ON TRANSFORMATION SEMIGROUPS WHICH ARE $\mathcal{BQ}$ -SEMIGROUPS

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A semigroup whose bi-ideals and quasi-ideals coincide is called a $\mathcal{BQ}$ -semigroup. The full transformation semigroup on a set X and the semigroup of all linear transformations of a vector space V over a field F into itself are denoted, respectively, by $T(X)$ and $L_F(V)$. It is known that every regular semigroup is a $\mathcal{BQ}$ -semigroup. Then both $T(X)$ and $L_F(V)$ are $\mathcal{BQ}$ -semigroups. In 1966, Magill introduced and studied the subsemigroup $\overline{T}(X, Y)$ of $T(X)$, where $\emptyset \neq Y \subseteq X$ and $\overline{T}(X, Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\}$. If W is a subspace of V , the subsemigroup $\overline{L}_F(V, W)$ of $L_F(V)$ will be defined analogously. In this paper, it is shown that $\overline{T}(X, Y)$ is a $\mathcal{BQ}$ -semigroup if and only if $Y = X$, $|Y| = 1$, or $|X| \leq 3$, and $\overline{L}_F(V, W)$ is a $\mathcal{BQ}$ -semigroup if and only if (i) $W = V$, (ii) $W = \{0\}$, or (iii) $F = \mathbb{Z}_2$, $\dim_F V = 2$, and $\dim_F W = 1$.

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1. Introduction

The cardinality of a set A is denoted by $|A|$. The image of a map α at x in the domain of α will be written by $x\alpha$.

An element a of a semigroup S is said to be *regular* if $a = aba$ for some $b \in S$, and S is called a *regular semigroup* if every element of S is regular. The set of all regular elements of S is denoted by $\text{Reg}(S)$.

The full transformation semigroup on a nonempty set X is denoted by $T(X)$, that is, $T(X)$ is the semigroup of all mappings $\alpha : X \rightarrow X$ under composition. The semigroup $T(X)$ is known to be regular [4, page 4]. Magill [9] introduced and studied the subsemigroup

$$\overline{T}(X, Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\} \quad (1.1)$$

of $T(X)$, where $\emptyset \neq Y \subseteq X$. Note that 1_X , the identity map on X , belongs to $\overline{T}(X, Y)$ and $\overline{T}(X, Y)$ contains $T(X, Y)$ as a subsemigroup, where $T(X, Y) = \{\alpha \in T(X) \mid \text{ran } \alpha \subseteq Y\}$ and $\text{ran } \alpha$ denotes the range of α . The semigroup $T(X, Y)$ was introduced and studied by Symons [13].

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For a vector space V over a field F , let $L_F(V)$ be the semigroup of all linear transformations $\alpha: V \rightarrow V$ under composition. It is known that $L_F(V)$ is a regular semigroup [5, page 63]. For a subspace W of V , we define the subsemigroup $\bar{L}_F(V, W)$ of $L_F(V)$ analogously, that is,

$$\bar{L}_F(V, W) = \{\alpha \in L_F(V) \mid W\alpha \subseteq W\}. \quad (1.2)$$

Clearly, $1_V \in \bar{L}_F(V, W)$ and 0 , the zero map on V , also belongs to $\bar{L}_F(V, W)$. In addition, $\bar{L}_F(V, W)$ contains $L_F(V, W) = \{\alpha \in L_F(V) \mid \text{ran } \alpha \subseteq W\}$ as a subsemigroup.

A subsemigroup Q of a semigroup S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$, and a *bi-ideal* of S is a subsemigroup B of S such that $BSB \subseteq B$. The notions of quasi-ideal and bi-ideal for semigroups were introduced by Steinfield [11] and Good and Hughes [3], respectively. Both quasi-ideals and bi-ideals are generalizations of one-sided ideals, and bi-ideals also generalize quasi-ideals. For a nonempty subset A of S , let $(A)_q$ and $(A)_b$ be the quasi-ideal and the bi-ideal of S generated by A , respectively, that is, $(A)_q[(A)_b]$ is the intersection of all quasi-ideals (bi-ideals) of S containing A [12, pages 10, 12]. Observe that $(A)_b \subseteq (A)_q$.

PROPOSITION 1.1 [2, pages 84, 85]. *For a nonempty subset A of a semigroup S ,*

- (i) $(A)_q = S^1 A \cap A S^1$,
- (ii) $(A)_b = A S^1 A \cup A$.

Kapp [6] used $\mathcal{BQ}$ to denote the class of all semigroups whose bi-ideals and quasi-ideals coincide and Mielke [10] called a semigroup in $\mathcal{BQ}$ a *$\mathcal{BQ}$ -semigroup*. Important $\mathcal{BQ}$ -semigroups are the following ones.

PROPOSITION 1.2 [8]. *Every regular semigroup is a $\mathcal{BQ}$ -semigroup.*

PROPOSITION 1.3 [6]. *Every left (right) simple semigroup or every left (right) 0-simple semigroup is a $\mathcal{BQ}$ -semigroup.*

Recall that a semigroup S is *left (right) simple* if S has no proper left (right) ideal, and a semigroup S with 0 is called *left (right) 0-simple* if $S^2 \neq \{0\}$ and S has no proper nonzero left (right) ideal. Kemprasit showed in [7] that if X is an infinite set, then the subsemigroup $\{\alpha \in T(X) \mid X \setminus \text{ran } \alpha \text{ is infinite}\}$ of $T(X)$ is a $\mathcal{BQ}$ -semigroup but it is neither regular nor left (right) simple. In fact, $\mathcal{BQ}$ -semigroups have been characterized by Calais [1] as follows.

PROPOSITION 1.4 [1]. *A semigroup S is a $\mathcal{BQ}$ -semigroup if and only if $(x, y)_b = (x, y)_q$ for all $x, y \in S$.*

Every bi-ideal of a regular semigroup is a $\mathcal{BQ}$ -semigroup. The proof is rather simple and is as follows: let T be a bi-ideal of a regular semigroup S and B a bi-ideal of T . Then $TST \subseteq T$ and $BTB \subseteq B$. Let $x \in TB \cap BT$. Since S is regular, $x = xsx$ for some $s \in S$ which implies that $x = xsx \in BTsTB \subseteq BTSTB \subseteq BTB \subseteq B$. Thus $TB \cap BT \subseteq B$. Hence B is a quasi-ideal of T , as desired. Since $T(X, Y)$ and $L_F(V, W)$ are left ideals of $T(X)$ and $L_F(V)$, respectively, it follows that $T(X, Y)$ and $L_F(V, W)$ are always $\mathcal{BQ}$ -semigroups. However, the semigroups $\bar{T}(X, Y)$ and $\bar{L}_F(V, W)$ need not be $\mathcal{BQ}$ -semigroups. Notice

that if X is infinite, then the semigroup $\{\alpha \in T(X) \mid X \setminus \text{ran } \alpha \text{ is infinite}\}$ is a left ideal of $T(X)$. Similarly, if V has infinite dimension over F , then the semigroup $\{\alpha \in L_F(V) \mid \dim_F(V/\text{ran } \alpha) \text{ is infinite}\}$ is a left ideal of $L_F(V)$.

In Section 2, we give a necessary and sufficient condition for $\overline{T}(X, Y)$ to be a $\mathcal{B}\mathcal{Q}$ -semigroup in terms of $|X|$ and $|Y|$. In Section 3, a necessary and sufficient condition for $\overline{L}_F(V, W)$ to be a $\mathcal{B}\mathcal{Q}$ -semigroup is given in terms of $|F|$, $\dim_F V$, and $\dim_F W$.

In the remainder, let X be a nonempty set, $\emptyset \neq Y \subseteq X$, V a vector space over a field F , and W a subspace of V .

2. The semigroup $\overline{T}(X, Y)$

We begin this section by characterizing regular elements of the semigroup $\overline{T}(X, Y)$. Then it is shown that $\overline{T}(X, Y)$ is a regular semigroup if and only if $Y = X$ or Y contains only one element.

PROPOSITION 2.1. *The following statements hold for the semigroup $\overline{T}(X, Y)$.*

- (i) *For $\alpha \in \overline{T}(X, Y)$, $\alpha \in \text{Reg}(\overline{T}(X, Y))$ if and only if $\text{ran } \alpha \cap Y = Y\alpha$.*
- (ii) *The semigroup $\overline{T}(X, Y)$ is regular if and only if either $Y = X$ or $|Y| = 1$.*

Proof. (i) Since $Y\alpha \subseteq Y$, we have $Y\alpha \subseteq \text{ran } \alpha \cap Y$. Assume that $\alpha = \alpha\beta\alpha$ for some $\beta \in \overline{T}(X, Y)$. If $x \in \text{ran } \alpha \cap Y$, then $x \in Y$ and $x = a\alpha$ for some $a \in X$ which imply that $x = a\alpha = a\alpha\beta\alpha = x\beta\alpha \in Y\beta\alpha \subseteq Y\alpha$. Hence we have $\text{ran } \alpha \cap Y = Y\alpha$.

Conversely, assume that $\text{ran } \alpha \cap Y = Y\alpha$. Then for each $x \in \text{ran } \alpha \cap Y$, we have $x\alpha^{-1} \cap Y \neq \emptyset$. We choose an element $x' \in x\alpha^{-1} \cap Y$ for each $x \in \text{ran } \alpha \cap Y$. Also, for $x \in \text{ran } \alpha \setminus Y$, choose an element $\bar{x} \in x\alpha^{-1}$. Then $x'\alpha = x$ for all $x \in \text{ran } \alpha \cap Y$ and $\bar{x}\alpha = x$ for all $x \in \text{ran } \alpha \setminus Y$. Let a be a fixed element in Y and define $\beta : X \rightarrow X$ by a bracket notation as follows:

$$\beta = \left[\begin{array}{cc|c} x & t & X \setminus \text{ran } \alpha \\ \hline x' & \bar{t} & a \end{array} \right]_{\substack{x \in \text{ran } \alpha \cap Y \\ t \in \text{ran } \alpha \setminus Y}} \quad (2.1)$$

Then $Y\beta \subseteq \{x' \mid x \in \text{ran } \alpha \cap Y\} \cup \{a\} \subseteq Y$, and for $x \in X$,

$$x\alpha\beta\alpha = (x\alpha)\beta\alpha = \begin{cases} (x\alpha)'\alpha = x\alpha & \text{if } x\alpha \in \text{ran } \alpha \cap Y, \\ (\bar{x}\alpha)\alpha = x\alpha & \text{if } x\alpha \in \text{ran } \alpha \setminus Y. \end{cases} \quad (2.2)$$

Hence $\beta \in \overline{T}(X, Y)$ and $\alpha = \alpha\beta\alpha$.

(ii) Suppose that $Y \subsetneq X$ and $|Y| > 1$. Let a and b be two distinct elements of Y . Define $\alpha : X \rightarrow X$ by

$$\alpha = \left[\begin{array}{cc} Y & X \setminus Y \\ a & b \end{array} \right]. \quad (2.3)$$

Then $\text{ran } \alpha = \{a, b\} \subseteq Y$, so $\alpha \in \overline{T}(X, Y)$ and $\text{ran } \alpha \cap Y = \{a, b\} \neq \{a\} = Y\alpha$. It follows from (i) that $\alpha \notin \text{Reg}(\overline{T}(X, Y))$. Hence $\overline{T}(X, Y)$ is not a regular semigroup.

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If $Y = X$, then $\overline{T}(X, Y) = T(X)$ which is regular. Next, assume that $Y = \{c\}$. In this case, $\overline{T}(X, Y)$ is isomorphic to the semigroup $P(X \setminus Y)$ consisting of all partial transformations of $X \setminus Y$, via the map $P(X \setminus Y) \rightarrow \overline{T}(X, Y)$, $\alpha \mapsto \overline{\alpha}$, where

$$\overline{\alpha} = \begin{bmatrix} x & X \setminus \text{dom } \alpha \\ x\alpha & c \end{bmatrix}_{x \in \text{dom } \alpha} \quad (2.4)$$

It is well known that $P(X \setminus Y)$ is regular [4, page 4]. Hence $\overline{T}(X, Y)$ is a regular semigroup, as required. $\square$

To characterize when $\overline{T}(X, Y)$ is a $\mathcal{BQ}$ -semigroup, Propositions 1.1, 1.2, 1.4, and 2.1 and the following three lemmas are needed.

LEMMA 2.2. *Let S be a semigroup. If $\emptyset \neq A \subseteq \text{Reg}(S)$, then $(A)_b = (A)_q$.*

Proof. We know that $(A)_b \subseteq (A)_q$. Let $x \in (A)_q$. By Proposition 1.1(i), $x = sa = bt$ for some $s, t \in S^1$ and $a, b \in A$. Since $a \in \text{Reg}(S)$, $a = aa'a$ for some $a' \in S$. Then

$$x = sa = saa'a = bta'a \in ASA \subseteq (A)_b \quad (2.5)$$

by Proposition 1.1(ii). Hence we have $(A)_b = (A)_q$, as desired. $\square$

LEMMA 2.3. *Let S be a semigroup, let $\emptyset \neq A \subseteq S$, and let $B \subseteq \text{Reg}(S)$. If $(A)_b = (A)_q$, then $(A \cup B)_b = (A \cup B)_q$.*

Proof. We first show that $S^1A \cap BS^1$ and $S^1B \cap AS^1$ are subsets of $(A \cup B)_b$. Let $x \in S^1A \cap BS^1$. Then $x = sa = bt$ for some $s, t \in S^1$, $a \in A$, and $b \in B$. Since $b \in \text{Reg}(S)$, $b = bb'b$ for some $b' \in S$. It follows that

$$x = bt = bb'bt = bb'sa \in BSA \subseteq (A \cup B)S(A \cup B) \subseteq (A \cup B)_b. \quad (2.6)$$

This shows that $S^1A \cap BS^1 \subseteq (A \cup B)_b$. It can be shown similarly that $S^1B \cap AS^1 \subseteq (A \cup B)_b$. Consequently,

$$\begin{aligned} (A \cup B)_q &= S^1(A \cup B) \cap (A \cup B)S^1 \\ &= (S^1A \cup S^1B) \cap (AS^1 \cup BS^1) \\ &= (S^1A \cap AS^1) \cup (S^1A \cap BS^1) \cup (S^1B \cap AS^1) \cup (S^1B \cap BS^1) \\ &= (A)_q \cup (S^1A \cap BS^1) \cup (S^1B \cap AS^1) \cup (B)_q \\ &= (A)_b \cup (S^1A \cap BS^1) \cup (S^1B \cap AS^1) \cup (B)_b, \\ &\quad \text{from the assumption and Lemma 2.2,} \\ &\subseteq (A)_b \cup (A \cup B)_b \cup (A \cup B)_b \cup (B)_b = (A \cup B)_b. \end{aligned} \quad (2.7)$$

But $(A \cup B)_b \subseteq (A \cup B)_q$, so $(A \cup B)_b = (A \cup B)_q$. $\square$

LEMMA 2.4. If $|X| = 3$ and $|Y| = 2$, then for all $\alpha, \beta \in \overline{T}(X, Y)$, $(\alpha, \beta)_b = (\alpha, \beta)_q$ in $\overline{T}(X, Y)$.

Proof. For convenience, let X_a denote the constant map whose domain and range are X and $\{a\}$, respectively.

Assume that $X = \{a, b, c\}$ and $Y = \{a, b\}$. Clearly,

$$\begin{aligned} \overline{T}(X, Y) = \bigg\{ 1_X, X_a, X_b, \begin{bmatrix} a & b & c \\ a & a & b \end{bmatrix}, \begin{bmatrix} a & b & c \\ a & a & c \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & b & a \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & b & c \end{bmatrix}, \\ \begin{bmatrix} a & b & c \\ a & b & a \end{bmatrix}, \begin{bmatrix} a & b & c \\ a & b & b \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & a & a \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & a & b \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & a & c \end{bmatrix} \bigg\}. \end{aligned} \quad (2.8)$$

By Proposition 2.1(i), $\overline{T}(X, Y) \setminus \text{Reg}(\overline{T}(X, Y)) = \{ \begin{bmatrix} a & b & c \\ a & a & b \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & b & a \end{bmatrix} \}$. Let $\lambda = \begin{bmatrix} a & b & c \\ a & a & b \end{bmatrix}$ and $\eta = \begin{bmatrix} a & b & c \\ b & b & a \end{bmatrix}$. Note that $\lambda^2 = X_a = \eta\lambda$ and $\eta^2 = X_b = \lambda\eta$. To show that $(\alpha, \beta)_b = (\alpha, \beta)_q$ for all $\alpha, \beta \in \overline{T}(X, Y)$, by Lemma 2.3, it suffices to show that $(\lambda)_b = (\lambda)_q$, $(\eta)_b = (\eta)_q$, and $(\lambda, \eta)_b = (\lambda, \eta)_q$. By direct multiplication, we have

$$\begin{aligned} \overline{T}(X, Y)\lambda &= \{\lambda, X_a\}, & \lambda\overline{T}(X, Y) &= \{\lambda, X_a, X_b, \eta\}, & \lambda\overline{T}(X, Y)\lambda &= \{X_a\}, \\ \overline{T}(X, Y)\eta &= \{\eta, X_b\}, & \eta\overline{T}(X, Y) &= \{\eta, X_a, X_b, \lambda\}, & \eta\overline{T}(X, Y)\eta &= \{X_b\}, \\ \lambda\overline{T}(X, Y)\eta &= \{X_b\}, & \eta\overline{T}(X, Y)\lambda &= \{X_a\}. \end{aligned} \quad (2.9)$$

Hence

$$\begin{aligned} (\lambda)_b &= \lambda\overline{T}(X, Y)\lambda \cup \{\lambda\} = \{X_a, \lambda\} = \overline{T}(X, Y)\lambda \cap \lambda\overline{T}(X, Y) = (\lambda)_q, \\ (\eta)_b &= \eta\overline{T}(X, Y)\eta \cup \{\eta\} = \{X_b, \eta\} = \overline{T}(X, Y)\eta \cap \eta\overline{T}(X, Y) = (\eta)_q, \\ (\lambda, \eta)_b &= \{\lambda, \eta\}\overline{T}(X, Y)\{\lambda, \eta\} \cup \{\lambda, \eta\} \\ &= \lambda\overline{T}(X, Y)\lambda \cup \lambda\overline{T}(X, Y)\eta \cup \eta\overline{T}(X, Y)\lambda \cup \eta\overline{T}(X, Y)\eta \cup \{\lambda, \eta\} \\ &= \{X_a, X_b, \lambda, \eta\}, \\ (\lambda, \eta)_q &= \overline{T}(X, Y)\{\lambda, \eta\} \cap \{\lambda, \eta\}\overline{T}(X, Y) \\ &= (\overline{T}(X, Y)\lambda \cup \overline{T}(X, Y)\eta) \cap (\lambda\overline{T}(X, Y) \cup \eta\overline{T}(X, Y)) \\ &= \{\lambda, X_a, \eta, X_b\} = (\lambda, \eta)_b. \end{aligned} \quad (2.10)$$

□

THEOREM 2.5. The semigroup $\overline{T}(X, Y)$ is a $\mathcal{B}\mathcal{Q}$ -semigroup if and only if one of the following statements holds.

- (i) $Y = X$.
- (ii) $|Y| = 1$.
- (iii) $|X| \leq 3$.

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Proof. Assume that (i), (ii), and (iii) are false. Then $X \setminus Y \neq \emptyset$, $|Y| > 1$, and $|X| > 3$.

Case 1 ($(|Y| = 2)$). Let $Y = \{a, b\}$. Since $|X| > 3$, $|X \setminus Y| > 1$. Let $c \in X \setminus Y$. Then $X \setminus \{a, b, c\} \neq \emptyset$. Define $\alpha, \beta, \gamma \in \overline{T}(X, Y)$ by

$$\alpha = \begin{bmatrix} a & b & c & X \setminus \{a, b, c\} \\ b & b & a & c \end{bmatrix}, \quad \beta = \begin{bmatrix} c & x \\ a & x \end{bmatrix}_{x \in X \setminus \{c\}}, \quad \gamma = \begin{bmatrix} a & b & X \setminus \{a, b\} \\ b & b & c \end{bmatrix}. \quad (2.11)$$

Then $\alpha\alpha\beta = b = a\gamma\alpha$, $b\alpha\beta = b = b\gamma\alpha$, $c\alpha\beta = a = c\gamma\alpha$, and $(X \setminus \{a, b, c\})\alpha\beta = \{a\} = (X \setminus \{a, b, c\})\gamma\alpha \neq (X \setminus \{a, b, c\})\alpha$, so $\alpha \neq \alpha\beta = \gamma\alpha \in (\alpha)_q$ by Proposition 1.1(i). If $\alpha\beta \in (\alpha)_b$, then by Proposition 1.1(ii), $\alpha\beta = \alpha\eta\alpha$ for some $\eta \in \overline{T}(X, Y)$. Hence we have $a = c\alpha\beta = c\alpha\eta\alpha = (a\eta)\alpha$. This implies that $a\eta = c$ which is contrary to $a \in Y$ and $c \in X \setminus Y$. Thus $(\alpha)_b \neq (\alpha)_q$, so by Proposition 1.4, $\overline{T}(X, Y)$ is not a $\mathcal{BQ}$ -semigroup.

Case 2 ($(|Y| > 2)$). Let a, b, c be distinct elements of Y . Let $\alpha, \beta, \gamma \in \overline{T}(X, Y)$ be defined by

$$\alpha = \begin{bmatrix} a & Y \setminus \{a\} & X \setminus Y \\ b & a & c \end{bmatrix}, \quad \beta = \begin{bmatrix} a & b & x \\ b & a & x \end{bmatrix}_{x \in X \setminus \{a, b\}}, \quad (2.12)$$

$$\gamma = \begin{bmatrix} a & Y \setminus \{a\} & x \\ c & a & x \end{bmatrix}_{x \in X \setminus Y}.$$

Then $\alpha\alpha\beta = a = a\gamma\alpha \neq a\alpha$, $(Y \setminus \{a\})\alpha\beta = \{b\} = (Y \setminus \{a\})\gamma\alpha$, and $(X \setminus Y)\alpha\beta = \{c\} = (X \setminus Y)\gamma\alpha$. Thus $\alpha \neq \alpha\beta = \gamma\alpha \in (\alpha)_q$. If $\alpha\beta \in (\alpha)_b$, then $\alpha\beta = \alpha\eta\alpha$ for some $\eta \in \overline{T}(X, Y)$. Therefore we have for every $x \in X \setminus Y$, $c = x\alpha\beta = x\alpha\eta\alpha = (c\eta)\alpha$ which implies that $c\eta \in X \setminus Y$. This is a contradiction since $c \in Y$. Hence $(\alpha)_b \neq (\alpha)_q$, and so by Proposition 1.4, $\overline{T}(X, Y)$ is not a $\mathcal{BQ}$ -semigroup.

If $Y = X$ or $|Y| = 1$, then $\overline{T}(X, Y)$ is regular by Proposition 2.1(ii) which implies by Proposition 1.2 that $\overline{T}(X, Y)$ is a $\mathcal{BQ}$ -semigroup. If $|X| = 3$ and $|Y| = 2$, then by Lemma 2.4 and Proposition 1.4, $\overline{T}(X, Y)$ is a $\mathcal{BQ}$ -semigroup.

Hence the theorem is completely proved. $\square$

Two direct consequences of Propositions 1.2, 2.1(ii), Theorem 2.5, and the proof of Lemma 2.4 are as follows.

COROLLARY 2.6. *If $|X| \neq 3$, then the following statements are equivalent.*

- (i) $\overline{T}(X, Y)$ is a $\mathcal{BQ}$ -semigroup.
- (ii) $Y = X$ or $|Y| = 1$.
- (iii) $\overline{T}(X, Y)$ is a regular semigroup.

COROLLARY 2.7. *The semigroup $\overline{T}(X, Y)$ is a nonregular $\mathcal{BQ}$ -semigroup if and only if $|X| = 3$ and $|Y| = 2$. Hence for each set X with $|X| = 3$, there are exactly 3 semigroups $\overline{T}(X, Y)$ which are nonregular $\mathcal{BQ}$ -semigroups, and each of such $\overline{T}(X, Y)$ contains 12 elements.*

Remark 2.8. We have mentioned that $T(X, Y)$ is a left ideal of $T(X)$. But for $\alpha \in T(X, Y)$ and $\beta \in \overline{T}(X, Y)$, $X\alpha\beta \subseteq Y\beta \subseteq Y$, so $T(X, Y)$ is an ideal of $\overline{T}(X, Y)$. We have $1_X \in \overline{T}(X, Y) \setminus T(X, Y)$ if $Y \neq X$. Hence if $Y \neq X$, then $\overline{T}(X, Y)$ is neither left nor right simple.

Therefore we deduce from Corollary 2.7 that if $|X| = 3$ and $|Y| = 2$, then $\overline{T}(X, Y)$ is an example of $\mathcal{BQ}$ -semigroup which is neither regular nor left (right) simple (see Propositions 1.2 and 1.3).

3. The semigroup $\overline{L}_F(V, W)$

In this section, we give a necessary and sufficient condition for $\overline{L}_F(V, W)$ to be a $\mathcal{BQ}$ -semigroup. We first provide the conditions of the regularity of elements of $\overline{L}_F(V, W)$ and of the semigroup $\overline{L}_F(V, W)$. The following facts about vector spaces and linear transformations will be used. If U_1 and U_2 are subspaces of V , B_1 is a basis of the subspace $U_1 \cap U_2$, $B_2 \subseteq U_1$ and $B_3 \subseteq U_2$ are such that $B_1 \cup B_2$ and $B_1 \cup B_3$ are bases of U_1 and U_2 , respectively, then $B_1 \cup B_2 \cup B_3$ is a basis of the subspace $U_1 + U_2$ of V . If $\alpha \in L_F(V)$, B_1 is a basis of $\ker \alpha$, B_2 is a basis of $\text{ran } \alpha$, and choose an element $u' \in u\alpha^{-1}$ for every $u \in B_2$, then $B_1 \cup \{u' \mid u \in B_2\}$ is a basis of V .

PROPOSITION 3.1. *The following statements hold for the semigroup $\overline{L}_F(V, W)$.*

- (i) *For $\alpha \in \overline{L}_F(V, W)$, $\alpha \in \text{Reg}(\overline{L}_F(V, W))$ if and only if $\text{ran } \alpha \cap W = W\alpha$.*
- (ii) *The semigroup $\overline{L}_F(V, W)$ is regular if and only if either $W = V$ or $W = \{0\}$.*

Proof. (i) The proof that $\alpha \in \text{Reg}(\overline{L}_F(V, W))$ implies $\text{ran } \alpha \cap W = W\alpha$ is analogous to the proof of the “only if” part of Proposition 2.1(i).

Conversely, assume that $\text{ran } \alpha \cap W = W\alpha$. Let B_1 be a basis of $\text{ran } \alpha \cap W$, $B_2 \subseteq \text{ran } \alpha \setminus B_1$, and $B_3 \subseteq W \setminus B_1$ such that $B_1 \cup B_2$ and $B_1 \cup B_3$ are bases of $\text{ran } \alpha$ and W , respectively. Then $B_1 \cup B_2 \cup B_3$ is a basis of $\text{ran } \alpha + W$. Let $B_4 \subseteq V \setminus (B_1 \cup B_2 \cup B_3)$ be such that $B_1 \cup B_2 \cup B_3 \cup B_4$ is a basis of V . Since $B_1 \subseteq \text{ran } \alpha \cap W = W\alpha$, we have $u\alpha^{-1} \cap W \neq \emptyset$ for every $u \in B_1$. For each $u \in B_1$, choose an element $u' \in u\alpha^{-1} \cap W$. Since $B_2 \subseteq \text{ran } \alpha$, for each $v \in B_2$, $v\alpha^{-1} \neq \emptyset$, so choose an element $\bar{v} \in v\alpha^{-1}$. Define $\beta \in L_F(V)$ on the basis $B_1 \cup B_2 \cup B_3 \cup B_4$ by

$$\beta = \begin{bmatrix} u & v & B_3 \cup B_4 \\ u' & \bar{v} & 0 \end{bmatrix}_{\substack{u \in B_1 \\ v \in B_2}} \quad (3.1)$$

It follows that $W\beta = \langle B_1 \cup B_3 \rangle \beta = \langle \{u' \mid u \in B_1\} \rangle \subseteq W$, so $\beta \in \overline{L}_F(V, W)$. Let B_0 be a basis of $\ker \alpha$. Then $B_0 \cup \{u' \mid u \in B_1\} \cup \{\bar{v} \mid v \in B_2\}$ is a basis of V . Since

$$\begin{aligned} B_0\alpha\beta\alpha &= \{0\} = B_0\alpha, & u'\alpha\beta\alpha &= u\beta\alpha = u'\alpha \quad \forall u \in B_1, \\ \bar{v}\alpha\beta\alpha &= v\beta\alpha = \bar{v}\alpha \quad \forall v \in B_2, \end{aligned} \quad (3.2)$$

we have $\alpha = \alpha\beta\alpha$, so α is a regular element of $\overline{L}_F(V, W)$.

(ii) Assume that $\{0\} \neq W \subsetneq V$. Let B_1 be a basis of W and B a basis of V containing B_1 . Then $B_1 \neq \emptyset \neq B \setminus B_1$. Let $w \in B_1$ and $u \in B \setminus B_1$. Define $\alpha \in L_F(V)$ by

$$\alpha = \begin{bmatrix} u & B \setminus \{u\} \\ w & 0 \end{bmatrix}. \quad (3.3)$$

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Then $W\alpha = \langle B_1 \rangle \alpha \subseteq \langle B \setminus \{u\} \rangle \alpha = \{0\}$, so $\alpha \in \bar{L}_F(V, W)$. Since $\text{ran } \alpha \cap W = \langle w \rangle \neq \{0\} = W\alpha$, by (i), we deduce that α is not a regular element of $\bar{L}_F(V, W)$. Hence $\bar{L}_F(V, W)$ is not a regular semigroup.

Since $\bar{L}_F(V, V) = L_F(V) = \bar{L}_F(V, \{0\})$, the converse holds. $\square$

To prove the main theorem, the following lemma is also needed. Lemma 2.3 and Proposition 3.1(i) are useful to obtain this result.

LEMMA 3.2. *If $F = \mathbb{Z}_2$, $\dim_F V = 2$, and $\dim_F W = 1$, then for all $\alpha, \beta \in \bar{L}_F(V, W)$, $(\alpha, \beta)_b = (\alpha, \beta)_q$ in $\bar{L}_F(V, W)$.*

Proof. Let $\{w\}$ be a basis of W and $\{w, u\}$ a basis of V . Since $F = \mathbb{Z}_2$, it follows that $W = \{0, w\}$ and $V = \{0, w, u, u + w\}$. Clearly, both $\{u, u + w\}$ and $\{w, u + w\}$ are also bases of V . Thus $\langle w \rangle \cap \langle u \rangle = \langle w \rangle \cap \langle u + w \rangle = \langle u \rangle \cap \langle u + w \rangle = \{0\}$. All elements of $\bar{L}_F(V, W)$ defined on the basis $\{w, u\}$ of V can be given as follows:

$$\begin{aligned} \bar{L}_F(V, W) = \left\{ 0, 1_V, \begin{bmatrix} w & u \\ 0 & w \end{bmatrix}, \begin{bmatrix} w & u \\ 0 & u \end{bmatrix}, \begin{bmatrix} w & u \\ 0 & w+u \end{bmatrix}, \right. \\ \left. \begin{bmatrix} w & u \\ w & 0 \end{bmatrix}, \begin{bmatrix} w & u \\ w & w \end{bmatrix}, \begin{bmatrix} w & u \\ w & w+u \end{bmatrix} \right\}. \end{aligned} \quad (3.4)$$

By Proposition 3.1(i), $\bar{L}_F(V, W) \setminus \text{Reg}(\bar{L}_F(V, W)) = \{ \begin{bmatrix} w & u \\ 0 & w \end{bmatrix} \}$. Let $\lambda = \begin{bmatrix} w & u \\ 0 & w \end{bmatrix}$. Note that $\lambda^2 = 0$. To prove the lemma, by Lemma 2.3, it suffices to show that $(\lambda)_b = (\lambda)_q$. By direct multiplication, we have

$$\bar{L}_F(V, W)\lambda = \{0, \lambda\}, \quad \lambda\bar{L}_F(V, W) = \{0, \lambda\}, \quad \lambda\bar{L}_F(V, W)\lambda = \{0\}. \quad (3.5)$$

Consequently, $(\lambda)_b = \lambda\bar{L}_F(V, W)\lambda \cup \{\lambda\} = \{0, \lambda\} = \bar{L}_F(V, W)\lambda \cap \lambda\bar{L}_F(V, W) = (\lambda)_q$. $\square$

THEOREM 3.3. *The semigroup $\bar{L}_F(V, W)$ is a $\mathcal{BQ}$ -semigroup if and only if one of the following statements holds.*

- (i) $W = V$.
- (ii) $W = \{0\}$.
- (iii) $F = \mathbb{Z}_2$, $\dim_F V = 2$, and $\dim_F W = 1$.

Proof. Assume that (i), (ii), and (iii) are false. Then (1) $\{0\} \neq W \subsetneq V$ and (2) $F \neq \mathbb{Z}_2$, $\dim_F V > 2$, or $\dim_F W > 1$. Let B_1 be a basis of W and B a basis of V containing B_1 . Then $B_1 \neq \emptyset$ and $B \setminus B_1 \neq \emptyset$.

Case 1 ($(F \neq \mathbb{Z}_2)$). Let $a \in F \setminus \{0, 1\}$, $w \in B_1$, and $u \in B \setminus B_1$. Define $\alpha, \beta, \gamma \in \bar{L}_F(V, W)$ by

$$\alpha = \begin{bmatrix} u & B \setminus \{u\} \\ w & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} w & B \setminus \{w\} \\ aw & 0 \end{bmatrix}, \quad \gamma = \begin{bmatrix} u & B \setminus \{u\} \\ au & 0 \end{bmatrix}. \quad (3.6)$$

Then we have $\alpha\beta = \begin{bmatrix} u & B \setminus \{u\} \\ aw & 0 \end{bmatrix} = \gamma\alpha$. Since $a \neq 1$, we have $\alpha\beta \neq \alpha$. By Proposition 1.1(i), $\alpha\beta \in \bar{L}_F(V, W)(\alpha)_q$. Suppose that $\alpha\beta \in \bar{L}_F(V, W)(\alpha)_b$. By Proposition 1.1(ii), $\alpha\beta = \alpha\eta\alpha$

for some $\eta \in \bar{L}_F(V, W)$. Then $aw = u\alpha\beta = u\alpha\eta\alpha = (w\eta)\alpha$. But $w\eta \in W$ and $W\alpha = \langle B_1 \rangle\alpha \subseteq \langle B \setminus \{u\} \rangle\alpha = \{0\}$, so $aw = 0$ which is contrary to $a \neq 0$. Thus $(\alpha)_q \neq (\alpha)_b$, so $\bar{L}_F(V, W)$ is not a $\mathcal{BQ}$ -semigroup by Proposition 1.4.

Case 2 ($(\dim_F W > 1)$). Then $|B_1| > 1$. Let $w_1, w_2 \in B_1$ be such that $w_1 \neq w_2$ and $u \in B \setminus B_1$. Define $\alpha, \beta, \gamma \in \bar{L}_F(V, W)$ by

$$\alpha = \begin{bmatrix} w_1 & u & B \setminus \{w_1, u\} \\ w_2 & w_1 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} w_1 & B \setminus \{w_1\} \\ w_1 & 0 \end{bmatrix}, \quad \gamma = \begin{bmatrix} u & B \setminus \{u\} \\ u & 0 \end{bmatrix}. \quad (3.7)$$

Then $\alpha\beta = \begin{bmatrix} u & B \setminus \{u\} \\ w_1 & 0 \end{bmatrix} = \gamma\alpha \neq \alpha$, so $\alpha\beta \in (\alpha)_q$. If $\alpha\beta \in (\alpha)_b$, then $\alpha\beta = \alpha\eta\alpha$ for some $\eta \in \bar{L}_F(V, W)$. Thus $w_1 = u\alpha\beta = u\alpha\eta\alpha = (w_1\eta)\alpha$. Since $w_1\eta \in W = \langle B_1 \rangle$, we have $w_1\eta = aw_1 + v$ for some $a \in F$ and $v \in \langle B_1 \setminus \{w_1\} \rangle$. But $B_1 \setminus \{w_1\} \subseteq B \setminus \{w_1, u\}$, so $v\alpha = 0$. Consequently, $w_1 = (aw_1 + v)\alpha = aw_2$ which is contrary to the independence of w_1 and w_2 . By Proposition 1.4, $\bar{L}_F(V, W)$ is not a $\mathcal{BQ}$ -semigroup.

Case 3 ($(\dim_F V > 2 \text{ and } \dim_F W = 1)$). Then $|B_1| = 1$ and $|B \setminus B_1| > 1$. Let $B_1 = \{w\}$ and $u_1, u_2 \in B \setminus B_1$ be such that $u_1 \neq u_2$. Let $\alpha, \beta, \gamma \in \bar{L}_F(V, W)$ be defined by

$$\alpha = \begin{bmatrix} u_1 & u_2 & B \setminus \{u_1, u_2\} \\ w & u_1 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} w & B \setminus \{w\} \\ w & 0 \end{bmatrix}, \quad \gamma = \begin{bmatrix} u_1 & B \setminus \{u_1\} \\ u_1 & 0 \end{bmatrix}. \quad (3.8)$$

Then we have $\alpha\beta = \begin{bmatrix} u_1 & B \setminus \{u_1\} \\ w & 0 \end{bmatrix} = \gamma\alpha \neq \alpha$, so $\alpha\beta \in (\alpha)_q$. Suppose that $\alpha\beta \in (\alpha)_b$. It follows that $\alpha\beta = \alpha\eta\alpha$ for some $\eta \in \bar{L}_F(V, W)$. Thus $w = u_1\alpha\beta = u_1\alpha\eta\alpha = (w\eta)\alpha$. But $w\eta \in W = \langle w \rangle$ and $w\alpha = 0$, so $w = (w\eta)\alpha = 0$, a contradiction. Hence $(\alpha)_q \neq (\alpha)_b$, so $\bar{L}_F(V, W)$ is not a $\mathcal{BQ}$ -semigroup, as before.

For the converse, if (i) or (ii) holds, then $\bar{L}_F(V, W) = L_F(V)$ which is a $\mathcal{BQ}$ -semigroup by Proposition 1.2. If (iii) holds, then $\bar{L}_F(V, W)$ is a $\mathcal{BQ}$ -semigroup by Proposition 1.4 and Lemma 3.2. $\square$

The following corollaries follow directly from Propositions 1.2, 3.1(ii), Theorem 3.3, and the proof of Lemma 3.2.

COROLLARY 3.4. *If $F \neq \mathbb{Z}_2$ or $\dim_F V \neq 2$, then the following statements are equivalent.*

- (i) $\bar{L}_F(V, W)$ is a $\mathcal{BQ}$ -semigroup.
- (ii) $W = V$ or $W = \{0\}$.
- (iii) $\bar{L}_F(V, W)$ is a regular semigroup.

COROLLARY 3.5. *The semigroup $\bar{L}_F(V, W)$ is a nonregular $\mathcal{BQ}$ -semigroup if and only if $F = \mathbb{Z}_2$, $\dim_F V = 2$, and $\dim_F W = 1$. Hence if $F = \mathbb{Z}_2$ and $\dim_F V = 2$, there are exactly 3 semigroups $\bar{L}_F(V, W)$ which are nonregular $\mathcal{BQ}$ -semigroups, and each of such $\bar{L}_F(V, W)$ contains 8 elements.*

Remark 3.6. We also have that $L_F(V, W)$ is an ideal of $\bar{L}_F(V, W)$ (see Remark 2.8). Consequently, if $\{0\} \neq W \subsetneq V$, then $\bar{L}_F(V, W)$ is neither left nor right 0-simple. Hence if $F = \mathbb{Z}_2$, $\dim_F V = 2$, and $\dim_F W = 1$, then $\bar{L}_F(V, W)$ is a $\mathcal{BQ}$ -semigroup which is neither regular nor left (right) 0-simple.

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