EXPANSION OF $\alpha$-OPEN SETS AND DECOMPOSITION OF $\alpha$-CONTINUOUS MAPPINGS

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We introduce the notions of expansion $\mathcal{A}_a$ of $\alpha$-open sets and $\mathcal{A}_a$-expansion $\alpha$-continuous mappings in topological spaces. The main result of this paper is that a map $f$ is $\alpha$-continuous if and only if it is $\mathcal{A}_a$-expansion $\alpha$-continuous and $\mathcal{B}_a$-expansion $\alpha$-continuous, where $\mathcal{A}_a, \mathcal{B}_a$ are two mutually dual expansions.

1. Introduction

In 1965, Njastad [2] introduced the notion of $\alpha$-sets in topological space. In 1983, Mashhour et al. [1] introduced, with the help of $\alpha$-sets, a weak form of continuity which they termed as $\alpha$-continuity. Noiri [3] introduced the same concept, but under the name strong semicontinuity. Noiri [4] defined with the aid of $\alpha$-sets a new weakened form of continuous mapping called weakly $\alpha$-continuous mapping. Sen and Bhattacharyya [5] introduced another new weakened form of continuity called weak $^*\alpha$-continuity and proved that a mapping is $\alpha$-continuous if and only if it is weakly $\alpha$-continuous and weak $^*\alpha$-continuous.

In this paper, we give a general setting for such decompositions of $\alpha$-continuity by using expansion of $\alpha$-open sets, whereas in [6], Tong used expansion of open sets to give a general setting for the decomposition of continuous mapping into weakly continuous and weak $^*\alpha$-continuous mappings.

2. Preliminaries

Throughout this paper, $(X, \tau), (Y, \sigma)$, and so forth (or simply $X, Y$, etc.) will always denote topological spaces. The family of all $\alpha$-open sets in $X$ is denoted by $\tau_a$.

We recall the definition of weakly $\alpha$-continuous and weak $^*\alpha$-continuous mappings.

Definition 2.1 [1]. A mapping $f : X \rightarrow Y$ is said to be $\alpha$-continuous if for each open set $V$ in $Y$, $f^{-1}(V) \in \tau_a$.

Definition 2.2 [3]. A mapping $f : X \rightarrow Y$ is said to be weakly $\alpha$-continuous if for each $x$ in $X$ and for each open set $V$ in $Y$ containing $f(x)$, there exists a set $U \in \tau_a$ containing $x$ such that $f(U) \subseteq \text{Cl} V$, where $\text{Cl} V$ means the closure of $V$. 
Proposition 2.3 [5]. A mapping \( f : X \to Y \) is weakly \( \alpha \)-continuous if and only if \( f^{-1}(V) \subseteq \alpha \text{int}[f^{-1}(\text{Cl} V)] \), for every open set \( V \) in \( Y \), where \( \alpha \text{int}(A) \) means \( \alpha \)-interior of \( A \).

Definition 2.4 [5]. A mapping \( f : X \to Y \) is said to be weak* \( \alpha \)-continuous if and only if for every open set \( V \subseteq Y \), \( f^{-1}(\text{Fr} V) \) is \( \alpha \)-closed in \( X \), where \( \text{Fr} V = \text{Cl} V \setminus V \) is the boundary operator for open sets.

3. Decompositions of \( \alpha \)-continuity

Definition 3.1. Let \((X, \tau)\) be a topological space, let \(2^X\) be the set of all subsets in \( X \). A mapping \( \mathcal{A}_\alpha : \tau_\alpha \to 2^X \) is said to be an expansion on \( X \) if \( U \subseteq \mathcal{A}_\alpha U \) for each \( U \in \tau_\alpha \).

Remark 3.2. If \( \gamma_\alpha \) is the identity expansion, then \( \gamma_\alpha \) is defined by \( \gamma_\alpha U = U \). \( \mu_\alpha \) is defined by \( \mu_\alpha U = (\alpha \text{int} U \cap U^c)^c \) is an expansion. \( \mathcal{C}_\alpha \) is defined by \( \mathcal{C}_\alpha U = \text{Cl} U \) and \( \mathcal{F}_\alpha U \) is defined by \( \mathcal{F}_\alpha U = (\text{Fr} U)^c \) are expansions.

Definition 3.3 [6]. Let \((X, \tau)\) be a topological space. A pair of expansions \( \mathcal{A} \) and \( \mathcal{B} \) on \( X \) is said to be mutually dual if \( \mathcal{A} U \cap \mathcal{B} U = U \) for each \( U \in \tau \).

Remark 3.4. Let \((X, \tau)\) be a topological space. Then \( \mathcal{C}_\alpha \) and \( \mathcal{F}_\alpha \) are mutually dual. This follows from [6, Proposition 2].

Example 3.5. Let \( X = \{a, b, c\} \) with topologies \( \tau = \{\phi, \{a\}, X\}, \tau_\alpha = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\} \). \( \mathcal{A}_\alpha(\phi) = \phi \), \( \mathcal{A}_\alpha(a) = \{a\}, \mathcal{A}_\alpha(a, b) = \{a, b\}, \mathcal{A}_\alpha(a, c) = X \), and \( \mathcal{A}_\alpha(X) = X \). Then \( \mathcal{A}_\alpha \) is an expansion. Let \( \mathcal{B}_1(\phi) = X \), \( \mathcal{B}_1(a) = \{a, c\} \), \( \mathcal{B}_1(a, b) = X \), \( \mathcal{B}_1(a, c) = \{a, c\} \), and \( \mathcal{B}_1(X) = X \). \( \mathcal{B}_2(\phi) = X \), \( \mathcal{B}_2(a) = \{a, b\}, \mathcal{B}_2(a, b) = \{a, b\}, \mathcal{B}_2(a, c) = \{a, c\} \), and \( \mathcal{B}_2(X) = X \). Then \( \mathcal{B}_1, \mathcal{B}_2 \) are both mutually dual to \( \mathcal{A}_\alpha \).

Proposition 3.6. Let \((X, \tau)\) be a topological space. Then \( \gamma_\alpha \) and \( \mu_\alpha \) are mutually dual.

Proof.

\[
(\gamma_\alpha U) \cap (\mu_\alpha U) = U \cap (\alpha \text{int} U \cap U^c)^c \\
= (\alpha \text{int} U) \cap (\alpha \text{int} U \cap U^c)^c \\
= (\alpha \text{int} U) \cap ((\alpha \text{int} U)^c \cup U) \\
= ((\alpha \text{int} U) \cap (\alpha \text{int} U)^c) \cup (\alpha \text{int} U \cap U) \\
\phi \cup U = U. 
\]

Definition 3.7. Let \((X, \tau)\) and \((Y, \sigma)\) be two topological spaces and let \( \mathcal{A}_\alpha \) be an expansion on \( Y \). Then the mapping \( f : X \to Y \) is said to be \( \mathcal{A}_\alpha \)-expansion \( \alpha \)-continuous if \( f^{-1}(V) \subseteq \alpha \text{int}[f^{-1}(\mathcal{A}_\alpha V)] \), for each \( V \in \sigma \).

Remark 3.8. A weakly \( \alpha \)-continuous mapping \( f : X \to Y \) can be renamed as \( \mathcal{C}_\alpha \)-expansion \( \alpha \)-continuous mapping.

Theorem 3.9. Let \((X, \tau)\) and \((Y, \sigma)\) be two topological spaces and \( \mathcal{A}_\alpha, \mathcal{B}_\alpha \) are two mutually dual expansions on \( Y \). Then the mapping \( f : X \to Y \) is \( \alpha \)-continuous if and only if \( f \) is \( \mathcal{A}_\alpha \)-expansion \( \alpha \)-continuous and \( \mathcal{B}_\alpha \)-expansion \( \alpha \)-continuous.
Proof. Necessity. Suppose that \( f \) is \( \alpha \)-continuous. Since \( \mathcal{A}_\alpha, \mathcal{B}_\alpha \) are mutually dual on \( Y \), \( \mathcal{A}_\alpha V \cap \mathcal{B}_\alpha V = V \) for each \( V \in \sigma \).

Then
\[
f^{-1}(V) = f^{-1}(\mathcal{A}_\alpha V \cap \mathcal{B}_\alpha V) = f^{-1}(\mathcal{A}_\alpha V) \cap f^{-1}(\mathcal{B}_\alpha V). \tag{3.2}
\]

Since \( f \) is \( \alpha \)-continuous, \( f^{-1}(V) = \alpha \text{int} f^{-1}(V) \).

Therefore,
\[
f^{-1}(V) = \alpha \text{int} f^{-1}(V) = \alpha \text{int} (f^{-1}(\mathcal{A}_\alpha V \cap \mathcal{B}_\alpha V)) = \alpha \text{int} f^{-1}(\mathcal{A}_\alpha V) \cap \alpha \text{int} f^{-1}(\mathcal{B}_\alpha V). \tag{3.3}
\]

This implies that \( f^{-1}(V) \subseteq \alpha \text{int} f^{-1}(\mathcal{A}_\alpha V) \) and \( f^{-1}(V) \subseteq \alpha \text{int} f^{-1}(\mathcal{B}_\alpha V) \). This shows that \( f \) is \( \mathcal{A}_\alpha \)-expansion \( \alpha \)-continuous and \( \mathcal{B}_\alpha \)-expansion \( \alpha \)-continuous.

Sufficiency. Since \( f \) is \( \mathcal{A}_\alpha \)-expansion \( \alpha \)-continuous, \( f^{-1}(V) \subseteq \alpha \text{int} f^{-1}(\mathcal{A}_\alpha V) \) for each \( V \in \sigma \). Since \( f \) is \( \mathcal{B}_\alpha \)-expansion \( \alpha \)-continuous, \( f^{-1}(V) \subseteq \alpha \text{int} f^{-1}(\mathcal{B}_\alpha V) \) for each \( V \in \sigma \). As \( \mathcal{A}_\alpha \) and \( \mathcal{B}_\alpha \) are two mutually dual expansions on \( Y \), \( \mathcal{A}_\alpha V \cap \mathcal{B}_\alpha V = V \),
\[
f^{-1}(V) = f^{-1}(\mathcal{A}_\alpha V \cap \mathcal{B}_\alpha V) = f^{-1}(\mathcal{A}_\alpha V) \cap f^{-1}(\mathcal{B}_\alpha V),
\]

\[
\alpha \text{int} f^{-1}(V) = \alpha \text{int} f^{-1}(\mathcal{A}_\alpha V) \cap \alpha \text{int} f^{-1}(\mathcal{B}_\alpha V) \supseteq f^{-1}(V) \cap f^{-1}(V) = f^{-1}(V). \tag{3.4}
\]

This implies that \( f^{-1}(V) \subseteq \alpha \text{int} f^{-1}(V) \). Always, \( \alpha \text{int} f^{-1}(V) \subseteq f^{-1}(V) \). So \( f^{-1}(V) = \alpha \text{int} f^{-1}(V) \). Therefore, \( f^{-1}(V) \) is an \( \alpha \)-open set in \( X \) for each \( V \in \sigma \). Hence \( f \) is \( \alpha \)-continuous. \( \square \)

Definition 3.10. Let \( (X, \tau) \) and \( (Y, \sigma) \) be two topological spaces, \( \mathcal{B}_\alpha \) an expansion on \( Y \). Then a mapping \( f : X \to Y \) is said to be \( \alpha \)-closed \( \mathcal{B}_\alpha \)-continuous if \( f^{-1}((\mathcal{B}_\alpha V)^c) \) is an \( \alpha \)-closed set in \( X \) for each \( V \in \sigma \).

Remark 3.11. A weak *\( \alpha \)-continuous mapping can be renamed as \( \alpha \)-closed \( \mathcal{F}_\alpha \)-continuous mapping since \( (\mathcal{F}_\alpha V)^c = (\text{Fr} V)^c = \text{Fr} V \).

Proposition 3.12. An \( \alpha \)-closed \( \mathcal{B}_\alpha \)-continuous mapping is \( \mathcal{B}_\alpha \)-expansion \( \alpha \)-continuous.

Proof. First, we prove that \( (f^{-1}((\mathcal{B}_\alpha V)^c))^c = f^{-1}(\mathcal{B}_\alpha V) \).

Let \( x \in (f^{-1}((\mathcal{B}_\alpha V)^c))^c \). Then \( x \notin f^{-1}(\mathcal{B}_\alpha V)^c \). Hence \( f(x) \notin (\mathcal{B}_\alpha V)^c \), \( f(x) \in \mathcal{B}_\alpha V \), and \( x \in f^{-1}(\mathcal{B}_\alpha V) \).

Conversely, if \( x \in f^{-1}(\mathcal{B}_\alpha V) \), then \( f(x) \in \mathcal{B}_\alpha V \). Hence \( f(x) \notin (\mathcal{B}_\alpha V)^c \), \( x \notin f^{-1}(\mathcal{B}_\alpha V)^c \). Therefore, \( (f^{-1}((\mathcal{B}_\alpha V)^c))^c = f^{-1}(\mathcal{B}_\alpha V) \).

Since \( f^{-1}((\mathcal{B}_\alpha V)^c) \) is an \( \alpha \)-closed set in \( X \), \( (f^{-1}((\mathcal{B}_\alpha V)^c))^c \) is an \( \alpha \)-open set in \( X \). Hence \( f^{-1}(\mathcal{B}_\alpha V) \) is an \( \alpha \)-open in \( X \) and \( f^{-1}(\mathcal{B}_\alpha V) = \alpha \text{int} f^{-1}(\mathcal{B}_\alpha V) \).

Since \( \mathcal{B}_\alpha \) is an expansion on \( Y \), \( V \subseteq \mathcal{B}_\alpha V \), we have \( f^{-1}(V) \subseteq f^{-1}(\mathcal{B}_\alpha V) = \alpha \text{int} f^{-1}(\mathcal{B}_\alpha V) \). Therefore, \( f \) is \( \mathcal{B}_\alpha \)-expansion \( \alpha \)-continuous. \( \square \)

By Theorem 3.9 and Proposition 3.12, we have the following corollary.
Corollary 3.13. Let \((X, \tau)\) and \((Y, \alpha)\) be two topological spaces and \(\mathcal{A}_\alpha, \mathcal{B}_\alpha\) are two mutually dual expansions on \(Y\). Then a mapping \(f : X \rightarrow Y\) is \(\alpha\)-continuous if and only if \(f\) is \(\mathcal{A}_\alpha\)-expansion \(\alpha\)-continuous, and \(\alpha\)-closed \(\mathcal{B}_\alpha\)-continuous.

By Remarks 3.8, 3.11, and by the above corollary, we have the following corollary.

Corollary 3.14 [5]. A mapping is \(\alpha\)-continuous if and only if it is weakly \(\alpha\)-continuous and \(\alpha\)-closed.

References


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