STRONG CONVERGENCE OF APPROXIMATION FIXED POINTS FOR NONEXPANSIVE NONSELF-MAPPING

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Let C be a closed convex subset of a uniformly smooth Banach space E , and $T : C \rightarrow E$ a nonexpansive nonself-mapping satisfying the weakly inwardness condition such that $F(T) \neq \emptyset$, and $f : C \rightarrow C$ a fixed contractive mapping. For $t \in (0, 1)$, the implicit iterative sequence $\{x_t\}$ is defined by $x_t = P(tf(x_t) + (1 - t)Tx_t)$, the explicit iterative sequence $\{x_n\}$ is given by $x_{n+1} = P(\alpha_n f(x_n) + (1 - \alpha_n)Tx_n)$, where $\alpha_n \in (0, 1)$ and P is a sunny nonexpansive retraction of E onto C . We prove that $\{x_t\}$ strongly converges to a fixed point of T as $t \rightarrow 0$, and $\{x_n\}$ strongly converges to a fixed point of T as α_n satisfying appropriate conditions. The results presented extend and improve the corresponding results of Hong-Kun Xu (2004) and Yisheng Song and Rudong Chen (2006).

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1. Introduction

Let C be a nonempty closed convex subset of a Banach space E , and Let $T : C \rightarrow C$ be a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$). We use $\text{Fix}(T)$ to denote the set of fixed points of T ; that is, $\text{Fix}(T) = \{x \in C : x = Tx\}$. Recall that a self-mapping $f : C \rightarrow C$ is a contraction on C if there exists a constant $\beta \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \beta \|x - y\|, \quad x, y \in C. \quad (1.1)$$

Xu (see [6]) defined the following two viscosity iterations for nonexpansive mappings:

$$x_t = tf(x_t) + (1 - t)Tx_t, \quad x \in C, \quad (1.2)$$

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad (1.3)$$

where α_n is a sequence in $(0, 1)$. Xu proved the strong convergence of $\{x_t\}$ defined by (1.2) as $t \rightarrow 0$ and $\{x_n\}$ defined by (1.3) in both Hilbert space and uniformly smooth Banach space.

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Recently, Song and Chen [2] proved if C is a closed subset of a real reflexive Banach space E which admits a weakly sequentially continuous duality mapping from E to E^* , and if $T : C \rightarrow E$ is a nonexpansive nonself-mapping satisfying the weakly inward condition, $F(T) \neq \emptyset$, $f : C \rightarrow C$ is a fixed contractive mapping, and P is a sunny nonexpansive retraction of E onto C , then the sequences $\{x_t\}$ and $\{x_n\}$ defined by

$$x_t = P(tf(x_t) + (1-t)Tx_t), \quad (1.4)$$

$$x_{n+1} = P(\alpha_n f(x_n) + (1-\alpha_n)Tx_n) \quad (1.5)$$

strongly converge to a fixed point of T .

In this paper, we establish the strong convergence of both $\{x_t\}$ defined by (1.4) and $\{x_n\}$ defined by (1.5) for a nonexpansive nonself-mapping T in a uniformly smooth Banach space. Our results extend and improve the results in [2, 6].

2. Preliminaries

Let E be a real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\| \|f\|, \|x\| = \|f\|\} \quad \forall x \in E, \quad (2.1)$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the sequence, we will denote the single-valued duality mapping by j , and $x_n \rightarrow x$ will denote strong convergence of the sequence $\{x_n\}$ to x . In Banach space E , the following result is well known [1, 3] for all $x, y \in E$, for all $j(x+y) \in J(x+y)$, for all $j(x) \in J(x)$,

$$\|x\|^2 + 2\langle y, j(x) \rangle \leq \|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y) \rangle. \quad (2.2)$$

Recall that the norm of E is said to be Gâteaux differentiable (and E is said to be smooth) if

$$\lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t} \quad (2.3)$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be uniformly Gâteaux differentiable if, for each $y \in U$, this limit is attained uniformly for $x \in U$. Finally, the norm is said to be uniformly Fréchet differentiable (and E is said to be uniformly smooth) if the limit in (2.3) is attained uniformly for $(x, y) \in U \times U$. A Banach space E is said to be smooth if and only if J is single valued. It is also well known that if E is uniformly smooth, J is uniformly norm-to-norm continuous. These concepts may be found in [3].

If C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a mapping $P : C \rightarrow D$ is called a retraction from C to D if $P^2 = P$. It is easily known that a mapping $P : C \rightarrow D$ is retraction, then $Px = x$, for all $x \in D$. A mapping $P : C \rightarrow D$ is called sunny if

$$P(Px + t(x - Px)) = Px \quad \forall x \in C, \quad (2.4)$$

whenever $Px + t(x - Px) \in C$ and $t > 0$. A subset D of C is said to be a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction of C onto D . For more detail, see [1, 3–5].

The following lemma is well known [3].

LEMMA 2.1. *Let C be a nonempty convex subset of a smooth Banach space E , $D \in C$, $J : E \rightarrow E^*$ the (normalized) duality mapping of E , and $P : C \rightarrow D$ a retraction. Then the following are equivalent:*

- (i) $\langle x - Px, j(y - Px) \rangle \leq 0$ for all $x \in C$ and $y \in D$;
- (ii) P is both sunny and nonexpansive.

Let C be a nonempty convex subset of a Banach space E , then for $x \in C$, we define the inward set [4, 5]:

$$I_C(x) = \{y \in E : y = x + \lambda(z - x), z \in C \text{ and } \lambda \geq 0\}. \quad (2.5)$$

A mapping $T : C \rightarrow E$ is said to be satisfying the inward condition if $Tx \in I_C(x)$ for all $x \in C$. T is also said to be satisfying the weakly inward condition if for each $x \in C$, $Tx \in \overline{I_C(x)}$ ($\overline{I_C(x)}$ is the closure of $I_C(x)$). Clearly $C \subset I_C(x)$ and it is not hard to show that $I_C(x)$ is a convex set as C is. Using above these results and definitions, we can easily show the following lemma.

LEMMA 2.2 ([2], Lemma 1.2). *Let C be a nonempty closed subset of a smooth Banach space E , let $T : C \rightarrow E$ be nonexpansive nonself-mapping satisfying the weakly inward condition, and let P be a sunny nonexpansive retraction of E onto C . Then $F(T) = F(PT)$.*

LEMMA 2.3 ([2], Lemma 2.1). *Let E be a Banach space and let C be a nonempty closed convex subset of E . Suppose that $T : C \rightarrow E$ is a nonexpansive mapping such that for each fixed contractive mapping $f : C \rightarrow C$, and P is a sunny nonexpansive retraction of E onto C . For each $t \in (0, 1)$, $\{x_t\}$ is defined by (1.4). Suppose $u \in C$ is a fixed point of T , then*

- (i) $\langle x_t - f(x_t), j(x_t - u) \rangle \leq 0$;
- (ii) $\{x_t\}$ is bounded.

Definition 2.4. μ is called a Banach limit if μ is a continuous linear functional on l^∞ satisfying

- (i) $\|\mu(e)\| = 1 = \mu(1)$, $e = (1, 1, 1, \dots)$;
- (ii) $\mu_n(a_n) = \mu_n(a_{n+1})$, for all $a_n \in (a_0, a_1, \dots) \in l^\infty$;
- (iii) $\liminf_{n \rightarrow \infty} a_n \leq \mu(a_n) \leq \limsup_{n \rightarrow \infty} a_n$, for all $a_n \in (a_0, a_1, \dots) \in l^\infty$.

According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a_0, a_1, \dots)$.

Further, we know the following result.

LEMMA 2.5 ([3], Lemma 4.5.4). *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm and let $\{x_n\}$ be a bounded sequence in E . Let μ be a Banach limit and $u \in C$. Then*

$$\mu_n \|x_n - u\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2 \quad (2.6)$$

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if and only if

$$\mu_n \langle x - u, J(x_n - u) \rangle \leq 0 \quad (2.7)$$

for all $x \in C$.

3. Main results

THEOREM 3.1. *Let E be a uniformly smooth Banach, suppose that C is a nonempty closed convex subset of E and $T : C \rightarrow E$ is a nonexpansive nonself-mapping satisfying the weakly inward condition and $F(T) \neq \emptyset$. Let $f : C \rightarrow C$ be a fixed contractive mapping, and let $\{x_t\}$ be defined by (1.4), where P is a sunny nonexpansive retraction of E onto C . Then as $t \rightarrow 0$ $\{x_t\}$ converges strongly to some fixed point q of T that q is the unique solution in $F(T)$ to the following variational inequality:*

$$\langle (I - f)q, j(q - u) \rangle \leq 0 \quad \forall u \in F(T). \quad (3.1)$$

Proof. For all $u \in F(T)$ by Lemma 2.3(ii), $\{x_t\}$ is bounded, therefore the sets $\{Tx_t : t \in (0, 1)\}$ and $\{f(x_t) : t \in (0, 1)\}$ are also bounded. From $x_t = P(tf(x_t) + (1 - t)Tx_t)$, we have

$$\begin{aligned} \|x_t - PTx_t\| &= \|P(tf(x_t) + (1 - t)Tx_t) - PTx_t\| \\ &\leq \|tf(x_t) + (1 - t)Tx_t - Tx_t\| \\ &= t\|Tx_t - f(x_t)\| \rightarrow 0 \quad \text{as } t \rightarrow 0. \end{aligned} \quad (3.2)$$

This implies that

$$\lim_{t \rightarrow 0} \|x_t - PTx_t\| = 0. \quad (3.3)$$

Assume $t_n \rightarrow 0$, set $x_n := x_{t_n}$, and define $g : C \rightarrow \mathbb{R}$ by $g(x) = \mu_n \|x_n - x\|^2$, $x \in C$, where μ_n is a Banach limit on ℓ^∞ . Let

$$K = \left\{ x \in C : g(x) = \min_{y \in C} \mu_n \|x_n - y\|^2 \right\}. \quad (3.4)$$

It is easily seen that K is a nonempty closed convex bounded subset of E , since (note $\|x_n - Tx_n\| \rightarrow 0$)

$$g(Tx) = \mu_n \|x_n - Tx\|^2 = \mu_n \|Tx_n - Tx\|^2 \leq \mu_n \|x_n - x\|^2 = g(x). \quad (3.5)$$

It follows that $T(K) \subset K$, that is, K is invariant under T . Since a uniformly smooth Banach space has the fixed point property for nonexpansive mappings, T has a fixed point, say q , in K . From Lemma 2.5 we get

$$\mu_n \langle x - q, j(x_n - q) \rangle \leq 0, \quad x \in C. \quad (3.6)$$

For all $q \in F(T)$, we have $tf(x_t) + (1-t)q = P[tf(x_t) + (1-t)q]$, then

$$\begin{aligned} & \|x_t - [tf(x_t) + (1-t)q]\| \\ &= \|P[tf(x_t) + (1-t)Tx_t] - P[tf(x_t) + (1-t)q]\| \\ &\leq \|(1-t)(Tx_t - q)\| \leq (1-t)\|x_t - q\|. \end{aligned} \quad (3.7)$$

Hence from (2.2) and the above inequality we get

$$\begin{aligned} & \|x_t - [tf(x_t) + (1-t)q]\|^2 \\ &= \|(1-t)(x_t - q) + t(x_t - f(x_t))\|^2 \\ &\geq (1-t)^2\|x_t - q\|^2 + 2t(1-t)\langle x_t - f(x_t), j(x_t - q) \rangle. \end{aligned} \quad (3.8)$$

Therefore

$$\langle x_t - f(x_t), j(x_t - q) \rangle \leq 0. \quad (3.9)$$

Then

$$\begin{aligned} 0 &\geq \langle x_t - f(x_t), j(x_t - q) \rangle \\ &= \|x_t - q\|^2 + \langle q - f(q), j(x_t - q) \rangle + \langle f(q) - f(x_t), j(x_t - q) \rangle \\ &\geq (1-\beta)\|x_t - q\|^2 + \langle q - f(q), j(x_t - q) \rangle. \end{aligned} \quad (3.10)$$

We get

$$\|x_t - q\|^2 \leq \frac{1}{1-\beta} \langle f(q) - q, j(x_t - q) \rangle. \quad (3.11)$$

Now applying Banach limit to the above inequality, we get

$$\mu_n \|x_t - q\|^2 \leq \mu_n \left(\frac{1}{1-\beta} \langle f(q) - q, j(x_t - q) \rangle \right). \quad (3.12)$$

Let $x = f(q)$ in (3.6), and noting (3.12), we have

$$\mu_n \|x_t - q\|^2 \leq 0, \quad (3.13)$$

that is,

$$\mu_n \|x_n - q\|^2 = 0 \quad (3.14)$$

and then exists a subsequence which is still denoted by $\{x_n\}$ such that

$$x_n \longrightarrow q, \quad n \longrightarrow \infty. \quad (3.15)$$

We have proved that for any sequence $\{x_{t_n}\}$ in $\{x_t : t \in (0,1)\}$, there exists a subsequence which is still denoted by $\{x_{t_n}\}$ that converges to some point q of T . To prove that

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the entire net $\{x_t\}$ converges to q , suppose that there exists another sequence $\{x_{s_k}\} \subset \{x_t\}$ such that $x_{s_k} \rightarrow p$, as $s_k \rightarrow 0$, then we also have $p \in F(T)$ (using $\lim_{t \rightarrow 0} \|x_t - PTx_t\| = 0$). Next we show $p = q$ and q is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I - f)q, j(q - u) \rangle \quad \forall u \in F(T). \quad (3.16)$$

Since the sets $\{x_t - u\}$ and $\{x_t - f(x_t)\}$ are bounded and the uniform smoothness of E implies that the duality map J is norm-to-norm uniformly continuous on bounded sets of E , for any $u \in F(T)$, by $x_{s_k} \rightarrow p$ ($s_k \rightarrow 0$), we have

$$\begin{aligned} & \| (I - f)x_{s_k} - (I - f)p \| \rightarrow 0, \quad s_k \rightarrow 0, \\ & | \langle x_{s_k} - f(x_{s_k}), j(x_{s_k} - u) \rangle - \langle (I - f)p, j(p - u) \rangle | \\ &= | \langle x_{s_k} - f(x_{s_k}) - (I - f)p, j(x_{s_k} - u) \rangle - \langle (I - f)p, j(x_{s_k} - u) - j(p - u) \rangle | \\ &\leq \| (I - f)x_{s_k} - (I - f)p \| \| x_{s_k} - u \| \\ &\quad + | \langle (I - f)p, j(x_{s_k} - u) - j(p - u) \rangle | \rightarrow 0 \quad \text{as } s_k \rightarrow 0. \end{aligned} \quad (3.17)$$

Therefore, noting Lemma 2.3(i), for any $u \in F(T)$, we get

$$\langle (I - f)p, j(p - u) \rangle = \lim_{s_k \rightarrow 0} \langle x_{s_k} - f(x_{s_k}), j(x_{s_k} - u) \rangle \leq 0. \quad (3.18)$$

Similarly, we also can show

$$\langle (I - f)q, j(q - u) \rangle = \langle x_{t_n} - f(x_{t_n}), j(x_{t_n} - u) \rangle \leq 0. \quad (3.19)$$

Interchange q and u to obtain

$$\langle (I - f)p, j(p - q) \rangle \leq 0. \quad (3.20)$$

Interchange p and u to obtain

$$\langle (I - f)q, j(q - p) \rangle \leq 0. \quad (3.21)$$

This implies that

$$\langle (p - q) - (f(p) - f(q)), j(p - q) \rangle \leq 0, \quad (3.22)$$

that is,

$$\|p - q\|^2 \leq \beta \|p - q\|^2. \quad (3.23)$$

This is a contradiction, so we must have $q = p$.

The proof is complete. $\square$

From Theorem 3.1 we can get the following corollary directly.

COROLLARY 3.2. *Let E be a uniformly smooth space, suppose C is a nonempty closed convex subset of E , $T : C \rightarrow E$ is a nonexpansive mapping satisfying the weakly inward condition, and $F(T) \neq \emptyset$. Let $f : C \rightarrow C$ be a fixed contractive mapping from C to C . $\{x_t\}$ is defined by*

$$x_t = tf(x_t) + (1-t)PTx_t, \quad (3.24)$$

where P is a sunny nonexpansive retraction of E onto C , then x_t converges strongly to some fixed point q of T as $t \rightarrow 0$ and q is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I-f)q, j(q-u) \rangle \quad \forall u \in F(T). \quad (3.25)$$

LEMMA 3.3 ([6], Lemma 2.1). *Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n \quad \forall n \geq 0, \quad (3.26)$$

where $\{\gamma_n\} \in (0,1)$ and δ_n is a sequence in $\mathbb{R}$ such that:

- (i) $\lim_{n \rightarrow \infty} \gamma_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$;
- (ii) either $\sum_{n=0}^{\infty} \delta_n < +\infty$ or $\limsup_{n \rightarrow \infty} (\delta_n/\gamma_n) \leq 0$,

then $\lim_{n \rightarrow \infty} \alpha_n = 0$.

THEOREM 3.4. *Let E be a uniformly smooth Banach space, suppose that C is a nonempty closed convex subset of E , $T : C \rightarrow E$ is a nonexpansive nonself-mapping satisfying the weakly inward condition, and $F(T) \neq \emptyset$. Let $f : C \rightarrow C$ be a fixed contractive mapping, and $\{x_n\}$ is defined by (1.5), where P is a sunny nonexpansive retraction of E onto C , and $\alpha_n \in (0,1)$ satisfies the following conditions:*

- (i) $\alpha_n \rightarrow 0$, as $n \rightarrow \infty$;
- (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (iii) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n \rightarrow \infty} (\alpha_{n+1}/\alpha_n) = 1$.

Then x_n converges strongly to a fixed point q of T such that q is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I-f)q, j(q-u) \rangle \leq 0 \quad \forall u \in F(T). \quad (3.27)$$

Proof. First we show $\{x_n\}$ is bounded. Take $u \in F(T)$, it follows that

$$\begin{aligned} \|x_{n+1} - u\| &= \|P((1-\alpha_n)Tx_n + \alpha_n f(x_n)) - Pu\| \\ &\leq \|(1-\alpha_n)Tx_n + \alpha_n f(x_n) - u\| \\ &\leq (1-\alpha_n)\|Tx_n - u\| + \alpha_n(\|f(x_n) - f(u)\| + \|f(u) - u\|) \\ &\leq (1-\alpha_n)\|x_n - u\| + \alpha_n(\beta\|x_n - u\| + \|f(u) - u\|) \\ &= (1-(1-\beta)\alpha_n)\|x_n - u\| + \alpha_n\|f(u) - u\| \\ &\leq \max \left\{ \|x_n - u\|, \frac{1}{1-\beta}\|f(u) - u\| \right\}. \end{aligned} \quad (3.28)$$

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By induction,

$$\|x_n - u\| \leq \max \left\{ \|x_0 - u\|, \frac{1}{1-\beta} \|f(u) - u\| \right\}, \quad n \geq 0, \quad (3.29)$$

and $\{x_n\}$ is bounded, so are $\{Tx_n\}$ and $\{f(x_n)\}$. We claim that

$$x_{n+1} - x_n \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.30)$$

Indeed we have (for some appropriate constant $M > 0$)

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|P(\alpha_n f(x_n) + (1 - \alpha_n)Tx_n) - P(\alpha_{n-1}f(x_{n-1}) + (1 - \alpha_{n-1})Tx_{n-1})\| \\ &\leq \|\alpha_n f(x_n) + (1 - \alpha_n)Tx_n - \alpha_{n-1}f(x_{n-1}) - (1 - \alpha_{n-1})Tx_{n-1}\| \\ &\leq \|(1 - \alpha_n)(Tx_n - Tx_{n-1}) + (\alpha_n - \alpha_{n-1})(f(x_{n-1}) - Tx_{n-1})\| \\ &\quad + \alpha_n \|f(x_n) - f(x_{n-1})\| \\ &\leq (1 - \alpha_n)\|x_n - x_{n-1}\|[3pt] + M|\alpha_n - \alpha_{n-1}| + \beta\alpha_n\|x_n - x_{n-1}\| \\ &= (1 - (1 - \beta)\alpha_n)\|x_n - x_{n-1}\|[3pt] + M|\alpha_n - \alpha_{n-1}|. \end{aligned} \quad (3.31)$$

By Lemma 3.3 we have $\|x_{n+1} - x_n\| \rightarrow 0$, as $n \rightarrow \infty$. We now show that

$$\|x_n - PTx_n\| \rightarrow 0. \quad (3.32)$$

In fact,

$$\begin{aligned} \|x_{n+1} - PTx_n\| &= \|P(\alpha_n f(x_n) + (1 - \alpha_n)Tx_n) - PTx_n\| \\ &\leq \alpha_n \|f(x_n) - Tx_n\|. \end{aligned} \quad (3.33)$$

This follows from (3.30) that

$$\begin{aligned} \|x_n - PTx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - PTx_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - Tx_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.34)$$

Let $q = \lim_{t \rightarrow 0} x_t$, where $\{x_t\}$ is defined in Corollary 3.2, we get that q is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I - f)q, j(q - u) \rangle \leq 0 \quad \forall u \in F(T). \quad (3.35)$$

We next show that

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, j(x_n - q) \rangle \leq 0. \quad (3.36)$$

Form Corollary 3.2, let $x_t = tf(x_t) + (1-t)PTx_t$, indeed we can write

$$x_t - x_n = t(f(x_t) - x_n) + (1-t)(PTx_t - x_n). \quad (3.37)$$

Noting (3.32), putting

$$a_n(t) = \|x_n - PTx_n\|(\|x_n - PTx_n\| + 2\|x_n - x_t\|) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \quad (3.38)$$

and using (2.2), we obtain

$$\begin{aligned} & \|x_t - x_n\|^2 \\ & \leq (1-t)^2 \|PTx_t - x_n\|^2 + 2t \langle f(x_t) - x_n, j(x_t - x_n) \rangle \\ & \leq (1-t)^2 \|PTx_t - PTx_n + PTx_n - x_n\|^2 + 2t \langle f(x_t) - x_t, j(x_t - x_n) \rangle \\ & \quad + 2t \|x_t - x_n\|^2 \leq (1-t)^2 \|x_t - x_n\|^2 + (1-t)^2 \|x_n - PTx_n\|^2 \\ & \quad + 2(1-t)^2 \|PTx_n - x_n\| \|x_t - x_n\| + 2t \langle f(x_t) - x_t, j(x_t - x_n) \rangle + 2t \|x_t - x_n\|^2 \\ & \leq (1+t^2) \|x_t - x_n\|^2 + a_n(t) + 2t \langle f(x_t) - x_t, j(x_t - x_n) \rangle. \end{aligned} \quad (3.39)$$

The last inequality implies

$$\langle f(x_t) - x_t, j(x_n - x_t) \rangle \leq \frac{t}{2} \|x_t - x_n\|^2 + \frac{1}{2t} a_n(t). \quad (3.40)$$

From $a_n(t) \rightarrow 0$ as $n \rightarrow \infty$ we get

$$\limsup_{n \rightarrow \infty} \langle f(x_t) - x_t, j(x_n - x_t) \rangle \leq M \cdot \frac{t}{2}, \quad (3.41)$$

where $M > 0$ is a constant such that $M \geq \|x_t - x_n\|^2$ for all $n \geq 0$ and $t \in (0, 1)$. By letting $t \rightarrow 0$ in (3.41) we have

$$\lim_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \langle f(x_t) - x_t, j(x_n - x_t) \rangle \leq 0. \quad (3.42)$$

On the one hand, for all $\varepsilon > 0$, $\exists \delta_1$ such that $t \in (0, \delta_1)$,

$$\limsup_{n \rightarrow \infty} \langle f(x_t) - x_t, j(x_n - x_t) \rangle \leq \frac{\varepsilon}{2}. \quad (**)$$

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On the other hand, $\{x_t\}$ strongly converges to q , as $t \rightarrow 0$, the set $\{x_t - x_n\}$ is bounded, and the duality map J is norm-to-norm uniformly continuous on bounded sets of uniformly smooth space E ; from $x_t \rightarrow q$ ($t \rightarrow 0$), we get

$$\begin{aligned}
 & \|f(q) - q - (f(x_t) - x_t)\| \rightarrow 0, \quad t \rightarrow 0, \\
 & \| \langle f(q) - q, j(x_n - q) \rangle - \langle f(x_t) - x_t, j(x_n - x_t) \rangle \| \\
 &= \| \langle f(q) - q, j(x_n - q) - j(x_n - x_t) \rangle + \langle f(q) - q - (f(x_t) - x_t), j(x_n - x_t) \rangle \| \\
 &\leq \|f(q) - q\| \|j(x_n - q) - j(x_n - x_t)\| \\
 &\quad + \|f(q) - q - (f(x_t) - x_t)\| \|x_n - x_t\| \rightarrow 0, \quad t \rightarrow 0.
 \end{aligned} \tag{3.43}$$

Hence for the above $\varepsilon > 0$, $\exists \delta_2$, such that for all $t \in (0, \delta_2)$, for all n , we have

$$\| \langle f(q) - q, j(x_n - q) \rangle - \langle f(x_t) - x_t, j(x_n - x_t) \rangle \| \leq \frac{\varepsilon}{2}. \tag{3.44}$$

Therefore, we have

$$\langle f(q) - q, j(x_n - q) \rangle \leq \langle f(x_t) - x_t, j(x_n - x_t) \rangle + \frac{\varepsilon}{2}. \tag{3.45}$$

Noting (**) and taking $\delta = \min\{\delta_1, \delta_2\}$, for all $t \in (0, \delta)$, we have

$$\begin{aligned}
 & \limsup_{n \rightarrow \infty} \langle f(q) - q, j(x_n - q) \rangle \\
 &\leq \limsup_{n \rightarrow \infty} \left(\langle f(x_t) - x_t, j(x_n - x_t) \rangle + \frac{\varepsilon}{2} \right) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
 \end{aligned} \tag{3.46}$$

Since ε is arbitrary, we get

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, j(x_n - q) \rangle \leq 0. \tag{3.47}$$

Finally we show $x_n \rightarrow q$. Indeed

$$x_{n+1} - (\alpha_n f(x_n) + (1 - \alpha_n)q) = (x_{n+1} - q) - \alpha_n (f(x_n) - q). \tag{3.48}$$

By (2.2) we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|x_{n+1} - (\alpha_n f(x_n) + (1 - \alpha_n)q) + \alpha_n(f(x_n) - q)\|^2 \\
&\leq \|x_{n+1} - P(\alpha_n f(x_n) + (1 - \alpha_n)q)\|^2 + 2\alpha_n \langle f(x_n) - q, j(x_{n+1} - q) \rangle \\
&\leq \|P[\alpha_n f(x_n) + (1 - \alpha_n)Tx_n] - P(\alpha_n f(x_n) + (1 - \alpha_n)q)\|^2 \\
&\quad + 2\alpha_n \langle f(x_n) - q, j(x_{n+1} - q) \rangle \\
&\leq (1 - \alpha_n)^2 \|Tx_n - q\|^2 + 2\alpha_n \langle f(x_n) - f(q), j(x_{n+1} - q) \rangle \\
&\quad + 2\alpha_n \langle f(q) - q, j(x_{n+1} - q) \rangle \\
&\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \|f(q) - f(x_n)\| \|x_{n+1} - q\| \\
&\quad + 2\alpha_n \langle f(q) - q, j(x_{n+1} - q) \rangle \\
&\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + \alpha_n (\|f(q) - f(x_n)\|^2 + \|x_{n+1} - q\|^2) \\
&\quad + 2\alpha_n \langle f(q) - q, j(x_{n+1} - q) \rangle.
\end{aligned} \tag{3.49}$$

Therefore, we have

$$\begin{aligned}
(1 - \alpha_n) \|x_{n+1} - q\|^2 \\
\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + \alpha_n \beta^2 \|x_n - q\|^2 + 2\alpha_n \langle f(q) - q, j(x_{n+1} - q) \rangle.
\end{aligned} \tag{3.50}$$

That is,

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \left(1 - \frac{1 - \beta^2}{1 - \alpha_n} \alpha_n\right) \|x_n - q\|^2 + \frac{\alpha_n^2}{1 - \alpha_n} \|x_n - q\|^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha_n} \langle f(q) - q, j(x_{n+1} - q) \rangle \\
&\leq (1 - \gamma_n) \|x_n - q\|^2 + \lambda \gamma_n \alpha_n + \frac{2}{1 - \beta^2} \gamma_n \langle f(q) - q, j(x_{n+1} - q) \rangle,
\end{aligned} \tag{3.51}$$

where $\gamma_n = ((1 - \beta^2)/(1 - \alpha_n))\alpha_n$ and λ is a constant such that $\lambda > (1/(1 - \beta^2))\|x_n - q\|^2$.

Hence,

$$\|x_{n+1} - q\|^2 \leq (1 - \gamma_n) \|x_n - q\|^2 + \gamma_n \left(\lambda \alpha_n + \frac{2}{1 - \beta^2} \langle f(q) - q, j(x_{n+1} - q) \rangle \right). \tag{3.52}$$

It is easily seen that $\gamma_n \rightarrow 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$, and (noting (3.36))

$$\limsup_{n \rightarrow \infty} \left(\lambda \alpha_n + \frac{2}{1 - \beta^2} \langle f(q) - q, j(x_{n+1} - q) \rangle \right) \leq 0. \tag{3.53}$$

Applying Lemma 3.3 onto (3.52), we have $x_n \rightarrow q$.

The proof is complete. $\square$

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