ANALYTICAL SOLUTIONS FOR THE NEUTRON TRANSPORT USING THE SPECTRAL METHODS

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We present a method for solving the two-dimensional equation of transfer. The method can be extended easily to the general linear transport problem. The used technique allows us to reduce the two-dimensional equation to a system of one-dimensional equations. The idea of using the spectral method for searching for solutions to the multidimensional transport problems leads us to a solution for all values of the independant variables, the proposed method reduces the solution of the multidimensional problems into a set of one-dimensional ones that have well-established deterministic solutions. The procedure is based on the development of the angular flux in truncated series of Chebyshev polynomials which will permit us to transform the two-dimensional problem into a set of one-dimensional problems.

1. Introduction

The neutron transport equation is a linearized version of the Boltzmann equation with wide applications in physics, geophysics, and astrophysics. The neutron transport equation models the transport of neutral particles in a scattering, fission, and absorption events with no self-interactions [11]. It is used in radiation shielding and reactor core calculations, as well as in radiative transfer of stellar and planetary atmospheres and it also describes dispersion of light, the passage of γ-rays through dispersive media, and so forth.

The resolution of problems dealing with transport phenomena is the subject of several works, especially in the context of transfer multidimensional problems based on analytical and numerical approaches. One can refer, for example, to Fourier transform [3, 4, 10, 13, 18, 22, 24] and many others using the Laplace transform [8, 9].

Chebyshev spectral methods for radiative transfer problems are also studied, for example, by Kim and Ishimaru in [19], by Kim and Moscoso in [20], by Asadzadeh and Kadem in [2], and by Kadem in [15–17]. For more detailed study on Chebyshev spectral method

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and also approximations by the spectral methods, we refer the reader to monographs by Boyd [6] and Bernardi and Maday [5].

The neutron transport problem was studied analytically by, for example, [12, 23]. The stationary monoenergetic transport of neutrons in a domain Ω surrounded by vacuum can be represented by the following integrodifferential equation. Given a source S and the coefficients α and σ, find the angular flux f such that for $\mu \in S^2$,

$$\begin{gathered}\mu \cdot \nabla u(x,u) + \int_{S^2} \sigma(x,\mu,\eta) u(x,\eta) d\eta + S(x,\mu), \quad \text{for } x \in \Omega, \\ u(x,\mu) = 0, \quad \text{for } x \in \Gamma_- = \{x \in \Gamma = \partial\Omega : \mu \cdot n(x) < 0\},\end{gathered} \tag{1.1}$$

where σ is the transfer kernel (collision function), α is the total cross-section, $S^2 = \{\mu \in R^3 : |\mu| = 1\}$, $n(x)$ is the outward unit normal to Γ at $x \in \Gamma$, and $\mu \cdot \nabla = \sum_{i=1}^{3} \mu_i (\partial/\partial x_i)$, $i = 1,2,3$.

There also exist analogues of (1.1) in different geometries. We have, for example, the following case.

Slab geometry $x \in (0,a)$, $\mu \in (-1,1)$, $a > 0$,

$$\begin{gathered}\mu \frac{\partial u}{\partial x}(x,\mu) + \alpha(x,\mu) u(x,\mu) = \int_{-1}^{1} \sigma(x,\mu,\eta) u(x,\eta) d\eta + S(x,\mu), \\ u(x,\mu) = 0, \quad \text{for } \mu > 0, \quad u(a,\mu) = 0, \quad \text{for } \mu < 0.\end{gathered} \tag{1.2}$$

Two-dimensional geometry: $x \in \Omega \subset R^2$, $\mu \in S = (\mu \subset R^2 : |\mu| = 1)$,

$$\begin{gathered}\mu \cdot \nabla u(x,\mu) + \alpha(x,\mu) u(x,\mu) = \int_{S} \sigma(x,\mu,\eta) u(x,\eta) d\eta + S(x,\mu), \\ u(x,\mu) = 0, \quad \text{for } x \in \Gamma_\mu^-.\end{gathered} \tag{1.3}$$

Cylindrical domain with functions being constant in the x_3-direction:

$$\begin{gathered}x \in C_R = \Omega \times R, \quad \Omega \subset R^2, \ \mu \in D = \left(\mu \in R^2 : |\mu| \le 1\right), \\ \mu \cdot \nabla u(x,\mu) + \alpha(x) u(x,\mu) = \int_{D} u(x,\eta) \left(1 - |\eta|^2\right)^{-1/2} \sigma(x,\mu,\eta) d\eta + S(x,\mu), \\ u(x,\mu) = 0, \quad \text{for } x \in \Gamma_\mu^-.\end{gathered} \tag{1.4}$$

A variety of numerical methods for the neutron transport equation have been proposed in the literature. In general, these methods are based on using quadrature formulas for the discretization of the angular variable (discrete-ordinates method), any other classical method, such as the doubling method can be used, and finite element or finite difference methods for the discretization of the spatial variable. Most of the previous theoretical convergence analysis of such numerical methods is concerned with the slab case (1.2).

In this paper, based on analytical procedures, we use the spectral method [14] to decompose the two-dimensional problem into a set of one-dimensional problems that can

be solved by many methods, among others, we mention the discrete-ordinates method [11, 21]. The method is based on the expansion of the flux in a truncated series of Chebyshev polynomials [1].

Truncating and replacing this expansion into the transport equation, it is then expected that the coefficients of the expansion must satisfy one-dimensional transport equations, the convergence is then studied.

2. The two-dimensional spectral solution

Consider the following problem:

$$\begin{gathered} Tf(r,\Psi) = Kf(r,\Psi) + S(r,\Psi), \quad (r,\Psi) \in Q = R \times D, \\ f(r,\Psi) = 0 \quad \text{for } (r,\Psi) \in \Gamma_-, \end{gathered} \tag{2.1}$$

where $R =]0,a] \times]0,b]$, $D = \{\Psi \in R^2,\ \|\Psi\|_2 < 1\}$, $\Gamma_- \stackrel{\text{def}}{=} \{(r,\Psi) \in \partial R \times D,\ \Psi \cdot n(r) < 0\}$, and $n(r)$ designates the normal outside vector at ∂R in r. We suppose furthermore that for $x = 0$, $f(0,y,-\mu,\eta) = g_1(y;\mu,\eta)$; and for $y = 0$, $f(x,0,-\mu,\eta) = g_2(x;\mu,\eta)$. Here the functions $g_1(y,\mu,\eta)$ and $g_2(x,\mu,\eta)$ describing some incidental flux are specified.

The operator of collision K is defined by $Kf(r,\Psi) = \int_{\bar{D}} \vartheta_s(\Psi,\Psi')f(r,\Psi')d\Psi'$, where ϑ_s is a positive bounded kernel.

The operator of transport T is defined by $Tf(r,\Psi) = \Psi \cdot \nabla f(r,\Psi) + \sigma f(r,\Psi)$ with $D(T) \stackrel{\text{def}}{=} \{f \in L^p(Q),\ \Psi \cdot \nabla f \in L^p(Q),\ f(r,\Psi) = 0 \text{ on } \Gamma_-,\ 1 < p < +\infty\}$. S denotes a positive term source, σ denotes the total effective section supposed to be constant, and f represents the flux of neutrons to be determined.

In the considered geometry, one has $\Psi \cdot \nabla f(r,\Psi) = \mu(\partial f/\partial x) + \eta(\partial f/\partial y)$, $r = (x,y)$, and $\Psi = (\mu,\eta)$, (2.1) spells then

$$\begin{aligned} &\mu \frac{\partial}{\partial x} f(x,y,\mu,\eta) + \eta \frac{\partial}{\partial y} f(x,y,\mu,\eta) + \sigma f(x,y,\mu,\eta) \\ &\quad = \iint_{-1}^{1} \vartheta_s(\mu,\eta;\mu',\eta') f(x,y;\mu',\eta') d\mu' d\eta' + S(r,\Psi) \end{aligned} \tag{2.2}$$

for $0 < x \le a$, and $-1 < y \le 1$. Here $f(x,y,\mu,\eta)$ is the flux of neutrons in the direction defined by $\mu \in [-1,1]$ and $\eta \in [-1,1]$; μ and η, respectively, denote the cosine and the sine of the polar angle.

We seek a solution of (2.2) subject to the boundary conditions of the form

$$\begin{gathered} f(0,y;-\mu,\eta) = g_1(y;\mu,\eta), \\ f(a,y;-\mu,\eta) = 0 \end{gathered} \tag{2.3}$$

for $0 < \mu \le 1$, $-1 \le \eta \le 1$, and

$$\begin{gathered} f(x,0;\mu,-\eta) = g_2(x;\mu,\eta), \\ f(x,b;\mu,\eta) = 0 \end{gathered} \tag{2.4}$$

for $-1 < \mu < 1$, $-1 < \eta \le 1$, and $0 < b < 1$.

We first expand the flux $f(x,y;\mu,\eta)$ via a truncated series of Chebyshev polynomials in the y variable:

$$f(x,y;\mu,\eta) = \sum_{i=0}^{I} f_i(x,\mu,\eta)T_i(y). \tag{2.5}$$

We then determine the components $f_i(x,\mu,\eta)$, for $i = 1,\dots,I$, to define the flux given by (2.5).

For the first component $f_0(x,\mu,\eta)$, we substitute (2.5) into the boundary conditions at $y = 0$ and $y = b$, given by (2.4), to find

$$\begin{aligned} f_0(x,\mu,\eta) &= g_2(x,\mu,\eta) - \sum_{i=1}^{I} \frac{T_i(0)}{T_0(0)} f_i(x,\mu,\eta), \quad -1<\mu<1,\ -1<\eta<1, \\ f_0(x,\mu,\eta) &= -\sum_{i=1}^{I} \frac{T_i(b)}{T_0(b)} f_i(x,\mu,\eta), \quad -1<\mu<1,\ -1<\eta<1,\ 0<b<1, \end{aligned} \tag{2.6}$$

where $0 < x \le a$, $-1 \le \mu \le 1$, and assuming that $T_i(0)$ and $T_i(b)$ are different from zero.

We now substitute (2.5) into (2.2), multiply the resulting equation by $T_k(y)/\sqrt{1-y^2}$, and integrate in the y variable in the interval $(-1,1)$ with $k = 1,\dots,I$, to get the following one-dimensional transport problems:

$$\mu\frac{\partial}{\partial x} f_k(x,\mu,\eta) + \sigma f_k(x,\mu,\eta) = \iint_{-1}^{1} \vartheta_s(\mu,\eta;\mu',\eta') f_k(x;\mu',\eta') d\mu' d\eta' + G_k(x;\mu,\eta), \tag{2.7}$$

with

$$G_k(x;\mu,\eta) = S_k(x,\mu,\eta) - \eta \sum_{i=k+1}^{I} A_i^k f_k(x,\mu,\eta), \tag{2.8}$$

where

$$A_i^k = \frac{2-\delta_{k,0}}{\pi} \int_{-1}^{1} \frac{d}{dy} \frac{T_i(y)}{\sqrt{1-y^2}} T_k(y) dy, \tag{2.9}$$

$$S_k(x,\mu,\eta) = \frac{2-\delta_{k,0}}{\pi} \int_{-1}^{1} S(x,y,\mu,\eta) \frac{T_k(y)}{\sqrt{1-y^2}} dy, \tag{2.10}$$

and where $\delta_{k,0}$ denotes the Kronecker delta.

The components $f_k(x;\mu,\eta)$ satisfy the following boundary conditions at $x = 0$, and at $x = a$:

$$\begin{aligned} f_k(0,-\mu,\eta) &= \frac{2-\delta_{k,0}}{\pi} \int_{-1}^{1} g_1(y;\mu,\eta) \frac{T_k(y)}{\sqrt{1-y^2}} dy, \\ f_k(a,-\mu,\eta) &= 0, \end{aligned} \tag{2.11}$$

for $0 < \mu \le 1$ and $0 \le \eta \le 1$.

In addition, by taking into account the summation term in (2.8), one can see that one can resolve the one-dimensional transfer problem given by (2.7)–(2.10) to define the components $f_k(x,\mu,\phi)$ for $k = I,\dots,1$, in this decreasing order to avoid the coupling of the equations; once this is done, the flux of neutrons is completely determined by (2.5).

3. Specific application of the method

Consider the three-dimensional neutron transport equation written as

$$\begin{aligned}\mu\frac{\partial}{\partial x}\Psi(\mathbf{x},\mu,\theta)+\sqrt{1-\mu^2}\left(\cos\theta\frac{\partial}{\partial y}\Psi(\mathbf{x},\mu,\theta)+\sin\theta\frac{\partial}{\partial z}\Psi(\mathbf{x},\mu,\theta)\right)+\sigma_t\Psi(\mathbf{x},\mu,\theta)\\ =\int_{-1}^{1}\int_{0}^{2\pi}\sigma_s(\mu',\theta'\longrightarrow\mu,\theta)\Psi(\mathbf{x},\mu',\theta')\,d\theta'\,d\mu'+S(\mathbf{x},\mu,\theta),\end{aligned}\tag{3.1}$$

where we assume that the spatial variable $\mathbf{x}:=(x,y,z)$ varies in the cubic domain $\Omega:=\{(x,y,z):-1\le x,y,z\le 1\}$, and $\Psi(\mathbf{x},\ \mu,\theta):=\Psi(x,y,z,\mu,\theta)$ is the angular flux in the direction defined by $\mu\in[-1,1]$ and $\theta\in[0,2\pi]$.

We seek for a solution of (3.1) satisfying the following boundary conditions.

For the boundary terms in x, for $0\le\theta\le 2\pi$,

$$\Psi(x=\pm1,y,z,\mu,\theta)=\begin{cases}f_1(y,z,\mu,\theta), & x=-1,\ 0<\mu\le 1,\\ 0, & x=1,\ -1\le\mu<0.\end{cases}\tag{3.2}$$

For the boundary terms in y and for $-1\le\mu<1$,

$$\Psi(x,y=\pm1,z,\mu,\theta)=\begin{cases}f_2(x,z,\mu,\theta), & y=-1,\ 0<\cos\theta\le 1,\\ 0, & y=1,\ -1\le\cos\theta<0.\end{cases}\tag{3.3}$$

Finally, for the boundary terms in z, for $-1\le\mu<1$,

$$\Psi(x,y,z=\pm1,\mu,\theta)=\begin{cases}f_3(x,y,\mu,\theta), & z=-1,\ 0\le\theta<\pi,\\ 0, & z=1,\ \pi<\theta\le 2\pi.\end{cases}\tag{3.4}$$

Here we assume that $f_1(y,z,\mu,\phi)$, $f_2(x,z,\mu,\phi)$, and $f_3(x,y,\mu,\phi)$ are given functions.

Expanding the angular flux $\Psi(x,y,z,\mu,\phi)$ in a truncated series of Chebyshev polynomials $T_i(y)$ and $R_j(z)$ leads to

$$\Psi(x,y,z,\mu,\theta)=\sum_{i=0}^{I}\sum_{j=0}^{J}\Psi_{i,j}(x,\mu,\theta)T_i(y)R_j(z).\tag{3.5}$$

Inserting (3.5) into (3.1), multiplying the resulting expressions by $(T_i(y)/\sqrt{1-y^2}) \times (R_j(z)/\sqrt{1-z^2})$, and integrating over y and z, we obtain $I \times J$ one-dimensional transport problems, namely,

$$\begin{aligned} &\mu \frac{\partial \Psi_{\alpha,\beta}}{\partial x}(x,\mu,\phi) + \sigma_t \Psi_{\alpha,\beta}(x,\mu,\phi) \\ &\quad = \iint_{-1}^{1} \sigma_s(\mu',\phi' \longrightarrow \mu,\phi)\Psi_{\alpha,\beta}(x,\mu',\phi')\,d\phi'\,d\mu' + G_{\alpha,\beta}(x;\mu,\phi), \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} G_{\alpha,\beta}(x;\mu,\eta) &= S_{\alpha,\beta}(x,\mu,\phi) - \sqrt{1-\mu^2} \\ &\quad \times \left[\cos\phi \sum_{\alpha=i+1}^{I} A_i^{\alpha} \Psi_{\alpha,j}(x,\mu,\phi) + \sin\phi \sum_{\beta=j+1}^{J} B_j^{\beta} \Psi_{i,\beta}(x,\mu,\phi) \right], \end{aligned} \tag{3.7}$$

with

$$\begin{aligned} S_{\alpha,\beta}(x,\mu,\phi) &= \frac{(2-\delta_{\alpha,0})(2-\delta_{\beta,0})}{\pi^2} \iiint_{-1}^{1} \frac{T_\alpha(\mu)}{\sqrt{1-\mu^2}} d\mu \frac{T_\alpha(y)T_\beta(z)}{\sqrt{(1-y^2)(1-z^2)}} S(x,y,z,\mu,\phi)\,dz\,dy, \\ A_i^{\alpha} &= \frac{2-\delta_{\alpha,0}}{\pi} \int_{-1}^{1} \frac{d}{dy}(T_\alpha(y)) \frac{T_i(y)}{\sqrt{1-y^2}} dy, \\ B_j^{\beta} &= \frac{2-\delta_{\beta,0}}{\pi} \int_{-1}^{1} \frac{d}{dz}(T_\beta(z)) \frac{T_j(z)}{\sqrt{1-z^2}} dz. \end{aligned} \tag{3.8}$$

The corresponding discrete-ordinates equation [11] is then

$$\begin{aligned} &\mu_m \frac{\partial \Psi_{\alpha,\beta}}{\partial x}(x,\mu_m,\phi_m) + \sigma_t \Psi_{\alpha,\beta}(x,\mu_m,\phi_m) \\ &\quad = \sum_{n=1}^{M} \omega_n \Psi_{\alpha,\beta}(x,\mu_m,\phi_m) + G_{\alpha,\beta}(x;\mu_m,\phi_m) \end{aligned} \tag{3.9}$$

and we also expand $\Psi_{\alpha,\beta}(x,\mu_m,\phi_m)$ in a truncated series of Chebyshev polynomials, that is,

$$\Psi_{\alpha,\beta}(x,\mu_m,\phi_m) = \sum_{k=0}^{M} \frac{C_k(x,\phi_m)T_k(\mu_m)}{\sqrt{1-\mu_m^2}} \tag{3.10}$$

bringing (3.10) in (3.9) to get

$$\mu_m \frac{\partial}{\partial x}\left[\sum_{k=0}^{M} \frac{C_k(x,\phi_m)T_k(\mu_m)}{\sqrt{1-\mu_m^2}}\right] + \sigma_t\left[\sum_{k=0}^{M} \frac{C_k(x,\phi_m)T_k(\mu_m)}{\sqrt{1-\mu_m^2}}\right] = \sum_{n=1}^{M} \omega_n \left[\sum_{k=0}^{M} \frac{C_k(x,\phi_m)T_k(\mu_m)}{\sqrt{1-\mu_m^2}}\right] + G_{\alpha,\beta}(x;\mu_m,\eta_m) \tag{3.11}$$

with

$$\begin{aligned} G_{\alpha,\beta}&(x;\mu_m,\eta_m) \\ &= S_{\alpha,\beta}(x,\mu,\phi) - \sqrt{1-\mu^2} \\ &\quad \times \left[\cos\phi \sum_{\alpha=i+1}^{I} A_i^{\alpha} \sum_{k=0}^{M} \frac{C_k(x,\phi_m)T_k(\mu_m)}{\sqrt{1-\mu_m^2}} + \sin\phi \sum_{\beta=j+1}^{J} B_j^{\beta} \sum_{k=0}^{M} \frac{C_k(x,\phi_m)T_k(\mu_m)}{\sqrt{1-\mu_m^2}}\right]. \end{aligned} \tag{3.12}$$

Multiplying (3.11) by $T_l(\mu_m)$ and integrating over $\mu_m \in [-1,1]$, we find

$$\begin{aligned} \mu_m \frac{\partial}{\partial x} \sum_{k=0}^{M} C_k(x,\phi_m) \int_{-1}^{1} \frac{T_k(\mu_m)T_l(\mu_m)}{\sqrt{1-\mu_m^2}} d\mu_m + \sigma_t \sum_{k=0}^{M} C_k(x,\phi_m) \int_{-1}^{1} \frac{T_k(\mu_m)T_l(\mu_m)}{\sqrt{1-\mu_m^2}} d\mu_m \\ = \sum_{n=0}^{M} \omega_n \sum_{k=0}^{M} C_k(x,\phi_m) \int_{-1}^{1} \frac{T_k(\mu_m)T_l(\mu_m)}{\sqrt{1-\mu_m^2}} d\mu_m + \int_{-1}^{1} G_{\alpha,\beta}(x;\mu_m,\eta_m)T_l(\mu_m)\,d\mu_m \end{aligned} \tag{3.13}$$

with

$$\begin{aligned} \int_{-1}^{1} &G_{\alpha,\beta}(x;\mu_m,\eta_m)T_l(\mu_m)\,d\mu_m \\ &= \int_{-1}^{1} S_{\alpha,\beta}(x,\mu_m,\phi_m)T_l(\mu_m)\,d\mu_m - \cos\phi_m\sqrt{1-\mu_m^2} \sum_{i=\alpha+1}^{I} A_i^{\alpha} \\ &\quad \times \sum_{k=0}^{M} C_k(x,\phi_m) \int_{-1}^{1} \frac{T_k(\mu_m)T_l(\mu_m)}{\sqrt{1-\mu_m^2}}\, d\mu_m + \sin\phi_m\sqrt{1-\mu_m^2} \sum_{j=\beta+1}^{J} B_j^{\beta} \\ &\quad \times \sum_{k=0}^{M} C_k(x,\phi_m) \int_{-1}^{1} \frac{T_k(\mu_m)T_l(\mu_m)}{\sqrt{1-\mu_m^2}}\, d\mu_m, \end{aligned} \tag{3.14}$$

where

$$A_i^\alpha = \frac{2-\delta_{\alpha,0}}{\pi}\iint_{-1}^{1}\frac{d}{dy}(T_\alpha(y))\frac{T_i(y)}{\sqrt{1-y^2}}T_l(\mu_m)\,dy\,d\mu_m,$$
$$B_j^\beta = \frac{2-\delta_{\beta,0}}{\pi}\iint_{-1}^{1}\frac{d}{dz}(T_\beta(z))\frac{T_j(z)}{\sqrt{1-z^2}}T_l(\mu_m)\,dz\,d\mu_m \tag{3.15}$$

by using the properties of Chebyshev polynomials to (3.14) to get

$$\begin{aligned}&\int_{-1}^{1}G_{\alpha,\beta}(x;\mu_m,\eta_m)T_l(\mu_m)\,d\mu_m\\&\quad=\int_{-1}^{1}S_{\alpha,\beta}(x,\mu_m,\phi_m)T_l(\mu_m)\,d\mu_m-\left[\frac{\pi}{2-\delta_{m,0}}\sqrt{1-\mu_m^2}C_k(x,\phi_m)\right]\\&\qquad\times\left[\sum_{i=\alpha+1}^{I}A_i^\alpha\cos\phi_m+\sum_{j=\beta+1}^{J}B_j^\beta\sin\phi_m\right],\end{aligned} \tag{3.16}$$

then (3.9) becomes

$$\begin{aligned}&\mu_m\frac{\partial C_m}{\partial x}+\left[\sigma_t-\sum_{m=0}^{M}\sqrt{1-\mu_m^2}\left(\sum_{i=\alpha+1}^{I}A_i^\alpha\cos\phi_m-\sum_{j=\beta+1}^{J}B_j^\beta\sin\phi_m\right)\right]C_m\\&\quad=\frac{\pi}{2-\delta_{m,0}}\int_{-1}^{1}S_{\alpha,\beta}(x,\mu_m,\phi_m)T_l(\mu_m)\,d\mu_m.\end{aligned} \tag{3.17}$$

After written in vector and matrix notation and regrouping the coefficients C_m together in (3.11), we can derive the following differential equation:

$$\frac{\partial C_m}{\partial x}+DC_m=E_m, \tag{3.18}$$

where $D_m=(1/\mu_m)B_m$ and $E_m=(1/\mu_m)A_m$ with

$$\begin{aligned}A_m&:=\frac{\pi}{2-\delta_{m,0}}\int_{-1}^{1}S_{\alpha,\beta}(x,\mu_m,\phi_m)T_l(\mu_m)\,d\mu_m,\\B_m&:=\left[\sigma_t-\sum_{m=0}^{M}\sqrt{1-\mu_m^2}\left(\sum_{i=\alpha+1}^{I}A_i^\alpha\cos\phi_m-\sum_{j=\beta+1}^{J}B_j^\beta\sin\phi_m\right)\right],\end{aligned} \tag{3.19}$$

the solution of differential equation for the vector C_m is thus constructed as follows:

$$C_m(x)=e^{-Dx}C_m(0)-\int_0^x e^{-(x-\xi)D}E_m(\xi)d\xi; \tag{3.20}$$

(3.20) depends on vector $C_m(0)$. Having established an analytical formulation for the exponential appearing in (3.20), the unknown components of vector $C_m(0)$ for the boundary problem (3.1) can be readily obtained applying the boundary conditions (3.2), (3.3), and (3.4).

To derive an analytical formulation for the exponential of matrix D, appearing in (3.20), we use the Sumudu transform method [15].

4. Study of the spectral approximation

Now we expand

$$\Psi_{\alpha,\beta,N}(x,\phi_m) = \sum_{m=0}^{(N)} C_m^{(N)} \cos(m\phi_m), \tag{4.1}$$

where $C_m^{(N)}$ is the approximation to the coefficient C_m by the consideration of the truncated series $\Psi_{\alpha,\beta,N}$.

From spectral analysis, we know that when a function is infinitely smooth and all its derivatives exist, then the coefficients appearing in its sine or cosine series go to zero faster than $1/n$. Moreover, if the function and all its derivatives are periodic, then the decay is faster than any power of $1/n$.

However, as indicated by Canuto et al. [7], in practice this decay cannot be observed before enough coefficients that represent the essential structures of the function are considered.

In the calculation, one can test the convergence of the cosine truncated series defined in (4.1) by evaluating

$$\sup_k \left[\frac{|\Psi_{N+1}(k) - \Psi_N(k)|}{\Psi_N(k)} \right] \leq \epsilon, \tag{4.2}$$

where ϵ is the required precision. In general, the few first coefficients of the series are enough to generate the angular flux.

If N is the chosen value, we can write

$$C_m^{(N)} = 0 \quad \forall n > N. \tag{4.3}$$

Combining therefore (4.3) and (3.20), we will now describe the necessary algorithm to obtain all the cosine coefficients $C_m^{(N)}$.

Step 0. $N = 0$; for $n = N = 0$,

$$C_0^{(0)}(x) = e^{-A^{-1}Bx} C_0^{(0)}(0) - \int_0^x e^{-A^{-1}B(x-s)} A_0(x)\,dx, \tag{4.4}$$

with

$$A_0 := \pi \int_{-1}^{1} S_{\alpha,\beta}(x,\mu_0,\phi_0)\, T_l(\mu_0)\, d\mu_0 \tag{4.5}$$

which is well known, and thus $C_0^{(0)}(x)$ is completely determined. To finish the step, we apply (4.1) to obtain the first approximation to the angular flux, that is, Ψ_0.

Step 1. $N = 1$; for $n = 0$,

$$C_0^{(1)}(x) = e^{-A^{-1}Bx}C_0^{(1)}(0) - \int_0^x e^{-A^{-1}B(x-s)}A_1(x)dx, \tag{4.6}$$

with

$$A_1 := \frac{\pi}{2}\int_{-1}^{1} S_{\alpha,\beta}(x,\mu_1,\phi_1)T_l(\mu_1)d\mu_1; \tag{4.7}$$

for $n = 1$

$$C_1^{(1)}(x) = e^{-A^{-1}Bx}C_1^{(1)}(0) - \int_0^x e^{-A^{-1}B(x-s)}A_1(x)dx, \tag{4.8}$$

with

$$A_1 := \frac{\pi}{2}\int_{-1}^{1} S_{\alpha,\beta}(x,\mu_1,\phi_1)T_l(\mu_1)d\mu_1. \tag{4.9}$$

Bringing the approximated solution for $C_0^{(0)}$ obtained at Step 0 inside (4.8) and iterating with (4.4), we obtain immediately the approximated coefficients $C_0^{(1)}$and $C_1^{(1)}$. To finish the step, we evaluate through (4.1) the new approximation Ψ_1 and perform the precision condition defined in (4.2). If (4.2) is verified, the calculation is stopped; if not, we go to Step 1 and do likewise until the convergence condition in (4.2) is fulfilled.

With the above algorithm, we only need knowledge of the operator $e^{-A^{-1}Bx}$ (the problem was solved previously by using the Sumudu transform [15]).

5. Conclusion

An adaptation of the method to study and to prove convergence of the spectral solution in the framework of the analytical solution may be possible. Just some preliminary results were obtained. In this context we study a quadrature approximation of weighted integrals for a class of functions relevant to our purpose problem and derive some quadrature error estimate. Our attention is now focused in this direction.

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