
\[
\langle F(x_1, p_1, t) - F(x_2, p_2, t), p_1 - p_2 \rangle + \langle G(x_1, p_1, t) - G(x_2, p_2, t), x_1 - x_2 \rangle \\
\leq -\alpha(\|x_1 - x_2\| + \|p_1 - p_2\|)
\] (1.7)
for all $x_1, p_1, x_2, p_2 \in X$ and $t \in [a, \infty)$.

2. Lemmas
Two lemmas are given in this section. They are essential to prove the main theorem.

**Lemma 2.1.** Consider the Hamiltonian system
\[
\dot{x}(t) = A(t)x(t) + F_\beta(x, p, t) + \varphi(t), \\
x(a) = x_0, \\
\dot{p}(t) = -A^*(t)p(t) + G_\beta(x, p, t) + \psi(t), \\
x(\cdot) \in L^2([a, \infty); X), \\
p(\cdot) \in L^2([a, \infty); X),
\] (2.1)
where $\varphi(\cdot), \psi(\cdot) \in L^2([a, \infty); X)$. The functions $F_\beta$ and $G_\beta$ are defined as
\[
F_\beta(x, p, t) := -(1 - \beta)\alpha p + \beta F(x, p, t), \\
G_\beta(x, p, t) := -(1 - \beta)\alpha x + \beta G(x, p, t). \\
\] (2.2)

Assume that (2.1) has a unique solution for some real number $\beta = \beta_0 \geq 0$ and any $\varphi(t), \psi(t)$. There exists a real number $\delta > 0$, which is independent of $\beta_0$, such that (2.1) has a unique solution for any $\varphi(t), \psi(t)$, and $\beta \in [\beta_0, \beta_0 + \delta]$.

**Proof.** For any given $\varphi(\cdot), \psi(\cdot), x(\cdot), p(\cdot) \in L^2([a, \infty); X)$ and $\delta > 0$, construct the following Hamiltonian system:
\[
\dot{X}(t) = A(t)X(t) + F_{\beta_0}(X, P, t) + F_{\beta_0 + \delta}(x, p, t) - F_{\beta_0}(x, p, t) + \varphi(t), \\
X(a) = x_0, \\
\dot{P}(t) = -A^*(t)P(t) + G_{\beta_0}(X, P, t) + G_{\beta_0 + \delta}(x, p, t) - G_{\beta_0}(x, p, t) + \psi(t), \\
X(\cdot) \in L^2([a, \infty); X), \\
P(\cdot) \in L^2([a, \infty); X). \\
\] (2.3)
Note that
\[
F_{\beta_0 + \delta}(x, p, t) - F_{\beta_0}(x, p, t) \\
= -(1 - \beta_0 - \delta)\alpha p + (\beta_0 + \delta)F(x, p, t) + (1 - \beta_0)\alpha p - \beta_0 F(x, p, t) \\
= \alpha\delta p + \delta F(x, p, t), \\
\] (2.4)
\[
G_{\beta_0 + \delta}(x, p, t) - G_{\beta_0}(x, p, t) \\
= -(1 - \beta_0 - \delta)\alpha x + (\beta_0 + \delta)G(x, p, t) + (1 - \beta_0)\alpha x - \beta_0 G(x, p, t) \\
= \alpha\delta x + \delta G(x, p, t).
\]
The assumption of Lemma 2.1 implies that (2.3) has a unique solution for each pair $(x(\cdot), p(\cdot)) \in L^2([a, \infty); X) \times L^2([a, \infty); X)$. Therefore, the mapping $J$,

\[
L^2([a, \infty); X) \times L^2([a, \infty); X) \longrightarrow L^2([a, \infty); X) \times L^2([a, \infty); X),
\]

given by

\[
J(x(\cdot), p(\cdot)) := (X(\cdot), P(\cdot))
\]

is well defined.

Let $J(x_1(\cdot), p_1(\cdot)) = (X_1(\cdot), P_1(\cdot))$ and $J(x_2(\cdot), p_2(\cdot)) = (X_2(\cdot), P_2(\cdot))$. Since $X_1(\cdot) - X_2(\cdot) \in L^2([a, \infty); X)$ and $P_1(\cdot) - P_2(\cdot) \in L^2([a, \infty); X)$, there exists a sequence of real numbers $a < t_1 < t_2 < \cdots < t_k < \cdots$ such that $t_k \to \infty$ as $k \to \infty$ and

\[
X_1(t_k) - X_2(t_k) \to 0, \quad P_1(t_k) - P_2(t_k) \to 0, \quad \text{as } k \to \infty.
\]

Note that

\[
\frac{d}{dt} \langle X_1(t) - X_2(t), P_1(t) - P_2(t) \rangle \\
= \langle F_{\beta_0}(X_1, P_1, t) - F_{\beta_0}(X_2, P_2, t) + \alpha \delta(x_1 - x_2) \rangle
+ \langle G_{\beta_0}(X_1, P_1, t) - G_{\beta_0}(X_2, P_2, t) + \delta(x_1 - x_2) \rangle
:= I_1 + I_2.
\]

Since

\[
F_{\beta_0}(X_1, P_1, t) - F_{\beta_0}(X_2, P_2, t) = -\alpha(1 - \beta_0)(P_1 - P_2) + \beta_0(F(X_1, P_1, t) - F(X_2, P_2, t))
\]

implies that

\[
I_1 = -\alpha(1 - \beta_0)\|P_1 - P_2\|^2 + \beta_0 \langle F(X_1, P_1, t) - F(X_2, P_2, t), P_1 - P_2 \rangle
+ \alpha \delta(x_1 - x_2) \langle P_1 - P_2, P_1 - P_2 \rangle + \delta \langle F(x_1, P_1, t) - F(x_2, P_2, t), P_1 - P_2 \rangle,
\]

similarly,

\[
G_{\beta_0}(X_1, P_1, t) - G_{\beta_0}(X_2, P_2, t) = -\alpha(1 - \beta_0)(X_1 - X_2) + \beta_0(G(X_1, P_1, t) - G(X_2, P_2, t))
\]

implies that

\[
I_2 = -\alpha(1 - \beta_0)\|X_1 - X_2\|^2 + \beta_0 \langle G(X_1, P_1, t) - G(X_2, P_2, t), X_1 - X_2 \rangle
+ \alpha \delta(x_1 - x_2) \langle X_1 - X_2, X_1 - X_2 \rangle + \delta \langle G(x_1, P_1, t) - G(x_2, P_2, t), X_1 - X_2 \rangle.
\]
It follows from the estimates for $I_1, I_2$, and the assumption (I) that

$$
I_1 + I_2 \leq -\alpha (\|X_1 - X_2\|^2 + \|P_1 - P_2\|^2) \\
+ \alpha \delta (\|P_1 - P_2\| \|P_1 - P_2\| + \|x_1 - x_2\| \|X_1 - X_2\|) \\
+ \delta \| F(x_1, p_1, t) - F(x_2, p_2, t) \| \|P_1 - P_2\| \\
+ \delta \| G(x_1, p_1, t) - G(x_2, p_2, t) \| \|X_1 - X_2\| \\
\leq -\alpha (\|X_1 - X_2\|^2 + \|P_1 - P_2\|^2) \\
+ \delta (2L + \alpha) (\|X_1 - X_2\|^2 + \|P_1 - P_2\|^2 + \|x_1 - x_2\|^2 + \|P_1 - P_2\|^2).
$$

Therefore,

$$
\frac{d}{dt} \langle X_1(t) - X_2(t), P_1(t) - P_2(t) \rangle \\
\leq -\alpha (\|X_1 - X_2\|^2 + \|P_1 - P_2\|^2) \\
+ \delta (2L + \alpha) (\|X_1 - X_2\|^2 + \|P_1 - P_2\|^2 + \|x_1 - x_2\|^2 + \|P_1 - P_2\|^2).
$$

Integrating between $a$ and $t_k$, we have

$$
\langle X_1(t_k) - X_2(t_k), P_1(t_k) - P_2(t_k) \rangle - \langle X_1(a) - X_2(a), P_1(a) - P_2(a) \rangle \\
\leq -\alpha \int_a^{t_k} (\|X_1 - X_2\|^2 + \|P_1 - P_2\|^2) dt + \delta (2L + \alpha) \\
\times \int_a^{t_k} (\|X_1 - X_2\|^2 + \|P_1 - P_2\|^2 + \|x_1 - x_2\|^2 + \|P_1 - P_2\|^2) dt.
$$

Letting $k \to \infty$ and noting that (2.7), we obtain

$$
\int_a^\infty (\|X_1 - X_2\|^2 + \|P_1 - P_2\|^2) dt \leq \frac{2\delta L + \delta \alpha}{\alpha - 2\delta L - \delta \alpha} \int_a^\infty (\|x_1 - x_2\|^2 + \|P_1 - P_2\|^2) dt.
$$

Choose a small $\delta$ (independent of $\beta_0$) such that

$$
\frac{2\delta L + \delta \alpha}{\alpha - 2\delta L - \delta \alpha} \leq \frac{1}{2},
$$

then $J$ is a contractive mapping and hence has a unique fixed point. Thus, (2.3) becomes

$$
\dot{x}(t) = A(t)x(t) + F_{\beta_0 + \delta}(x, p, t) + \varphi(t), \\
x(a) = x_0, \\
\dot{p}(t) = -A^*(t)p(t) + G_{\beta_0 + \delta}(x, p, t) + \psi(t), \\
x(\cdot) \in L^2([a, \infty); X), \\
p(\cdot) \in L^2([a, \infty); X).
$$

This shows that system (2.1) has a unique solution on $[a, \infty)$ for $\beta \in [\beta_0, \beta_0 + \delta]$. The proof is complete. \(\square\)
Remark 3.2. Consider system (1.5). Note that

\[ J = \int_a^\infty [\langle Qx(t), x(t) \rangle + \langle Ru(t), u(t) \rangle] dt, \]

where \( u(t) \) and \( x(t) \) take values in Hilbert spaces \( U \) and \( X \), where \( B \in \mathcal{L}[U, X] \), and where \( Q \in \mathcal{L}[X, X] \) and \( R \in \mathcal{L}[U, U] \) are selfadjoint operators.
From optimal control theory, the following Hamiltonian system is derived:

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) - BR^{-1}Bp(t), \\
x(a) &= x_0, \\
\dot{p}(t) &= -A^*(t)p(t) - Qx(t), \\
x(\cdot) &\in L^2([a, \infty); X), \\
p(\cdot) &\in L^2([a, \infty); X).
\end{align*}
\]  

(3.4)

This is a special case of system (1.5). Therefore, system (3.4) has a unique solution if both \(BR^{-1}B\) and \(Q\) are positive definite.

References


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