GENERALIZATIONS OF PRINCIPALLY QUASI-INJECTIVE MODULES AND QUASIPRINCIPALLY INJECTIVE MODULES

ZHU ZHANMIN, XIA ZHANGSHENG, AND TAN ZHISONG

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Let R be a ring and M a right R -module with $S = \text{End}(M_R)$. The module M is called almost principally quasi-injective (or APQ-injective for short) if, for any $m \in M$, there exists an S -submodule X_m of M such that $l_M r_R(m) = Sm \oplus X_m$. The module M is called almost quasiprincipally injective (or AQP-injective for short) if, for any $s \in S$, there exists a left ideal X_s of S such that $l_S(\ker(s)) = Ss \oplus X_s$. In this paper, we give some characterizations and properties of the two classes of modules. Some results on principally quasi-injective modules and quasiprincipally injective modules are extended to these modules, respectively. Specially in the case R_R , we obtain some results on AP-injective rings as corollaries.

1. Introduction

Throughout R is a ring with identity and M is a right R -module with $S = \text{End}(M_R)$. Recall a ring R is called right principally injective [5] (or right P-injective for short) if, every homomorphism from a principally right ideal of R to R can be extended to an endomorphism of R , or equivalently, $lr(a) = Ra$ for all $a \in R$. The notion of right P-injective rings has been generalized by many authors. For example, in [4, 8], right P-injective rings are generalized to modules in two ways, respectively. Following [4], the module M is called principally quasi-injective (or PQ-injective for short) if, each R -homomorphism from a principal submodule of M to M can be extended to an endomorphism of M . This is equivalent to saying that $l_M r_R(m) = Sm$ for all $m \in M$, where $l_M r_R(m)$ consists of all elements $z \in M$ such that $mx = 0$ implies $zx = 0$ for any $x \in R$. In [8], the module M is called quasiprincipally injective (or QP-injective for short) if, every homomorphism from an M -cyclic submodule of M to M can be extended to an endomorphism of M , or equivalently, $l_S(\ker(s)) = Ss$ for all $s \in S$. In [6], right P-injective rings are generalized to almost principally injective rings, that is, a ring R is said to be almost principally injective (or AP-injective for short) if, for any $a \in R$, there exists a left ideal X_a such that $lr(a) = Ra \oplus X_a$. The nice structure of PQ-injective modules, QP-injective modules, and AP-injective rings draws our attention to define almost PQ-injective modules and almost QP-injective modules in similar ways to AP-injective rings, and to investigate their characterizations and properties.

2. APQ-injective modules

Definition 2.1. Let M be a right R -module and let $S = \text{End}(M_R)$. The module M is called almost principally quasi-injective (briefly, APQ-injective) if, for any $m \in M$, there exists an S -submodule X_m of M such that $l_M r_R(m) = Sm \oplus X_m$.

The concept of APQ-injective modules is explained by the following lemma.

LEMMA 2.2. Let M_R be a module and let $S = \text{End}(M_R)$, and $m \in M$.

- (1) If $l_M r_R(m) = Sm \oplus X$ for some $X \subseteq M$ as left S -modules, then $\text{Hom}_R(mR, M) = S \oplus \Gamma$ as left S -modules, where $\Gamma = \{f \in \text{Hom}_R(mR, M) \mid f(m) \in X\}$.
- (2) If $\text{Hom}_R(mR, M) = S \oplus \Gamma$ as left S -modules, then $l_M(r_R(m)) = Sm \oplus X$ as left S -modules, where $X = \{f(m) \mid f \in \Gamma\}$.
- (3) Sm is a summand of $l_M(r_R(m))$ as left S -modules if and only if S is a summand of $\text{Hom}_R(mR, M)$ as left S -modules.

Proof. The map $\theta : l_M(r_R(m)) \rightarrow \text{Hom}_R(mR, M)$ with $\theta(a) = \lambda_a$ is a left S -isomorphism, where $\lambda_a : mR \rightarrow M$ is defined by $\lambda_a(mr) = ar$, so the lemma follows. Moreover, ${}_S(Sm)$ is nonsmall in $l_M(r_R(m))$ if and only if S is nonsmall in $\text{Hom}_R(mR, M)$. $\square$

From Lemma 2.2, the following corollary follows.

COROLLARY 2.3 [4, Lemma 1.1]. Let M_R be a right R -module with $S = \text{End}(M_R)$ and $m \in M$. Then $l_M(r_R(m)) = Sm$ if and only if every R -homomorphism of mR into M extends to M .

From Corollary 2.3, we see that all PQ-injective modules are APQ-injective. Since a ring R is right P-injective (resp., AP-injective) if and only if the right R -module R_R is PQ-injective (resp., APQ-injective), and Page and Zhou [6] have given three examples of rings which are right AP-injective but not right P-injective, so in general, APQ-injective modules need not be PQ-injective.

Recall that a ring R is called right QP-injective [6, Definition 2.1], if for any $0 \neq a \in R$, there exists a left ideal X_a such that $lr(a) = Ra + X_a$ with $a \notin X_a$. Now we extend this concept to modules.

Definition 2.4. Let M be a right R -module with $S = \text{End}(M_R)$, the module M is said to be QPQ-injective (i.e., quasiprincipally quasi-injective) if, for any nonzero element m of M , there exists an S -submodule X_m of M such that $l_M r_R(m) = Sm + X_m$ with $m \notin X_m$.

Clearly, right APQ-injective modules are QPQ-injective, but the reverse implication is not true. For example, Z_Z is QPQ-injective, but not APQ-injective.

Let M be a right R -module with $S = \text{End}(M_R)$, and $J(S)$ the Jacobson radical of S . Following [4], write $W(S) = \{w \in S \mid \ker(w) \subseteq^{\text{ess}} M\}$.

THEOREM 2.5. Let M_R be QPQ-injective with $S = \text{End}(M_R)$. Then

- (1) $J(S) \subseteq W(S)$,
- (2) $\text{Soc}(M_R) \subseteq r_M(J(S))$.

Proof. (1) Let $a \in J(S)$. If $a \notin W(S)$, then $\ker(a) \cap K = 0$ for some $0 \neq K \leq M_R$. Take $k \in K$ such that $ak \neq 0$, then $l_M(r_R(ak)) = S(ak) + X_{ak}$ with $ak \notin X_{ak}$. If $r \in r_R(ak)$, then $kr \in \ker(a) \cap K$, so $kr = 0$, and hence $r \in r_R(k)$. This shows that $r_R(ak) = r_R(k)$. Note that

$k \in l_M(r_R(k)) = l_M(r_R(ak)) = S(ak) + X_{ak}$, so we may write $k = b(ak) + x$, where $b \in S$ and $x \in X$. Then $(1 - ba)k = x$, and so $k = (1 - ba)^{-1}x$. Thus $ak = a(1 - ba)^{-1}x \in X_{ak}$, a contradiction.

(2) Let $mR \subseteq M$ be simple. Suppose $am \neq 0$ for some $a \in J(S)$. Then, since mR is simple, $r_R(am) = r_R(m)$. Since M_R is QPQ-injective, there is a left S -module X such that $am \notin X$ and $l_M r_R(am) = S(am) + X$. Note that $m \in l_M r_R(am)$, and so we may write $m = b(am) + x$, where $b \in S$ and $x \in X$. Then $(1 - ba)m = x$, so $m = (1 - ba)^{-1}x \in X$. This means that $am \in X$, a contradiction. $\square$

COROLLARY 2.6. *Let M_R be QPQ-injective with $S = \text{End}(M_R)$. If S is semilocal, then $\text{Soc}(M_R) \subseteq \text{Soc}({}_S M)$.*

Proof. This follows from Theorem 2.5(2) and [1, Proposition 15.17]. $\square$

LEMMA 2.7. *Let M_R be APQ-injective with $S = \text{End}(M_R)$. If $s \notin W(S)$, then the inclusion $\ker(s) \subset \ker(s - sts)$ is strict for some $t \in S$.*

Proof. If $s \notin W(S)$, then $\ker(s) \cap mR = 0$ for some $0 \neq m \in M$. Thus $r_R(m) = r_R(sm)$, and so $l_M r_R(m) = l_M r_R(sm) = S(sm) \oplus X_{sm}$ as left S -modules because M_R is APQ-injective. Write $m = t(sm) + x$, where $x \in X_{sm}$. Then $(s - sts)m = sx \in S(sm) \cap X_{sm}$, and hence $(s - sts)m = 0$. Therefore, the inclusion $\ker(s) \subset \ker(s - sts)$ is strict. $\square$

LEMMA 2.8. *Let M be a right R -module with $S = \text{End}(M_R)$. Suppose that for any sequence $\{s_1, s_2, \dots\} \subseteq S$, the chain $\ker(s_1) \subseteq \ker(s_2 s_1) \subseteq \dots$ terminates. Then*

- (1) $W(S)$ is right T -nilpotent,
- (2) $S/W(S)$ contains no infinite set of nonzero pairwise orthogonal idempotents.

Proof. This is a corollary of [2, Lemma 1.9]. $\square$

THEOREM 2.9. *Let M_R be APQ-injective with $S = \text{End}(M_R)$, then the following conditions are equivalent.*

- (1) S is right perfect.
- (2) For any sequence $\{s_1, s_2, \dots\} \subseteq S$, the chain $\ker(s_1) \subseteq \ker(s_2 s_1) \subseteq \dots$ terminates.

Proof. (1) $\Rightarrow$ (2). Let $s_i \in S, i = 1, 2, \dots$. Since S is right perfect, S satisfies DCC on principal left ideals. So the chain $Ss_1 \supseteq Ss_2 s_1 \supseteq \dots$ terminates. Thus there exists $n > 0$ such that $S(s_n \cdots s_1) = S(s_{n+1} s_n \cdots s_1) = \dots$. It follows that $\ker(s_n \cdots s_1) = \ker(s_{n+1} s_n \cdots s_1) = \dots$.

(2) $\Rightarrow$ (1). First we prove that $S/W(S)$ is von Neumann regular. Let $s_1 \notin W(S)$. Then $\ker(s_1)$ is not essential in M . By Lemma 2.7, there exists $t_1 \in S$ such that $\ker(s_1) \subset \ker(s_1 - s_1 t_1 s_1)$ is proper. Put $s_2 = s_1 - s_1 t_1 s_1$. If $s_2 \in W(S)$, then we have $\overline{s_1} = \overline{s_1} \cdot \overline{t_1} \cdot \overline{s_1}$ in the ring $S/W(S)$. If $s_2 \notin W(S)$, then there exists $s_3 \in S$ such that $\ker(s_2) \subset \ker(s_3)$ is proper, where $s_3 = s_2 - s_2 t_2 s_2$ for some $t_2 \in S$ by the preceding proof. Repeating the above process, we get a strictly ascending chain

$$\ker(s_1) \subset \ker(s_2) \subset \ker(s_3) \subset \dots, \quad (2.1)$$

where $s_{i+1} = s_i - s_i t_i s_i$ for some $t_i \in S, i = 1, 2, \dots$. Let $u_1 = s_1, u_2 = 1 - s_1 t_1, u_3 = 1 - s_2 t_2, \dots, u_{i+1} = 1 - s_i t_i, \dots$. Then $s_1 = u_1, s_2 = u_2 u_1, s_3 = u_3 u_2 u_1, \dots, s_{i+1} = u_{i+1} u_i \cdots u_2 u_1, \dots$,

whence we have the following strict ascending chain

$$\ker(u_1) \subset \ker(u_2 u_1) \subset \ker(u_3 u_2 u_1) \subset \cdots, \quad (2.2)$$

which contradicts the hypothesis. So there exists a positive integer n such that $s_{n+1} \in W(S)$. This shows that $\overline{s_n}$ is a regular element of $S/W(S)$, and hence $\overline{s_{n-1}}, \overline{s_{n-2}}, \dots, \overline{s_1}$ are regular elements of $S/W(S)$. Thus $S/W(S)$ is regular.

Note that since M_R is APQ-injective, $J(S) \subseteq W(S)$ by Theorem 2.5(1). Since the chain $\ker(s_1) \subseteq \ker(s_2 s_1) \subseteq \cdots$ terminates, by Lemma 2.8(1), $W(S)$ is right T -nilpotent, and so it follows that $W(S) \subseteq J(S)$, and thus $S/J(S)$ is regular. By Lemma 2.8, we get that S is right perfect. $\square$

By Lemma 2.8 (1) and [7, Remark 2], we have the following lemma.

LEMMA 2.10. *Let M be a right R -module with $S = \text{End}(M_R)$. If M_R satisfies ACC on $\{r_M(A) \mid A \subseteq S\}$, then $W(S)$ is nilpotent.*

The next corollary follows from Theorem 2.9 and Lemma 2.10.

COROLLARY 2.11. *Let M_R be APQ-injective with $S = \text{End}(M_R)$. If M_R satisfies ACC on $\{r_M(A) \mid A \subseteq S\}$, then S is semiprimary.*

For a module M_R , a submodule X of M is called a kernel submodule if $X = \ker(f)$ for some $f \in \text{End}(M_R)$, and X is called an annihilator submodule if $X = \bigcap_{f \in A} \ker(f)$ for some $A \subseteq \text{End}(M_R)$.

COROLLARY 2.12. *Let M_R be an APQ-injective module and $S = \text{End}(M_R)$. Then*

- (1) *if M_R satisfies ACC on kernel submodules, then S is right perfect,*
- (2) *if M_R satisfies ACC on annihilator submodules, then S is semiprimary.*

3. AQP-injective modules

In this section we study a generalization of quasiprincipally injective modules.

Definition 3.1. Let M be a right R -module with $S = \text{End}(M_R)$. Then M is said to be almost quasiprincipally injective (briefly, AQP-injective) if, for any $s \in S$, there exists a left ideal X_s of S such that $l_S(\ker(s)) = Ss \oplus X_s$ as left S -modules.

The next result gives the relationship between the AQP-injectivity of a module and the AP-injectivity of its endomorphism ring.

THEOREM 3.2. *Let M_R be a right R -module with $S = \text{End}(M_R)$. Then*

- (1) *if S is right AP-injective, then M_R is AQP-injective,*
- (2) *if M_R is AQP-injective and M generates $\ker(s)$ for each $s \in S$, then S is right AP-injective.*

Proof. (1) Let $s \in S$. Since S is right AP-injective, there exists a left ideal I_s such that $l_{Sr_S}(s) = Ss \oplus I_s$. If $a \in l_S(\ker(s))$ and $b \in r_S(s)$, then $sb = 0$, so $bM \subseteq \ker(s)$, and hence $abM = 0$, that is, $ab = 0$. It follows that $l_S(\ker(s)) \subseteq l_{Sr_S}(s)$. Thus, we have $Ss \subseteq l_S(\ker(s)) \subseteq Ss \oplus I_s$. This shows that $l_S(\ker(s)) = Ss \oplus l_S(\ker(s)) \cap I_s$, and (1) is proved.

(2) Let $0 \neq s \in S$. As M_R is AQP-injective, $l_S(\ker(s)) = Ss \oplus X_s$ for some left ideal X_s of S . Assume $a \in l_{Sr_S}(s)$. Since M generates $\ker(s)$, $\ker(s) = \sum_{t \in T} t(M)$ for some subset T of S . It is easy to see that $at = 0$ for each $t \in T$, thus $ax = 0$ for each $x \in \ker(s)$. This implies that $l_{Sr_S}(s) \subseteq l_S(\ker(s))$, from which we have $Ss \subseteq l_{Sr_S}(s) \subseteq Ss \oplus X_s$, and hence $l_{Sr_S}(s) = Ss \oplus (l_{Sr_S}(s) \cap X_s)$. Therefore, S is right AP-injective. $\square$

THEOREM 3.3. *Let M be a right R -module with $S = \text{End}(M_R)$. If M is an AQP-injective module which is a self-generator, then $J(S) = W(S)$.*

Proof. Let $s \in J(S)$. Then we will show that $s \in W(S)$. If not, then there exists a nonzero submodule K of M such that $\ker(s) \cap K = 0$. As M is a self-generator, $K = \sum_{t \in I} t(M)$ for some subset I of S , hence we have some $0 \neq t \in I$ such that $\ker(s) \cap t(M) = 0$. Clearly, $st \neq 0$ and $\ker(st) = \ker(t)$. Since M is AQP-injective, $l_S(\ker(st)) = S(st) \oplus X_{st}$ as left S -modules. Now $t \in l_S(\ker(t)) = l_S(\ker(st)) = S(st) \oplus X_{st}$. Write $t = u(st) + v$, where $u \in S$ and $v \in X_{st}$. Then $st - su(st) = sv \in S(st) \cap X_{st}$, hence $st - su(st) = 0$, that is, $(1 - su)st = 0$. Note that $1 - su$ is left invertible, so $st = 0$, a contradiction.

Conversely, let $s \in W(S)$. Then, for each $t \in S$, $ts \in W(S)$ and so $1 - ts \neq 0$. Since M_R is AQP-injective, $l_S(\ker(1 - ts)) = S(1 - ts) \oplus X_{1-ts}$ as left S -modules. Note that $\ker(ts) \cap \ker(1 - ts) = 0$, so we have $\ker(1 - ts) = 0$, thus $S = S(1 - ts) \oplus X_{1-ts}$, and then $1 = e + x$ for some $e \in S(1 - ts)$ and $x \in X$. It follows that $e^2 = e$ and $Se = S(1 - ts)$, and so $1 - ts = ue$ for some $u \in S$. Since $\ker(ts)$ is essential in M_R , if $e \neq 1$, there is a nonzero element $(1 - e)m \in (1 - e)M \cap \ker(ts)$. Then $(1 - ts)(1 - e)m = (1 - e)m$. But $(1 - ts)(1 - e)m = ue(1 - e)m = 0$. This is a contradiction. So $e = 1$ and hence $1 - ts$ is left invertible. The result follows. $\square$

Recall that a module M_R is said to satisfy the C_2 -condition if every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M . A module M is said to satisfy the C_3 -condition if whenever M_1 and M_2 are two summands of M and $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a summand of M . It is well known that the C_2 -condition implies the C_3 -condition.

THEOREM 3.4. *If M_R is an AQP-injective module, then it satisfies the C_2 -condition. In particular, right AP-injective rings are right C_2 -rings.*

Proof. Let A be a direct summand of M with $A \cong B$ and $S = \text{End}(M_R)$. Let $A = eM$, let $e^2 = e \in S$, and let $\varphi: eM \rightarrow B$ be an isomorphism. Then $B = bM$ with $b = se$ for some $s \in S$, and $\ker(e) = \ker(b)$. Thus, $e \in l_S(\ker(e)) = l_S(\ker(b)) = Sb \oplus X_b$ as M_R is AQP-injective, where X_b is a left S -module. Then $e = tb + x$ with $t \in S$ and $x \in X_b$. Hence we have $b = be = btb + bx$, and thus $b = btb$. Let $f = bt$. Then $f^2 = f$ and $bM = fM$. $\square$

COROLLARY 3.5. *Let M be a quasiprojective right R -module and let $S = \text{End}(M_R)$. Then S is regular if and only if M_R is AQP-injective and $\text{im}(s)$ are M -projective for every $s \in S$.*

Proof. By combining Theorems 3.2, 3.4, and [9, Theorem 37.7], one can complete the proof. $\square$

Recall that a ring R is called right P.P. if every principally right ideal of R is projective.

COROLLARY 3.6. *A ring R is regular if and only if R is right P.P. and right AP-injective.*

Following [3], a module M is said to be weakly injective if, for any finitely generated submodule $N \subseteq E(M)$, we have $N \subseteq X \cong M$ for some $X \subseteq E(M)$.

COROLLARY 3.7. *Let M_R be an f.g. module. If M is weakly injective and AQP-injective, then M is injective. In particular, if R is a right AP-injective and a right weakly injective ring, then R is right self-injective.*

Proof. Let $x \in E(M)$. Then there exists $X \subseteq E(M)$ such that $M + xR \subseteq X \cong M$, hence X is AQP-injective, and so $M \mid X$ by Theorem 3.4. This shows that $M = X$, so $x \in M$. $\square$

We let $S = \text{End}(M_R)$. Following [7], an element $u \in S$ is called a right uniform element of S if $u \neq 0$ and $u(M)$ is a uniform submodule of M . In the following, we generalize some results on maximal left ideals of the endomorphism rings of quasiprincipally injective modules and on maximal right ideals of right AP-injective rings to maximal left ideals of the endomorphism rings of AQP-injective modules.

LEMMA 3.8. *Let M_R be a module with $S = \text{End}(M_R)$. Given a set $\{X_s \mid s \in S\}$ of left ideals of S , the following are equivalent.*

- (1) $l_S(\ker(s)) = Ss \oplus X_s$ for all $s \in S$.
- (2) $l_S(tM \cap \ker(s)) = (X_{st} : t)_l + Ss$ and $(X_{st} : t)_l \cap Ss \subseteq l_S(t)$ for all $s, t \in S$, where $(X_{st} : t)_l = \{x \in S \mid xt \in X_{st}\}$.

Proof. (1) $\Rightarrow$ (2). Let $x \in l_S(tM \cap \ker(s))$. Then $\ker(st) \subseteq \ker(xt)$ and so $xt \in l_S(\ker(xt)) \subseteq l_S(\ker(st)) = S(st) \oplus X_{st}$. Write $xt = s_1(st) + y$, where $s_1 \in S$ and $y \in X_{st}$, then $(x - s_1s)t = y \in X_{st}$ and hence $x - s_1s \in (X_{st} : t)_l$. It follows that $x \in (X_{st} : t)_l + Ss$. Obviously, $Ss \subseteq l_S(tM \cap \ker(s))$. If $z \in (X_{st} : t)_l$, then $zt \in X_{st} \subseteq l_S(\ker(st))$. Let $tm \in tM \cap \ker(s)$, then $stm = 0$, hence $ztm = 0$. This shows that $z \in l_S(tM \cap \ker(s))$. Therefore, $l_S(tM \cap \ker(s)) = (X_{st} : t)_l + Ss$. If $s' \in (X_{st} : t)_l \cap Ss$, then $s'st \in X_{st} \cap S(st) = 0$, and thus $s's \in l_S(t)$.

(2) $\Rightarrow$ (1). Let $t = 1$. $\square$

LEMMA 3.9. *Let M_R be an AQP-injective module with $S = \text{End}(M_R)$ and an index set $\{X_s \mid s \in S\}$ of ideals such that $X_{st} = X_{ts}$ for all $s, t \in S$. If $0 \neq u(M)$ is a uniform submodule of M , define $M_u = \{s \in S \mid \ker(s) \cap u(M) \neq 0\}$. Then M_u is the unique maximal left ideal of S which contains $\sum_{s \in S} (X_{su} : u)_l$.*

Proof. It is easy to see that M_u is a left ideal. Let $t \in (X_{su} : u)_l$, then $tu \in X_{su}$, and thus $tus \in X_{su} \cap S(us) = X_{us} \cap S(us)$, since $X_{su} = X_{us}$ is an ideal. Then $tus = 0$ and so $t \in M_u$ if $us \neq 0$. If $us = 0$, then $l_S(\ker(us)) = 0$, and so $X_{su} = X_{us} = 0$. This shows that $tu = 0$ and hence $t \in M_u$. Consequently, $(X_{su} : u)_l \subseteq M_u$ for all $s \in S$. Now if $s \notin M_u$, then $\ker(s) \cap uM = 0$, and so $S = (X_{su} : u)_l + Ss$ by Lemma 3.8, hence $S = M_u + Ss$, showing that M_u is a maximal left ideal.

Finally, let L be a left ideal of S such that $\sum_{s \in S} (X_{su} : u)_l \subseteq L \neq M_u$. Then, as above, $S = (X_{su} : u)_l + Ss$ for any $s \in L - M_u$. Therefore, $L = S$. $\square$

LEMMA 3.10. *Let M_R be AQP-injective with $S = \text{End}(M_R)$ and an index set $\{X_s \mid s \in S\}$ of ideals such that $X_{st} = X_{ts}$ for all $s, t \in S$ and let $W = u_1M \oplus u_2M \oplus \cdots \oplus u_nM$ be a direct sum of uniform submodules u_iM of M , where each $u_i \in S$. If $T \subseteq S$ is a maximal left ideal*

not of the form M_u for any $u \in S$ such that uM is uniform, then there is $t \in T$ such that $\ker(1-t) \cap W$ is essential in W .

Proof. Since $T \neq M_{u_1}$, let $\ker(a) \cap u_1M = 0$, $a \in T$, then $\ker(au_1) \subseteq \ker(u_1)$, and so $u_1 \in l_S(\ker(au_1)) = S(au_1) \oplus X_{au_1}$. Thus, there exists $s \in S$ such that $(1-sa)u_1 \in X_{au_1}$, and so $1-sa \in (X_{au_1} : u_1)_l \subseteq M_{u_1}$. Let $a_1 = sa$. If $1-a_1 \in M_{u_i}$ for all i , we are done. If, say, $1-a_1 \notin M_2$, then $(1-a_1)u_2M$ is uniform (being isomorphic to u_2M), so, as above, $(1-a') \in M_{(1-a_1)u_2}$ for some $a' \in T$. Let $a_2 = a'ta_1 - a'a_1$, then $1-a_2 \in M_{u_1} \cap M_{u_2}$, continue in this way to obtain $t \in S$, such that $\ker(1-t) \cap u_iM \neq 0$ for each i , Lemma 3.10 follows. $\square$

THEOREM 3.11. *Let M_R be a self-generator with finite Goldie dimension and $S = \text{End}(M_R)$. If M_R is AQP-injective with an index set $\{X_s \mid s \in S\}$ of left ideals of S such that $X_{st} = X_{ts}$ for all $s, t \in S$, then*

(1) *if T is a maximal left ideal of S , then $T = M_u$ for some $u \in S$ such that uM is a uniform submodule of M ,*

(2) *$S/J(S)$ is semisimple.*

Proof. Since M is a self-generator, every uniform submodule of M contains an M -cyclic submodule. Therefore, we can assume that $W = u_1M \oplus u_2M \oplus \cdots \oplus u_nM$ is essential as M_R has finite Goldie dimension. If T is not of the form A_u for some right uniform element of $u \in S$, then by Lemma 3.10, there exists some $t \in T$ such that $\ker(1-t) \cap W$ is essential in W , so $\ker(1-t)$ is essential in M . By Theorem 3.3, $1-t \in J(S) \subseteq T$, a contradiction. This proves (1). As to (2), if $s \in M_{u_1} \cap \cdots \cap M_{u_n}$, then $\ker(s) \cap u_iM \neq 0$ for each i , whence $\ker(s)$ is essential in M . Hence, $s \in J(S)$, proving (2). $\square$

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Zhu Zhanmin: Department of Mathematics, Jiaxing University, Jiaxing, Zhejiang 314001, China
E-mail address: zhanmin_zhu@hotmail.com

Xia Zhangsheng: Department of Mathematics, Hubei Institute for Nationalities, Enshi, Hubei 445000, China
E-mail address: xzsw8577@163.com

Tan Zhisong: Department of Mathematics, Hubei Institute for Nationalities, Enshi, Hubei 445000, China
E-mail address: es_tzs@hotmail.com