ON TRANSFORMATION SEMIGROUPS WHICH ARE $\mathcal{B}_2$-SEMIGROUPS

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A semigroup whose bi-ideals and quasi-ideals coincide is called a $\mathcal{B}_2$-semigroup. The full transformation semigroup on a set $X$ and the semigroup of all linear transformations of a vector space $V$ over a field $F$ into itself are denoted, respectively, by $T(X)$ and $L_F(V)$. It is known that every regular semigroup is a $\mathcal{B}_2$-semigroup. Then both $T(X)$ and $L_F(V)$ are $\mathcal{B}_2$-semigroups. In 1966, Magill introduced and studied the subsemigroup $T(X, Y)$ of $T(X)$, where $\emptyset \neq Y \subseteq X$ and $T(X, Y) = \{ \alpha \in T(X) \mid Y \alpha \subseteq Y \}$. If $W$ is a subspace of $V$, the subsemigroup $L_F(V, W)$ of $L_F(V)$ will be defined analogously. In this paper, it is shown that $T(X, Y)$ is a $\mathcal{B}_2$-semigroup if and only if $Y = X$, $|Y| = 1$, or $|X| \leq 3$, and $L_F(V, W)$ is a $\mathcal{B}_2$-semigroup if and only if (i) $W = V$, (ii) $W = \{0\}$, or (iii) $F = \mathbb{Z}_2$, $\dim_F V = 2$, and $\dim_F W = 1$.

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1. Introduction

The cardinality of a set $A$ is denoted by $|A|$. The image of a map $\alpha$ at $x$ in the domain of $\alpha$ will be written by $x\alpha$.

An element $a$ of a semigroup $S$ is said to be regular if $a = aba$ for some $b \in S$, and $S$ is called a regular semigroup if every element of $S$ is regular. The set of all regular elements of $S$ is denoted by $\text{Reg}(S)$.

The full transformation semigroup on a nonempty set $X$ is denoted by $T(X)$, that is, $T(X)$ is the semigroup of all mappings $\alpha : X \to X$ under composition. The semigroup $T(X)$ is known to be regular [4, page 4]. Magill [9] introduced and studied the subsemigroup

$$T(X, Y) = \{ \alpha \in T(X) \mid Y \alpha \subseteq Y \}$$

(1.1)

of $T(X)$, where $\emptyset \neq Y \subseteq X$. Note that $1_X$, the identity map on $X$, belongs to $T(X, Y)$ and $T(X, Y)$ contains $T(X, Y)$ as a subsemigroup, where $T(X, Y) = \{ \alpha \in T(X) \mid \text{ran} \alpha \subseteq Y \}$ and $\text{ran} \alpha$ denotes the range of $\alpha$. The semigroup $T(X, Y)$ was introduced and studied by Symons [13].
On transformation semigroups which are \( \mathcal{B}2 \)-semigroups

For a vector space \( V \) over a field \( F \), let \( L_F(V) \) be the semigroup of all linear transformations \( \alpha : V \to V \) under composition. It is known that \( L_F(V) \) is a regular semigroup [5, page 63]. For a subspace \( W \) of \( V \), we define the subsemigroup \( L_F(V, W) \) of \( L_F(V) \) analogously, that is,

\[
\overline{L}_F(V, W) = \{ \alpha \in L_F(V) \mid W\alpha \subseteq W \}.
\]

Clearly, \( 1_V \in \overline{L}_F(V, W) \) and \( 0 \), the zero map on \( V \), also belongs to \( \overline{L}_F(V, W) \). In addition, \( \overline{L}_F(V, W) \) contains \( L_F(V, W) = \{ \alpha \in L_F(V) \mid \text{ran} \alpha \subseteq W \} \) as a subsemigroup.

A subsemigroup \( Q \) of a semigroup \( S \) is called a quasi-ideal of \( S \) if \( SQ \cap QS \subseteq Q \), and a bi-ideal of \( S \) is a subsemigroup \( B \) of \( S \) such that \( BSB \subseteq B \). The notions of quasi-ideal and bi-ideal for semigroups were introduced by Steinfield [11] and Good and Hughes [3], respectively. Both quasi-ideals and bi-ideals are generalizations of one-sided ideals, and bi-ideals also generalize quasi-ideals. For a nonempty subset \( A \) of \( S \), let \( (A)_q \) and \( (A)_b \) be the quasi-ideal and the bi-ideal of \( S \) generated by \( A \), respectively, that is, \( (A)_q \) is the intersection of all quasi-ideals (bi-ideals) of \( S \) containing \( A \) [12, pages 10, 12]. Observe that \( (A)_b \subseteq (A)_q \).

**Proposition 1.1** [2, pages 84, 85]. For a nonempty subset \( A \) of a semigroup \( S \),

(i) \( (A)_q = S^1A \cap AS^1 \),

(ii) \( (A)_b = AS^1A \cup A \).

Kapp [6] used \( \mathcal{B}2 \) to denote the class of all semigroups whose bi-ideals and quasi-ideals coincide and Mielke [10] called a semigroup in \( \mathcal{B}2 \) a \( \mathcal{B}2 \)-semigroup. Important \( \mathcal{B}2 \)-semigroups are the following ones.

**Proposition 1.2** [8]. Every regular semigroup is a \( \mathcal{B}2 \)-semigroup.

**Proposition 1.3** [6]. Every left (right) simple semigroup or every left (right) 0-simple semigroup is a \( \mathcal{B}2 \)-semigroup.

Recall that a semigroup \( S \) is left (right) simple if \( S \) has no proper left (right) ideal, and a semigroup \( S \) with \( 0 \) is called left (right) 0-simple if \( S^2 \neq \{0\} \) and \( S \) has no proper nonzero left (right) ideal. Kemprasit showed in [7] that if \( X \) is an infinite set, then the subsemigroup \( \{ \alpha \in T(X) \mid X \setminus \text{ran} \alpha \text{ is infinite} \} \) of \( T(X) \) is a \( \mathcal{B}2 \)-semigroup but it is neither regular nor left (right) simple. In fact, \( \mathcal{B}2 \)-semigroups have been characterized by Calais [1] as follows.

**Proposition 1.4** [1]. A semigroup \( S \) is a \( \mathcal{B}2 \)-semigroup if and only if \( (x, y)_b = (x, y)_q \) for all \( x, y \in S \).

Every bi-ideal of a regular semigroup is a \( \mathcal{B}2 \)-semigroup. The proof is rather simple and is as follows: let \( T \) be a bi-ideal of a regular semigroup \( S \) and \( B \) a bi-ideal of \( T \). Then \( TST \subseteq T \) and \( BTB \subseteq B \). Let \( x \in TB \cap BT \). Since \( S \) is regular, \( x = xsx \) for some \( s \in S \) which implies that \( x = xsx \in BTsTB \subseteq BTSTB \subseteq BTB \subseteq B \). Thus \( TB \cap BT \subseteq B \). Hence \( B \) is a quasi-ideal of \( T \), as desired. Since \( T(X, Y) \) and \( L_F(V, W) \) are left ideals of \( T(X) \) and \( L_F(V) \), respectively, it follows that \( T(X, Y) \) and \( L_F(V, W) \) are always \( \mathcal{B}2 \)-semigroups. However, the semigroups \( T(X, Y) \) and \( L_F(V, W) \) need not be \( \mathcal{B}2 \)-semigroups. Notice
that if $X$ is infinite, then the semigroup $\{\alpha \in T(X) \mid X \setminus \operatorname{ran} \alpha \text{ is infinite}\}$ is a left ideal of $T(X)$. Similarly, if $V$ has infinite dimension over $F$, then the semigroup $\{\alpha \in L_F(V) \mid \dim_F(V/\operatorname{ran} \alpha) \text{ is infinite}\}$ is a left ideal of $L_F(V)$.

In Section 2, we give a necessary and sufficient condition for $\overline{T}(X, Y)$ to be a $\mathcal{B}_\Omega$-semigroup in terms of $|X|$ and $|Y|$. In Section 3, a necessary and sufficient condition for $L_F(V, W)$ to be a $\mathcal{B}_\Omega$-semigroup is given in terms of $|F|$, $\dim_F V$, and $\dim_F W$.

In the remainder, let $X$ be a nonempty set, $\emptyset \neq Y \subseteq X$, $V$ a vector space over a field $F$, and $W$ a subspace of $V$.

2. The semigroup $\overline{T}(X, Y)$

We begin this section by characterizing regular elements of the semigroup $\overline{T}(X, Y)$. Then it is shown that $\overline{T}(X, Y)$ is a regular semigroup if and only if $Y = X$ or $Y$ contains only one element.

**Proposition 2.1.** The following statements hold for the semigroup $\overline{T}(X, Y)$.

(a) For $\alpha \in \overline{T}(X, Y)$, $\alpha \in \operatorname{Reg}(\overline{T}(X, Y))$ if and only if $\operatorname{ran} \alpha \cap Y = Y \alpha$.

(b) The semigroup $\overline{T}(X, Y)$ is regular if and only if either $Y = X$ or $|Y| = 1$.

**Proof.** (i) Since $Y \alpha \subseteq Y$, we have $Y \alpha \subseteq \operatorname{ran} \alpha \cap Y$. Assume that $\alpha = a \beta \alpha$ for some $\beta \in \overline{T}(X, Y)$. If $x \in \operatorname{ran} \alpha \cap Y$, then $x \in Y$ and $x = a \alpha$ for some $a \in X$ which imply that $x = a \alpha = a a \beta \alpha = x \beta \alpha \in Y \beta \alpha \subseteq Y \alpha$. Hence we have $\operatorname{ran} \alpha \cap Y = Y \alpha$.

Conversely, assume that $\operatorname{ran} \alpha \cap Y = Y \alpha$. Then for each $x \in \operatorname{ran} \alpha \cap Y$, we have $x \alpha^{-1} \cap Y \neq \emptyset$. We choose an element $x' \in x \alpha^{-1} \cap Y$ for each $x \in \operatorname{ran} \alpha \cap Y$. Also, for $x \in \operatorname{ran} \alpha \setminus Y$, choose an element $x' \in x \alpha^{-1}$. Then $x' \alpha = x$ for all $x \in \operatorname{ran} \alpha \cap Y$ and $x' \alpha = x$ for all $x \in \operatorname{ran} \alpha \setminus Y$. Let $a$ be a fixed element in $Y$ and define $\beta : X \to X$ by a bracket notation as follows:

$$
\beta = \begin{bmatrix} x & t & X \setminus \operatorname{ran} \alpha \\
 x' & T & a \\
 t & a & x \end{bmatrix}_{x \in \operatorname{ran} \alpha \cap Y, t \in \operatorname{ran} \alpha \setminus Y} 
$$  \hspace{1cm} (2.1)

Then $Y \beta \subseteq \{x' \mid x \in \operatorname{ran} \alpha \cap Y\} \cup \{a\} \subseteq Y$, and for $x \in X$,

$$
x \alpha \beta \alpha = (x \alpha) \beta \alpha = \begin{cases} (x \alpha)' \alpha = x \alpha & \text{if } x \alpha \in \operatorname{ran} \alpha \cap Y, \\ (x \alpha) \alpha \alpha = x \alpha & \text{if } x \alpha \in \operatorname{ran} \alpha \setminus Y. \end{cases} \hspace{1cm} (2.2)
$$

Hence $\beta \in \overline{T}(X, Y)$ and $\alpha = a \beta \alpha$.

(ii) Suppose that $Y \subseteq X$ and $|Y| > 1$. Let $a$ and $b$ be two distinct elements of $Y$. Define $\alpha : X \to X$ by

$$
\alpha = \begin{bmatrix} Y & X \setminus Y \\
 a & b \end{bmatrix}. \hspace{1cm} (2.3)
$$

Then $\operatorname{ran} \alpha = \{a, b\} \subseteq Y$, so $\alpha \in \overline{T}(X, Y)$ and $\operatorname{ran} \alpha \cap Y = \{a, b\} \neq \{a\} = Y \alpha$. It follows from (i) that $\alpha \notin \operatorname{Reg}(\overline{T}(X, Y))$. Hence $\overline{T}(X, Y)$ is not a regular semigroup.
On transformation semigroups which are $\mathcal{B}\mathcal{D}$-semigroups

If $Y = X$, then $T(X, Y) = T(X)$ which is regular. Next, assume that $Y = \{c\}$. In this case, $T(X, Y)$ is isomorphic to the semigroup $P(X \setminus Y)$ consisting of all partial transformations of $X \setminus Y$, via the map $P(X \setminus Y) \to T(X, Y)$, $\alpha \mapsto \pi$, where

$$
\pi = \begin{bmatrix}
x & X \setminus \text{dom } \alpha \\
\text{dom } \alpha & c
\end{bmatrix}
$$

It is well known that $P(X \setminus Y)$ is regular [4, page 4]. Hence $T(X, Y)$ is a regular semigroup, as required.

To characterize when $T(X, Y)$ is a $\mathcal{B}\mathcal{D}$-semigroup, Propositions 1.1, 1.2, 1.4, and 2.1 and the following three lemmas are needed.

**Lemma 2.2.** Let $S$ be a semigroup. If $\emptyset \neq A \subseteq \text{Reg}(S)$, then $(A)_b = (A)_q$.

**Proof.** We know that $(A)_b \subseteq (A)_q$. Let $x \in (A)_q$. By Proposition 1.1(i), $x = sa = bt$ for some $s, t \in S^1$ and $a, b \in A$. Since $a \in \text{Reg}(S)$, $a = aa' a$ for some $a' \in S$. Then

$$
x = sa = saa' a = bta' a \in ASA \subseteq (A)_b
$$

by Proposition 1.1(ii). Hence we have $(A)_b = (A)_q$, as desired.

**Lemma 2.3.** Let $S$ be a semigroup, let $\emptyset \neq A \subseteq S$, and let $B \subseteq \text{Reg}(S)$. If $(A)_b = (A)_q$, then $(A \cup B)_b = (A \cup B)_q$.

**Proof.** We first show that $S^1 A \cap BS^1$ and $S^1 B \cap AS^1$ are subsets of $(A \cup B)_b$. Let $x \in S^1 A \cap BS^1$. Then $x = sa = bt$ for some $s, t \in S^1$, $a \in A$, and $b \in B$. Since $b \in \text{Reg}(S)$, $b = bb' b$ for some $b' \in S$. It follows that

$$
x = bt = bb' bt = bb'sa \in BSA \subseteq (A \cup B)S(A \cup B) \subseteq (A \cup B)_b.
$$

This shows that $S^1 A \cap BS^1 \subseteq (A \cup B)_b$. It can be shown similarly that $S^1 B \cap AS^1 \subseteq (A \cup B)_b$. Consequently,

$$
(A \cup B)_q = S^1 (A \cup B) \cap (A \cup B)S^1
$$

$$
= (S^1 A \cup S^1 B) \cap (AS^1 \cup BS^1)
$$

$$
= (S^1 A \cap AS^1) \cup (S^1 A \cap BS^1) \cup (S^1 B \cap AS^1) \cup (S^1 B \cap BS^1)
$$

$$
= (A)_q \cup (S^1 A \cap BS^1) \cup (S^1 B \cap AS^1) \cup (B)_q
$$

$$
= (A)_b \cup (S^1 A \cap BS^1) \cup (S^1 B \cap AS^1) \cup (B)_b,
$$

from the assumption and Lemma 2.2,

$$
\subseteq (A)_b \cup (A \cup B)_b \cup (A \cup B)_b \cup (B)_b = (A \cup B)_b.
$$

But $(A \cup B)_b \subseteq (A \cup B)_q$, so $(A \cup B)_b = (A \cup B)_q$. □
Lemma 2.4. If $|X| = 3$ and $|Y| = 2$, then for all $\alpha, \beta \in \overline{T}(X, Y)$, $(\alpha, \beta)_b = (\alpha, \beta)_q$ in $\overline{T}(X, Y)$.

Proof. For convenience, let $X_a$ denote the constant map whose domain and range are $X$ and $\{a\}$, respectively.

Assume that $X = \{a, b, c\}$ and $Y = \{a, b\}$. Clearly,

$$\overline{T}(X, Y) = \left\{ 1_X, X_a, X_b, \left[ \begin{array}{ccc} a & b & c \\ a & a & b \\ a & a & c \end{array} \right], \left[ \begin{array}{ccc} a & b & c \\ a & a & c \\ b & b & a \end{array} \right], \left[ \begin{array}{ccc} a & b & c \\ a & a & c \\ b & b & c \end{array} \right], \left[ \begin{array}{ccc} a & b & c \\ a & b & c \\ b & a & c \end{array} \right], \left[ \begin{array}{ccc} a & b & c \\ a & b & c \\ b & a & c \end{array} \right] \right\} \right.$$

By Proposition 2.1(i), $\overline{T}(X, Y) \cap \text{Reg}(\overline{T}(X, Y)) = \{[a \ b \ c], [a \ b \ c] \}$. Let $\lambda = [a \ b \ c]$ and $\eta = [a \ b \ c]$. Note that $\lambda^2 = X_a = \eta\lambda$ and $\eta^2 = X_b = \lambda\eta$. To show that $(\alpha, \beta)_b = (\alpha, \beta)_q$ for all $\alpha, \beta \in \overline{T}(X, Y)$, by Lemma 2.3, it suffices to show that $(\lambda)_b = (\lambda)_q$, $(\eta)_b = (\eta)_q$, and $(\lambda, \eta)_b = (\lambda, \eta)_q$. By direct multiplication, we have

$$\overline{T}(X, Y)\lambda = \{\lambda, X_a\}, \quad \lambda\overline{T}(X, Y) = \{\lambda, X_a, X_b, \eta\}, \quad \lambda\overline{T}(X, Y)\lambda = \{X_a\},$$

$$\overline{T}(X, Y)\eta = \{\eta, X_b\}, \quad \eta\overline{T}(X, Y) = \{\eta, X_a, X_b, \lambda\}, \quad \eta\overline{T}(X, Y)\eta = \{X_b\},$$

$$(\lambda, \eta)_b = \{\lambda, \eta\} \overline{T}(X, Y) \{\lambda, \eta\} \cup \{\lambda, \eta\}$$

$$= \lambda \overline{T}(X, Y) \lambda \cup \lambda \overline{T}(X, Y) \eta \cup \eta \overline{T}(X, Y) \eta \cup \lambda \overline{T}(X, Y) \eta \cup \{\lambda, \eta\}$$

$$= \{X_a, X_b, \lambda, \eta\},$$

$$(\lambda, \eta)_q = \overline{T}(X, Y) \{\lambda, \eta\} \cap \{\lambda, \eta\} \overline{T}(X, Y)$$

$$= (\overline{T}(X, Y) \lambda \cup \overline{T}(X, Y) \eta) \cap (\lambda \overline{T}(X, Y) \cup \eta \overline{T}(X, Y))$$

$$= \{\lambda, X_a, \eta, X_b\} = (\lambda, \eta)_b.$$ 

\[\square\]

Theorem 2.5. The semigroup $\overline{T}(X, Y)$ is a $\mathcal{B\Delta}$-semigroup if and only if one of the following statements holds.

(i) $Y = X$.

(ii) $|Y| = 1$.

(iii) $|X| \leq 3$. 

Lemma 2.4 and Proposition 1.4, a \alpha \beta

Case 1

Then $Y$ and $\beta$

Therefore we have for every $x$

Remark 2.8. We have mentioned that $T(X, Y)$ is a left ideal of $T(X)$. But for $\alpha \in T(X, Y)$ and $\beta \in \overline{T}(X, Y)$, $X\alpha \beta \subseteq Y \beta \subseteq Y$, so $T(X, Y)$ is an ideal of $\overline{T}(X, Y)$. We have $1_X \in \overline{T}(X, Y)$ if $Y \neq X$. Hence if $Y \neq X$, then $\overline{T}(X, Y)$ is neither left nor right simple.
Therefore we deduce from Corollary 2.7 that if $|X| = 3$ and $|Y| = 2$, then $\overline{T}(X, Y)$ is an example of $\mathcal{B}\mathcal{I}$-semigroup which is neither regular nor left (right) simple (see Propositions 1.2 and 1.3).

3. The semigroup $\overline{L}_F(V, W)$

In this section, we give a necessary and sufficient condition for $\overline{L}_F(V, W)$ to be a $\mathcal{B}\mathcal{I}$-semigroup. We first provide the conditions of the regularity of elements of $\overline{L}_F(V, W)$ and of the semigroup $\overline{L}_F(V, W)$. The following facts about vector spaces and linear transformations will be used. If $U_1$ and $U_2$ are subspaces of $V$, $B_1$ is a basis of the subspace $U_1 \cap U_2$, $B_2 \subseteq U_1$ and $B_3 \subseteq U_2$ are such that $B_1 \cup B_2$ and $B_1 \cup B_3$ are bases of $U_1$ and $U_2$, respectively, then $B_1 \cup B_2 \cup B_3$ is a basis of the subspace $U_1 + U_2$ of $V$. If $\alpha \in L_F(V)$, $B_1$ is a basis of $\ker \alpha$, $B_2$ is a basis of $\text{ran} \alpha$, and choose an element $u' \in u\alpha^{-1}$ for every $u \in B_2$, then $B_1 \cup \{u' \mid u \in B_2\}$ is a basis of $V$.

**Proposition 3.1.** The following statements hold for the semigroup $\overline{L}_F(V, W)$.

(i) For $\alpha \in \overline{L}_F(V, W)$, $\alpha \in \text{Reg}(\overline{L}_F(V, W))$ if and only if $\text{ran} \alpha \cap W = W\alpha$.

(ii) The semigroup $\overline{L}_F(V, W)$ is regular if and only if either $W = V$ or $W = \{0\}$.

**Proof.** (i) The proof that $\alpha \in \text{Reg}(\overline{L}_F(V, W))$ implies $\text{ran} \alpha \cap W = W\alpha$ is analogous to the proof of the “only if” part of Proposition 2.1(i).

Conversely, assume that $\text{ran} \alpha \cap W = W\alpha$. Let $B_1$ be a basis of $\text{ran} \alpha \cap W$, $B_2 \subseteq \text{ran} \alpha \setminus B_1$, and $B_3 \subseteq W \setminus B_1$ such that $B_1 \cup B_2$ and $B_1 \cup B_3$ are bases of $\text{ran} \alpha$ and $W$, respectively. Then $B_1 \cup B_2 \cup B_3$ is a basis of $\text{ran} \alpha + W$. Let $B_4 \subseteq V \setminus (B_1 \cup B_2 \cup B_3)$ be such that $B_1 \cup B_2 \cup B_3 \cup B_4$ is a basis of $V$. Since $B_1 \subseteq \text{ran} \alpha \cap W = W\alpha$, we have $u\alpha^{-1} \cap W \neq \emptyset$ for every $u \in B_1$. For each $u \in B_1$, choose an element $u' \in u\alpha^{-1} \cap W$. Since $B_2 \subseteq \text{ran} \alpha$, for each $v \in B_2$, $v\alpha^{-1} \neq \emptyset$, so choose an element $v \in v\alpha^{-1}$. Define $\beta \in L_F(V)$ on the basis $B_1 \cup B_2 \cup B_3 \cup B_4$ by

$$
\beta = \begin{bmatrix}
  u & v & B_3 \cup B_4 \\
  u' & v & 0
\end{bmatrix}_{u \in B_1, \atop v \in B_2} \quad (3.1)
$$

It follows that $W\beta = \langle B_1 \cup B_3 \rangle \beta = \{ \{u' \mid u \in B_1\} \rangle \subseteq W$, so $\beta \in \overline{L}_F(V, W)$. Let $B_0$ be a basis of $\ker \alpha$. Then $B_0 \cup \{u' \mid u \in B_1\} \cup \{v \mid v \in B_2\}$ is a basis of $V$. Since

$$
B_0\alpha\beta\alpha = \{0\} = B_0\alpha, \quad u'\alpha\beta\alpha = u\beta\alpha = u'\alpha \quad \forall u \in B_1,
$$

$$
v\alpha\beta\alpha = v\beta\alpha = v\alpha \quad \forall v \in B_2, \quad (3.2)
$$

we have $\alpha = \alpha\beta\alpha$, so $\alpha$ is a regular element of $\overline{L}_F(V, W)$.

(ii) Assume that $\{0\} \neq W \subseteq V$. Let $B_1$ be a basis of $W$ and $B$ a basis of $V$ containing $B_1$. Then $B_1 \neq \emptyset \neq B \setminus B_1$. Let $w \in B_1$ and $u \in B \setminus B_1$. Define $\alpha \in L_F(V)$ by

$$
\alpha = \begin{bmatrix}
  u & B \setminus \{u\} \\
  w & 0
\end{bmatrix}. \quad (3.3)
$$
Then $W \alpha = \langle B \rangle \alpha \subseteq \langle B \setminus \{u\} \rangle \alpha = \{0\}$, so $\alpha \in \mathcal{L}_F(V, W)$. Since $\text{ran} \alpha \cap W = \langle w \rangle \neq \{0\} = W \alpha$, by (i), we deduce that $\alpha$ is not a regular element of $\mathcal{L}_F(V, W)$. Hence $\mathcal{L}_F(V, W)$ is not a regular semigroup.

Since $\mathcal{L}_F(V, V) = L_F(V) = \mathcal{L}_F(V, \{0\}),$ the converse holds. 

To prove the main theorem, the following lemma is also needed. Lemma 2.3 and Proposition 3.1(i) are useful to obtain this result.

**Lemma 3.2.** If $F = \mathbb{Z}_2$, $\dim_F V = 2$, and $\dim_F W = 1$, then for all $\alpha, \beta \in \mathcal{L}_F(V, W)$, $(\alpha, \beta)_b = (\alpha, \beta)_{q}$ in $\mathcal{L}_F(V, W)$.

**Proof.** Let $\{w\}$ be a basis of $W$ and $\{w, u\}$ a basis of $V$. Since $F = \mathbb{Z}_2$, it follows that $W = \{0, w\}$ and $V = \{0, w, u, u + w\}$. Clearly, both $\{u, u + w\}$ and $\{w, u + w\}$ are also bases of $V$. Thus $\langle w \rangle \cap \langle u \rangle = \langle w \rangle \cap \langle u + w \rangle = \langle u \rangle \cap \langle u + w \rangle = \{0\}$. All elements of $\mathcal{L}_F(V, W)$ defined on the basis $\{w, u\}$ of $V$ can be given as follows:

\[
\mathcal{L}_F(V, W) = \left\{0, 1, \begin{bmatrix} w & u \\ 0 & w \end{bmatrix}, \begin{bmatrix} w & u \\ 0 & u \end{bmatrix}, \begin{bmatrix} w & u \\ 0 & w + u \end{bmatrix}, \begin{bmatrix} w & u \\ w & w \end{bmatrix}, \begin{bmatrix} w & u \\ w & w + u \end{bmatrix} \right\}.
\]

(3.4)

By Proposition 3.1(i), $\mathcal{L}_F(V, W) \setminus \text{Reg}(\mathcal{L}_F(V, W)) = \left\{\begin{bmatrix} 0 & w \\ 0 & w \end{bmatrix}\right\}$. Let $\lambda = \begin{bmatrix} 0 & w \\ 0 & w \end{bmatrix}$. Note that $\lambda^2 = 0$. To prove the lemma, by Lemma 2.3, it suffices to show that $(\lambda)_b = (\lambda)_q$. By direct multiplication, we have

\[
\mathcal{L}_F(V, W)\lambda = \{0, \lambda\}, \quad \lambda \mathcal{L}_F(V, W) = \{0, \lambda\}, \quad \lambda \mathcal{L}_F(V, W)\lambda = \{0\}.
\]

Consequently, $(\lambda)_b = \lambda \mathcal{L}_F(V, W) \lambda \cup \{\lambda\} = \{0, \lambda\} = \mathcal{L}_F(V, W)\lambda \cap \lambda \mathcal{L}_F(V, W) = (\lambda)_q$. 

**Theorem 3.3.** The semigroup $\mathcal{L}_F(V, W)$ is a $\mathcal{B}_2$-semigroup if and only if one of the following statements holds.

(i) $W = V$.

(ii) $W = \{0\}$.

(iii) $F = \mathbb{Z}_2$, $\dim_F V = 2$, and $\dim_F W = 1$.

**Proof.** Assume that (i), (ii), and (iii) are false. Then (1) $\{0\} \neq W \subseteq V$ and (2) $F \neq \mathbb{Z}_2$, $\dim_F V > 2$, or $\dim_F W > 1$. Let $B_1$ be a basis of $W$ and $B$ a basis of $V$ containing $B_1$. Then $B_1 \neq \emptyset$ and $B \setminus B_1 \neq \emptyset$.

**Case 1** ($(F \neq \mathbb{Z}_2)$). Let $a \in F \setminus \{0, 1\}$, $w \in B_1$, and $u \in B \setminus B_1$. Define $\alpha, \beta, \gamma \in \mathcal{L}_F(V, W)$ by

\[
\alpha = \begin{bmatrix} u & B \setminus \{u\} \\ w & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} w & B \setminus \{w\} \\ aw & 0 \end{bmatrix}, \quad \gamma = \begin{bmatrix} u & B \setminus \{u\} \\ au & 0 \end{bmatrix}.
\]

(3.6)

Then we have $\alpha \beta = \begin{bmatrix} u & B \setminus \{u\} \\ aw & 0 \end{bmatrix} = \gamma \alpha$. Since $a \neq 1$, we have $\alpha \beta \neq \alpha$. By Proposition 1.1(i), $\alpha \beta \in \mathcal{L}_F(V, W)(\alpha)_q$. Suppose that $\alpha \beta \in \mathcal{L}_F(V, W)(\alpha)_b$. By Proposition 1.1(ii), $\alpha \beta = \alpha \eta \alpha$
for some \( \eta \in \mathbb{L}_F(V, W) \). Then \( aw = u\alpha\beta = u\alpha\eta\alpha = (w\eta)\alpha \). But \( w\eta \in W \) and \( W\alpha = \langle B_1 \rangle \alpha \subseteq \langle B \setminus \{u\} \rangle \alpha = \{0\} \), so \( aw = 0 \) which is contrary to \( a \neq 0 \). Thus \((\alpha)_q \neq (\alpha)_b\), so \( \mathbb{L}_F(V, W) \) is not a \( \mathcal{B}\mathcal{D} \)-semigroup by Proposition 1.4.

Case 2 ((\( \dim_F W > 1 \))). Then \( |B_1| > 1 \). Let \( w_1, w_2 \in B_1 \) be such that \( w_1 \neq w_2 \) and \( u \in B \setminus B_1 \). Define \( \alpha, \beta, \gamma \in \mathbb{L}_F(V, W) \) by

\[
\alpha = \begin{bmatrix} w_1 & u & B \setminus \{w_1, u\} \\ w_2 & w_1 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} w_1 & B \setminus \{w_1\} \\ w_1 & 0 \end{bmatrix}, \quad \gamma = \begin{bmatrix} u & B \setminus \{u\} \\ u & 0 \end{bmatrix}.
\]

(3.7)

Then \( \alpha\beta = [u_{w_1} B \setminus \{u\}] = \gamma\alpha \neq \alpha \), so \( \alpha\beta \in (\alpha)_q \). If \( \alpha\beta \in (\alpha)_b \), then \( \alpha\beta = \alpha\eta\alpha \) for some \( \eta \in \mathbb{L}_F(V, W) \). Thus \( w_1 = u\alpha\beta = u\alpha\eta\alpha = (w_1\eta)\alpha \). Since \( w_1\eta \in W = \langle B_1 \rangle \), we have \( w_1\eta = aw_1 + v \) for some \( a \in F \) and \( v \in \langle B_1 \setminus \{w_1\} \rangle \). But \( B_1 \setminus \{w_1\} \subseteq B \setminus \{w_1, u\} \), so \( v\alpha = 0 \). Consequently, \( w_1 = (aw_1 + v)\alpha = aw_2 \) which is contrary to the independence of \( w_1 \) and \( w_2 \). By Proposition 1.4, \( \mathbb{L}_F(V, W) \) is not a \( \mathcal{B}\mathcal{D} \)-semigroup.

Case 3 ((\( \dim_F V > 2 \) and \( \dim_F W = 1 \))). Then \( |B_1| = 1 \) and \( |B \setminus B_1| > 1 \). Let \( B_1 = \{w\} \) and \( u_1, u_2 \in B \setminus B_1 \) be such that \( u_1 \neq u_2 \). Let \( \alpha, \beta, \gamma \in \mathbb{L}_F(V, W) \) be defined by

\[
\alpha = \begin{bmatrix} u_1 & u_2 & B \setminus \{u_1, u_2\} \\ w & u_1 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} w & B \setminus \{w\} \\ w & 0 \end{bmatrix}, \quad \gamma = \begin{bmatrix} u_1 & B \setminus \{u_1\} \\ u_1 & 0 \end{bmatrix}.
\]

(3.8)

Then we have \( \alpha\beta = [u_{w_1} B \setminus \{u\}] = \gamma\alpha \neq \alpha \), so \( \alpha\beta \in (\alpha)_q \). Suppose that \( \alpha\beta \in (\alpha)_b \). It follows that \( \alpha\beta = \alpha\eta\alpha \) for some \( \eta \in \mathbb{L}_F(V, W) \). Thus \( w = u_1\alpha\beta = u_1\alpha\eta\alpha = (w\eta)\alpha \). But \( w\eta \in W = \langle w \rangle \) and \( w\alpha = 0 \), so \( w = (w\eta)\alpha = 0 \), a contradiction. Hence \((\alpha)_q \neq (\alpha)_b\), so \( \mathbb{L}_F(V, W) \) is not a \( \mathcal{B}\mathcal{D} \)-semigroup, as before.

For the converse, if (i) or (ii) holds, then \( \overline{L}_F(V, W) = L_F(V) \) which is a \( \mathcal{B}\mathcal{D} \)-semigroup by Proposition 1.2. If (iii) holds, then \( \overline{L}_F(V, W) \) is a \( \mathcal{B}\mathcal{D} \)-semigroup by Proposition 1.4 and Lemma 3.2.

\[ \square \]

The following corollaries follow directly from Propositions 1.2, 3.1(iii), Theorem 3.3, and the proof of Lemma 3.2.

**Corollary 3.4.** If \( F \neq \mathbb{Z}_2 \) or \( \dim_F V \neq 2 \), then the following statements are equivalent.

(i) \( \overline{L}_F(V, W) \) is a \( \mathcal{B}\mathcal{D} \)-semigroup.

(ii) \( W = V \) or \( W = \{0\} \).

(iii) \( \overline{L}_F(V, W) \) is a regular semigroup.

**Corollary 3.5.** The semigroup \( \overline{L}_F(V, W) \) is a nonregular \( \mathcal{B}\mathcal{D} \)-semigroup if and only if \( F = \mathbb{Z}_2 \), \( \dim_F V = 2 \), and \( \dim_F W = 1 \). Hence if \( F = \mathbb{Z}_2 \) and \( \dim_F V = 2 \), there are exactly 3 semigroups \( \overline{L}_F(V, W) \) which are nonregular \( \mathcal{B}\mathcal{D} \)-semigroups, and each of such \( \overline{L}_F(V, W) \) contains 8 elements.

**Remark 3.6.** We also have that \( L_F(V, W) \) is an ideal of \( \overline{L}_F(V, W) \) (see Remark 2.8). Consequently, if \( \{0\} \neq W \subseteq V \), then \( \overline{L}_F(V, W) \) is neither left nor right 0-simple. Hence if \( F = \mathbb{Z}_2 \), \( \dim_F V = 2 \), and \( \dim_F W = 1 \), then \( \overline{L}_F(V, W) \) is a \( \mathcal{B}\mathcal{D} \)-semigroup which is neither regular nor left (right) 0-simple.
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