SOME NEW GENERALIZATIONS OF HARDY'S INTEGRAL INEQUALITY

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We have studied some new generalizations of Hardy's integral inequality using the generalized Holder's inequality.

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1. Introduction

The classical Hardy's inequality [2] states that for $p > 1$, $1/p + 1/q = 1$, $f \geq 0$, and $0 < \int_0^\infty f^p(t)dt < \infty$, then

$$\int_0^\infty \left[\frac{1}{x} \int_0^x f(t)dt \right]^p dx < q^p \int_0^\infty f^p(t)dt, \quad (1.1)$$

where $q^p = (p/(p-1))^p$ is the best possible. This inequality plays important role in analysis. It is obvious that, for parameters a and b such that $0 < a < b < \infty$, the following inequality is also valid:

$$\int_a^b \left[\frac{1}{x} \int_a^x f(t)dt \right]^p dx < q^p \int_a^b f^p(t)dt, \quad (1.2)$$

where $0 < \int_a^b f^p(t)dt < \infty$.

Bicheng et al. [1] have given some improvements of (1.1) and (1.2) as follows.

Let $0 < a < b < \infty$, $p > 1$, $1/p + 1/q = 1$, $f \geq 0$, and $0 < \int_a^b f^p(t)dt < \infty$, then

$$\begin{aligned} \int_a^b \left[\frac{1}{x} \int_a^x f(t)dt \right]^p dx \\ < q^p \left[1 - \left(\frac{a}{b} \right)^{1/q} \right]^p \int_a^b f^p(t)dt. \end{aligned} \quad (1.3)$$

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Let $a > 0$, $p > 1$, $1/p + 1/q = 1$, $f \geq 0$, and $0 < \int_a^\infty f^p(t)dt < \infty$, then

$$\int_a^\infty \left[\frac{1}{x} \int_a^x f(t)dt \right]^p dx < q^p \int_a^\infty [1 - \theta_p(t)] f^p(t)dt, \quad (1.4)$$

where $\theta_p(t) = 1/p \sum_{k=1}^\infty \left(\frac{p}{k}\right) (-1)^{k-1} (a/t)^{k/q} > 0$ for $t > a$, and $\theta_p(a) = 1/q$.

Oguntuase and Imoru [3] generalized (1.3) and (1.4) as follows.

Let $0 < a < b < \infty$, $p > 1$, $1/p + 1/q = 1 - 1/r$, $f \geq 0$, $r > 1$, and $0 < \int_a^b f^p(t)dt < \infty$, then

$$\begin{aligned} \int_a^b \left[\frac{1}{x^{(1-1/r)}} \int_a^x f(t)dt \right]^p dx \\ < q^{(1-1/r)p} \left(1 - \frac{1}{r}\right)^{(1-1/r)p} \left[1 - \left(\frac{a}{b}\right)^{1/q}\right]^{(1-1/r)p} \int_a^b f^p(t)dt. \end{aligned} \quad (1.5)$$

Let $a > 0$, $p > 1$, $1/p + 1/q = 1 - 1/r$, $f \geq 0$, $r > 1$, and $0 < \int_a^\infty f^p(t)dt < \infty$, then

$$\int_a^\infty \left[\frac{1}{x^{(1-1/r)}} \int_a^x f(t)dt \right]^p dx < q^{(1-1/r)p} \left(1 - \frac{1}{r}\right)^{(1-1/r)p} \int_a^\infty [1 - \theta_p(t)] f^p(t)dt, \quad (1.6)$$

where $\theta_p(t) = 1/(1 - 1/r)p \sum_{k=1}^\infty \left(\frac{(1-1/r)p}{k+1}\right) (-1)^{k-1} (a/t)^{k/(1-1/r)q} > 0$ for $t > a$, and $\theta_p(a) = 1/(1 - 1/r)q$.

Definition 1.1. Let $1 \leq p < \infty$, then the function space L_p is given by

$$L_p = \left\{ f : \int_0^\infty |f(x)|^p dx < \infty \right\}. \quad (1.7)$$

The function space L_p has been generalized to $L(p)$ in the following manner.

Definition 1.2. Let p be a bounded measurable function, with $0 < p(x) \leq \sup p(x) = H < \infty$. Define

$$L(p) = \left\{ f : \int_0^\infty |f(x)|^{p(x)} dx < \infty \right\}. \quad (1.8)$$

Note that $L(p)$ is a linear topological space paranormed by $d(f)$,

$$d(f) = \left(\int_0^\infty |f(x)|^{p(x)} dx \right)^{1/M}, \quad (1.9)$$

where $M = \max(1, H)$.

In this paper, we have the generalized Holder's inequality in $L(p)$ space and the results of [1, 3].

2. Main results

LEMMA 2.1. (a) Let the functions p and q be such that $p(x)^{-1} + q(x)^{-1} = 1$ for all x . Let $f \in L(p)$, $g \in L(q)$. Let

$$A = \int_0^\infty |f(x)|^{p(x)} dx, \quad B = \int_0^\infty |g(x)|^{q(x)} dx. \quad (2.1)$$

Then for $p(x) > 1$, $fg \in L_1$ and

$$\int_0^\infty |f(x)g(x)| dx \leq \alpha\beta, \quad (2.2)$$

where $\alpha = \sup_x 1/p(x) + \sup_x 1/q(x)$, $\beta = \sup_x A^{1/p(x)} B^{1/q(x)}$.

(b) Let $0 < p(x) < 1$ and $p(x)^{-1} + q(x)^{-1} = 1$. If $f \in L(p)$ and $g \in L(q)$, then

$$\int_0^\infty |f(x)|^{p(x)} dx \leq \alpha \left[\sup_x p(x) + \sup_x (1 - p(x)) \right], \quad (2.3)$$

where $\alpha = \sup_x [(\int |g(x)|^{q(x)} dx)^{1-p(x)} (\int |f(x)g(x)| dx)^{p(x)}]$.

Proof. To prove (a), for $a, b > 0$, we have

$$ab \leq \frac{a^{p(x)}}{p(x)} + \frac{b^{q(x)}}{q(x)} \quad \forall x. \quad (2.4)$$

Using the above inequality, we have

$$\frac{|f(x)|}{A^{1/p(x)}} \frac{|g(x)|}{B^{1/q(x)}} \leq \frac{1}{p(x)} \frac{|f(x)|^{p(x)}}{A} + \frac{1}{q(x)} \frac{|g(x)|^{q(x)}}{B}. \quad (2.5)$$

Therefore,

$$\begin{aligned} \int_0^\infty \frac{|f(x)|}{A^{1/p(x)}} \frac{|g(x)|}{B^{1/q(x)}} dx &\leq \frac{1}{A} \int_0^\infty \frac{|f(x)|^{p(x)}}{p(x)} dx + \frac{1}{B} \int_0^\infty \frac{|g(x)|^{q(x)}}{q(x)} dx \\ &\leq \sup_x \frac{1}{p(x)} + \sup_x \frac{1}{q(x)}. \end{aligned} \quad (2.6)$$

Also

$$\frac{1}{\sup_x A^{1/p(x)} B^{1/q(x)}} \int_0^\infty |f(x)g(x)| dx \leq \int_0^\infty \frac{|f(x)g(x)|}{A^{1/p(x)} B^{1/q(x)}} dx. \quad (2.7)$$

From (2.6) and (2.7), we get (2.2).

To prove (b), let $p_1(x) = 1/p(x)$, so that $p_1(x) > 1$ for all x . Let

$$A(x) = |g(x)|^{-1/p_1(x)}, \quad B(x) = |f(x)g(x)|^{1/p_1(x)}. \quad (2.8)$$

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So

$$\begin{aligned} \int_0^\infty |f(x)|^{p(x)} dx &= \int_0^\infty |A(x)B(x)| dx \\ &\leq \sup_x \left[\left(\int_0^\infty |A(x)|^{q_1(x)} dx \right)^{1/q_1(x)} \left(\int_0^\infty |B(x)|^{p_1(x)} dx \right)^{1/p_1(x)} \right] \\ &\quad \times \left(\sup_x \frac{1}{p_1(x)} + \sup_x \frac{1}{q_1(x)} \right) \quad \text{by (2.2)}. \end{aligned} \quad (2.9)$$

Since $1/p_1(x) + 1/q_1(x) = 1$, so $1/q_1(x) = 1 - p(x)$ and $q_1(x) = 1/1 - p(x)$. Substituting the values, we get (2.3):

$$\begin{aligned} \int_0^\infty |f(x)|^{p(x)} dx &= \sup_x \left[\left(\int_0^\infty |g(x)|^{q(x)} dx \right)^{1-p(x)} \left(\int_0^\infty |f(x)g(x)| dx \right)^{p(x)} \right] \\ &\quad \times \left[\sup_x p(x) + \sup_x (1 - p(x)) \right] = \alpha \left[\sup_x p(x) + \sup_x (1 - p(x)) \right]. \end{aligned} \quad (2.10)$$

This completes the proof of the lemma. $\square$

LEMMA 2.2. *Let $0 < b \leq \infty$, for all $x \in (0, b)$, $p(x) > 1$, $p(x)^{-1} + q(x)^{-1} = 1$, $f \geq 0$, and $0 < \int_0^b f^{p(t)}(t) dt < \infty$. Then the following inequality holds:*

$$\int_0^x f(t) dt < \alpha \sup_{x \in (0, b)} \left\{ \{ \inf q(x) \}^{1/q(x)} x^{\sup 1/q(x)2} \left[\int_0^x t^{1/q(t)} f^{p(t)}(t) dt \right]^{1/p(x)} \right\}, \quad (2.11)$$

where $\alpha = \sup_x 1/p(x) + \sup_x 1/q(x)$.

Proof. For any $x \in (0, b)$, by the generalized Holder's inequality (2.2), we have

$$\begin{aligned} \int_0^x f(t) dt &= \int_0^x t^{1/p(t)q(t)} f(t) t^{-1/p(t)q(t)} dt \\ &\leq \alpha \sup_{x \in (0, b)} \left\{ \left(\int_0^x t^{1/q(t)} f^{p(t)}(t) dt \right)^{1/p(x)} \left(\int_0^x t^{-1/p(t)} dt \right)^{1/q(x)} \right\} \\ &= \alpha \sup_{x \in (0, b)} \left\{ \{ \inf q(x) \}^{1/q(x)} x^{\sup 1/q(x)2} \left[\int_0^x t^{1/q(t)} f^{p(t)}(t) dt \right]^{1/p(x)} \right\}. \end{aligned} \quad (2.12)$$

Strictness follows from [1, Lemma 2.1]. Thus (2.11) is valid. $\square$

LEMMA 2.3. *Let $a \geq 0$, for all $x \in (a, \infty)$, $p(x) > 1$, $p(x)^{-1} + q(x)^{-1} = 1$, $f \geq 0$, and $0 < \int_a^x f^{p(t)}(t) dt < \infty$. Then the following inequality is true:*

$$\int_a^x f(t) dt < \alpha \sup_{x \in (a, \infty)} \left\{ \{ \inf q(x) \}^{1/q(x)} (x^{\sup 1/q(x)} - a^{\sup 1/q(x)}) \left[\int_a^x t^{1/q(t)} f^{p(t)}(t) dt \right]^{1/p(x)} \right\}, \quad (2.13)$$

where $\alpha = \sup_x 1/p(x) + \sup_x 1/q(x)$.

Proof. For any $x \in (a, \infty)$, by the generalized Holder's inequality (2.2), we have

$$\begin{aligned} \int_a^x f(t)dt &\leq \alpha \sup_{x \in (a, \infty)} \left\{ \left(\int_a^x t^{1/q(t)} f^{p(t)}(t)dt \right)^{1/p(x)} \left(\int_a^x t^{-1/p(t)} dt \right)^{1/q(x)} \right\} \\ &\leq \alpha \sup_{x \in (a, \infty)} \left\{ \{\inf q(x)\}^{1/q(x)} (x^{\sup 1/q(x)} - a^{\sup 1/q(x)}) \left[\int_a^x t^{1/q(t)} f^{p(t)}(t)dt \right]^{1/p(x)} \right\}. \end{aligned} \quad (2.14)$$

Strictness follows from [1, Lemma 2.2]. This completes the proof of the lemma. $\square$

THEOREM 2.4. Let $0 < a < b < \infty$, for all $x \in (a, b)$, $p(x) > 1$, $p(x)^{-1} + q(x)^{-1} = 1$, $f \geq 0$, and $0 < \int_a^b f^{p(t)}(t)dt < \infty$. Then

$$\int_a^b \left(\frac{1}{x} \int_a^x f(t)dt \right)^{p(x)} dx < \alpha^M \{\inf q(t)\}^M \left[1 - \left(\frac{a}{b} \right)^{\sup 1/q(t)} \right]^M \int_a^b f^{p(t)}(t)dt, \quad (2.15)$$

where $M = \sup p(t)$, $\alpha = \sup \{p(t)^{-1}\} + \sup \{q(t)^{-1}\}$.

Proof. Using (2.13), we obtain

$$\begin{aligned} &\int_a^b \left(\frac{1}{x} \int_a^x f(t)dt \right)^{p(x)} dx \\ &< \alpha^{\sup p(x)} \int_a^b x^{-p(x)} (x^{\sup 1/q(x)} - a^{\sup 1/q(x)})^{p(x)/q(x)} \{\inf q(x)\}^{p(x)/q(x)} \\ &\quad \times \int_a^x t^{1/q(t)} f^{p(t)}(t)dt dx \leq \alpha^{\sup p(t)} \{\inf q(t)\}^{\sup p(t)/q(t)} \\ &\quad \times \int_a^b \left\{ \int_t^b x^{-p(x) + \sup(p(x)/q(x)^2)} \left(1 - \left(\frac{a}{x} \right)^{\sup(1/q(x))} \right)^{p(x)/q(x)} dx \right\} t^{1/q(t)} f^{p(t)}(t)dt \\ &\leq \alpha^M \{\inf q(t)\}^{\sup(p(t)-1)} \\ &\quad \times \int_a^b \left\{ \int_t^b x^{-1-1/q(x)} \left(1 - \left(\frac{a}{b} \right)^{\sup 1/q(x)} \right)^{p(x)/q(x)} dx \right\} t^{1/q(t)} f^{p(t)}(t)dt \\ &\leq \alpha^M \{\inf q(t)\}^{\sup p(t)} \left[1 - \left(\frac{a}{b} \right)^{\sup(1/q(t))} \right]^{\sup p(t)-1} \int_a^b \left[1 - \left(\frac{t}{b} \right)^{\sup(1/q(t))} \right] f^{p(t)}(t)dt \\ &\leq \alpha^M \{\inf q(t)\}^M \left[1 - \left(\frac{a}{b} \right)^{\sup(1/q(t))} \right]^M \int_a^b f^{p(t)}(t)dt. \end{aligned} \quad (2.16)$$

$\square$

THEOREM 2.5. Let $a > 0$, for all $x \in (a, \infty)$, $p(x) > 1$, $p(x)^{-1} + q(x)^{-1} = 1$, $f \geq 0$, and $0 < \int_a^\infty f^{p(t)}(t)dt < \infty$. Then

$$\int_a^\infty \left(\frac{1}{x} \int_a^x f(t)dt \right)^{p(x)} dx < \alpha^M \{\inf q(t)\}^M \int_a^\infty [1 - \theta_M(t)] f^{p(t)}(t)dt, \quad (2.17)$$

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where $\theta_M(t) = (1/M)[\sum_{k=1}^{\infty} (\Gamma(M+1)/\Gamma(k+2)\Gamma(M-k))(-1)^{k-1}(a/t)^{(k/\inf q(t))}]$, for $t > a > 0$, $M > 1$, and $\theta_M(a) = \sup 1/q(t)$, $M = \sup p(t)$, and $\alpha = \sup\{p(t)^{-1}\} + \sup\{q(t)^{-1}\}$.

Proof. In view of inequalities (2.13) and (2.15), we find

$$\begin{aligned}
 & \int_a^{\infty} \left(\frac{1}{x} \int_a^x f(t) dt \right)^{p(x)} dx \\
 & < \alpha^{\sup p(x)} \int_a^{\infty} x^{-p(x)} (x^{\sup 1/q(x)} - a^{\sup 1/q(x)})^{p(x)/q(x)} \{ \inf q(x) \}^{p(x)/q(x)} \\
 & \quad \times \int_a^x t^{1/q(t)} f^{p(t)}(t) dt dx \leq \alpha^{\sup p(t)} \{ \inf q(t) \}^{\sup(p(t)-1)} \\
 & \quad \times \int_a^{\infty} \left\{ \int_t^{\infty} x^{-1-1/q(x)} \left(1 - \left(\frac{a}{b} \right)^{\sup(1/q(x))} \right)^{p(x)/q(x)} dx \right\} t^{1/q(t)} f^{p(t)}(t) dt \\
 & \leq \alpha^M \{ \inf q(t) \}^{\sup(p(t)-1)} \\
 & \quad \times \int_a^{\infty} \left[\int_t^{\infty} \left(1 - \left(\frac{a}{b} \right)^{\sup(1/q(x))} \right)^{p(x)/q(x)} d \left(1 - \left(\frac{a}{b} \right)^{\sup(1/q(x))} \right) \right] \\
 & \quad \times \left(\frac{t}{a} \right)^{\sup(1/q(t))} f^{p(t)}(t) dt \leq \alpha^M \{ \inf q(t) \}^{\sup p(t)} \\
 & \quad \times \int_a^{\infty} \frac{1}{\sup p(t)} \left\{ 1 - \left[1 - \left(\frac{a}{t} \right)^{\sup(1/q(t))} \right]^{\sup p(t)} \right\} \left(\frac{t}{a} \right)^{\sup(1/q(t))} f^{p(t)}(t) dt \\
 & = \alpha^M \{ \inf q(t) \}^M \int_a^{\infty} [1 - \theta_M(t)] f^{p(t)}(t) dt,
 \end{aligned} \tag{2.18}$$

where

$$\begin{aligned}
 \theta_M(t) &= 1 - \frac{1}{M} \left\{ 1 - \left[1 - \left(\frac{a}{t} \right)^{\sup(1/q(t))} \right]^M \right\} \left(\frac{t}{a} \right)^{\sup(1/q(t))}, \quad t > a > 0, M > 1, \\
 \theta_M(a) &= \sup \frac{1}{q(t)}.
 \end{aligned} \tag{2.19}$$

Since

$$\begin{aligned}
 \left[1 - \left(\frac{a}{t} \right)^{\sup(1/q(t))} \right]^M &= \sum_{k=0}^{\infty} \frac{\Gamma(M+1)}{\Gamma(k+1)\Gamma(M-k+1)} (-1)^k \left(\frac{a}{t} \right)^{k/\inf q(t)}, \quad t > a > 0, M > 1, \\
 \theta_M(t) &= \frac{1}{M} \left[\sum_{k=1}^{\infty} \frac{\Gamma(M+1)}{\Gamma(k+2)\Gamma(M-k)} (-1)^{k-1} \left(\frac{a}{t} \right)^{k/\inf q(t)} \right], \quad t > a > 0, M > 1,
 \end{aligned} \tag{2.20}$$

the proof is complete. $\square$

Note. When $t > a > 0$, by Bernoulli's inequality (see [2, Chapter 2.4]), we obtain

$$1 - M \left(\frac{a}{t} \right)^{\sup 1/q(t)} < \left[1 - \left(\frac{a}{t} \right)^{\sup 1/q(t)} \right]^M, \quad (2.21)$$

$$\theta_M(t) > 1 - \frac{1}{M} \left[1 - \left\{ 1 - M \left(\frac{a}{t} \right)^{\sup 1/q(t)} \right\} \right] \left(\frac{t}{a} \right)^{\sup 1/q(t)} = 0.$$

Applications

THEOREM 2.6. Let $0 < b \leq \infty$, for all $x \in (0, \infty)$, $r \geq p(x) > 1$, $p(x)^{-1} + q(x)^{-1} = 1$, $f \geq 0$, and $0 < \int_0^b x^{-r+\sup p(x)} f^{p(x)}(x) < \infty$. Then
(i) for $b \in (0, \infty)$,

$$\begin{aligned} & \int_0^b x^{-r} \left(\int_0^x f(t) dt \right)^{p(x)} dx \\ & < \frac{\alpha^M \{ \inf q(t) \}^M}{\inf \{ (r - p(t))q(t) + 1 \}} \int_0^b \left[1 - \left(\frac{t}{b} \right)^{\sup \{ r - p(t) + 1/q(t) \}} \right] t^{-r+\sup p(t)} f^{p(t)}(t) dt, \end{aligned} \quad (2.22)$$

(ii) for $b = \infty$,

$$\int_0^\infty x^{-r} \left(\int_0^x f(t) dt \right)^{p(x)} dx < \frac{\alpha^M \{ \inf q(t) \}^M}{\inf \{ (r - p(t))q(t) + 1 \}} \int_0^\infty t^{-r+\sup p(t)} f^{p(t)}(t) dt, \quad (2.23)$$

where $M = \sup p(t)$ and $\alpha = \sup \{ p(t)^{-1} \} + \sup \{ q(t)^{-1} \}$.

Proof. For case (i), $b \in (0, \infty)$, we use (2.11) to obtain

$$\begin{aligned} & \int_0^b x^{-r} \left(\int_0^x f(t) dt \right)^{p(x)} dx \\ & < \alpha^M \int_0^b x^{-r+\sup \{ p(x)/q(x)2 \}} \{ \inf q(x) \}^{p(x)/q(x)} \int_0^x t^{1/q(t)} f^{p(t)}(t) dt dx \\ & \leq \alpha^M \{ \inf q(t) \}^{\sup \{ p(t)-1 \}} \int_0^b \left(\int_t^b x^{-r+\sup \{ p(x)-1-1/q(x) \}} dx \right) t^{1/q(t)} f^{p(t)}(t) dt \\ & \leq \frac{\alpha^M \{ \inf q(t) \}^M}{-r + \sup \{ p(t) - 1/q(t) \}} \int_0^b (b^{-r+\sup \{ p(t)-1/q(t) \}} - t^{-r+\sup \{ p(t)-1/q(t) \}}) \\ & \quad \times t^{1/q(t)} f^{p(t)}(t) dt \\ & \leq \frac{\alpha^M \{ \inf q(t) \}^M}{\inf \{ (r - p(t))q(t) + 1 \}} \int_0^b \left[1 - \left(\frac{t}{b} \right)^{\sup \{ r - p(t) + 1/q(t) \}} \right] t^{-r+\sup p(t)} f^{p(t)}(t) dt. \end{aligned} \quad (2.24)$$

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For case (ii), $b = \infty$, we use (2.11) to find

$$\begin{aligned}
 & \int_0^\infty x^{-r} \left(\int_0^x f(t) dt \right)^{p(x)} dx \\
 & < \alpha^M \{ \inf q(t) \}^{\sup(p(t)-1)} \int_0^\infty x^{-r+\sup\{p(x)/q(x)^2\}} \int_0^x t^{1/q(t)} f^{p(t)}(t) dt dx \\
 & = \alpha^M \{ \inf q(t) \}^{\sup(p(t)-1)} \int_0^\infty \left(\int_t^\infty x^{-r+\sup\{p(x)-1-1/q(x)\}} dx \right) t^{1/q(t)} f^{p(t)}(t) dt \\
 & = \frac{\alpha^M \{ \inf q(t) \}^M}{\inf \{ (r - p(t))q(t) + 1 \}} \int_0^\infty t^{-r+\sup\{p(t)\}} f^{p(t)}(t) dt.
 \end{aligned} \tag{2.25}$$

□

Remark 2.7. (a) If $p(x)$ and $q(x)$ are constants in Lemma 2.1, then (1.6) reduces to usual Holder's inequality in L_p space.

(b) If we take $p(x)$ and $q(x)$ constants in Lemmas 2.2 and 2.3 and Theorems 2.4 and 2.5, then our results reduce to the corresponding Lemmas 2.1 and 2.2 and Theorems 2.4 and 2.5 obtained in [1].

(c) When $p(x) = r$ and $q(x)$ is constant, inequality (2.23) reduces to (1.1).

3. Some more generalized results

In this section, we have generalized the results of [3]. We use the generalized form of Holder's inequality with $p(x), q(x), r(x) > 1$. The normalization $1/p(x) + 1/q(x) = 1$ in Holder's inequality is replaced by relation of the form $1/p(x) + 1/q(x) = 1 - 1/r(x)$.

LEMMA 3.1. *Let $0 < b \leq \infty$, for all $x \in (0, b)$, $p(x) > 1$, $p(x)^{-1} + q(x)^{-1} = 1 - r(x)^{-1}$, $f \geq 0$, $r(x) > 1$, and $0 < \int_0^b f^{p(t)}(t) dt < \infty$. Then the following inequality holds:*

$$\begin{aligned}
 \int_0^x f(t) dt & < \alpha \sup_{x \in (0, b)} \left\{ \left\{ \inf \left\{ \left(1 - \frac{1}{r(x)} \right) q(x) \right\} \right\}^{1/q(x)} x^{\sup 1/\{(1-1/r(x))q(x)^2\}} \right. \\
 & \quad \left. \times \left[\int_0^x t^{1/\{(1-1/r(t))q(t)\}} f^{p(t)}(t) dt \right]^{1/p(x)} \right\},
 \end{aligned} \tag{3.1}$$

where $\alpha = \sup_x 1/p(x) + \sup_x 1/q(x) + \sup_x 1/r(x)$.

Proof. For any $x \in (0, b)$, by the generalized Holder's inequality (2.2), we have

$$\begin{aligned}
 \int_0^x f(t) dt & = \int_0^x t^{1/\{(1-1/r(t))p(t)q(t)\}} f(t) t^{-1/\{(1-1/r(t))p(t)q(t)\}} dt \\
 & \leq \alpha \sup_{x \in (0, b)} \left\{ \left(\int_0^x t^{1/\{(1-1/r(t))q(t)\}} f^{p(t)}(t) dt \right)^{1/p(x)} \left(\int_0^x t^{-1/\{(1-1/r(t))p(t)\}} dt \right)^{1/q(x)} \right\}
 \end{aligned}$$

$$\begin{aligned}
&\leq \alpha \sup_{x \in (0, b)} \left\{ \left(\int_0^x t^{\sup\{-1/\{(1-1/r(t))p(t)\}\}} dt \right)^{1/q(x)} \right. \\
&\quad \times \left. \left(\int_0^x t^{1/\{(1-1/r(t))q(t)\}} f^{p(t)}(t) dt \right)^{1/p(x)} \right\} \\
&= \alpha \sup_{x \in (0, b)} \left\{ \left\{ \inf \left\{ \left(1 - \frac{1}{r(x)} \right) q(x) \right\} \right\}^{1/q(x)} x^{\sup 1/\{(1-1/r(x))q(x)^2\}} \right. \\
&\quad \times \left. \left[\int_0^x t^{\{1/(1-1/r(t))q(t)\}} f^{p(t)}(t) dt \right]^{1/p(x)} \right\}.
\end{aligned} \tag{3.2}$$

Strictness follows from [3, Lemma 2.1]. Thus (3.1) is valid. $\square$

LEMMA 3.2. *Let $a \geq 0$, for all $x \in (a, \infty)$, $p(x) > 1$, $p(x)^{-1} + q(x)^{-1} = 1 - r(x)^{-1}$, $f \geq 0$, $r(x) > 1$, and $0 < \int_a^x f^{p(t)}(t) dt < \infty$. Then the following inequality holds:*

$$\begin{aligned}
\int_a^x f(t) dt &< \alpha \sup_{x \in (a, \infty)} \left\{ \left\{ \inf \left\{ \left(1 - \frac{1}{r(x)} \right) q(x) \right\} \right\}^{1/q(x)} \right. \\
&\quad \times \left. \left(x^{\sup 1/\{(1-1/r(x))q(x)\}} - a^{\sup 1/\{(1-1/r(x))q(x)\}} \right)^{1/q(x)} \right. \\
&\quad \times \left. \left[\int_a^x t^{1/\{(1-1/r(t))q(t)\}} f^{p(t)}(t) dt \right]^{1/p(x)} \right\},
\end{aligned} \tag{3.3}$$

where $\alpha = \sup_x 1/p(x) + \sup_x 1/q(x) + \sup_x 1/r(x)$.

Proof. For any $x \in (a, \infty)$, by the generalized Holder's inequality (2.2), we have

$$\begin{aligned}
\int_a^x f(t) dt &\leq \alpha \sup_{x \in (a, \infty)} \left\{ \left(\int_a^x t^{\sup\{-1/\{(1-1/r(t))p(t)\}\}} dt \right)^{1/q(x)} \right. \\
&\quad \times \left. \left(\int_a^x t^{1/\{(1-1/r(t))q(t)\}} f^{p(t)}(t) dt \right)^{1/p(x)} \right\} \\
&\leq \alpha \sup_{x \in (a, \infty)} \left\{ \left\{ \inf \left\{ \left(1 - \frac{1}{r(x)} \right) q(x) \right\} \right\}^{1/q(x)} \right. \\
&\quad \times \left. \left(x^{\sup 1/\{(1-1/r(x))q(x)\}} - a^{\sup 1/\{(1-1/r(x))q(x)\}} \right)^{1/q(x)} \right. \\
&\quad \times \left. \left(\int_a^x t^{1/\{(1-1/r(t))q(t)\}} f^{p(t)}(t) dt \right)^{1/p(x)} \right\}.
\end{aligned} \tag{3.4}$$

Strictness follows from [3, Lemma 2.2]. This completes the proof of the lemma. $\square$

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THEOREM 3.3. *Let $0 < a < b < \infty$, for all $x \in (a, b)$, $p(x) > 1$, $p(x)^{-1} + q(x)^{-1} = 1 - r(x)^{-1}$, $f \geq 0$, $r(x) > 1$, and $0 < \int_a^b f^{p(t)}(t)dt < \infty$. Then*

$$\begin{aligned} & \int_a^b \left(\frac{1}{x^{(1-1/r(x))}} \int_a^x f(t)dt \right)^{p(x)} dx \\ & < \alpha^M \left\{ \inf \left\{ \left(1 - \frac{1}{r(t)} \right) q(t) \right\} \right\}^N \left[1 - \left(\frac{a}{b} \right)^{\sup 1/\{(1-1/r(t))q(t)\}} \right]^N \int_a^b f^{p(t)}(t)dt, \end{aligned} \quad (3.5)$$

where $M = \sup p(t)$, $\alpha = \sup 1/p(t) + \sup 1/q(t) + \sup 1/r(t)$, and $N = \sup \{p(t)(1 - 1/r(t))\}$.

Proof. Using (3.3), we obtain

$$\begin{aligned} & \int_a^b \left(\frac{1}{x^{(1-1/r(x))}} \int_a^x f(t)dt \right)^{p(x)} dx \\ & < \int_a^b \left[\frac{1}{x^{(1-1/r(x))}} \alpha \sup \left\{ \left\{ \inf \left\{ \left(1 - \frac{1}{r(x)} \right) qx \right\} \right\}^{1/q(x)} \right. \right. \\ & \quad \times \left(x^{\sup 1/\{(1-1/r(x))q(x)\}} - a^{\sup 1/\{(1-1/r(x))q(x)\}} \right)^{1/q(x)} \\ & \quad \times \left. \left. \left\{ \int_a^x t^{1/\{(1-1/r(t))q(t)\}} f^{p(t)}(t)dt \right\}^{1/p(x)} \right\}^{p(x)} dx \right] \\ & \leq \alpha^{\sup p(x)} \left\{ \inf \left\{ \left(1 - \frac{1}{r(x)} \right) qx \right\} \right\}^{\sup \{p(x)/q(x)\}} \\ & \quad \times \int_a^b x^{-(1-1/r(x))p(x)} \left(x^{\sup 1/\{(1-1/r(x))q(x)\}} - a^{\sup 1/\{(1-1/r(x))q(x)\}} \right)^{p(x)/q(x)} \\ & \quad \times \int_a^x t^{1/q(t)} f^{p(t)}(t)dt dx \\ & \leq \alpha^{\sup p(t)} \left\{ \inf \left\{ \left(1 - \frac{1}{r(t)} \right) q(t) \right\} \right\}^{\sup p(t)/q(t)} \\ & \quad \times \int_a^b \left\{ \int_t^b x^{-(1-1/r(x))p(x) + \sup(p(x)/(1-1/r(x))q(x)^2)} \right. \\ & \quad \times \left. \left(1 - \left(\frac{a}{x} \right)^{\sup(1/(1-1/r(x))q(x))} \right)^{p(x)/q(x)} dx \right\} \\ & \quad \times t^{1/\{(1-1/r(t))q(t)\}} f^{p(t)}(t)dt \leq \alpha^M \left\{ \inf \left\{ \left(1 - \frac{1}{r(t)} \right) q(t) \right\} \right\}^{\sup(p(t)-1)} \end{aligned}$$

$$\begin{aligned}
& \times \int_a^b \left\{ \int_t^b x^{-1-1/\{(1-1/r(x))q(x)\}} \left(1 - \left(\frac{a}{x} \right)^{\sup(1/(1-1/r(x))q(x))} \right)^{p(x)/q(x)} dx \right\} \\
& \times t^{1/\{(1-1/r(t))q(t)\}} f^{p(t)}(t) dt \\
& \leq \alpha^M \left\{ \inf \left\{ \left(1 - \frac{1}{r(t)} \right) q(t) \right\} \right\}^{\sup\{p(t)(1-1/r(t))\}} \\
& \times \left[1 - \left(\frac{a}{b} \right)^{\sup(1/(1-1/r(t))q(t))} \right]^{\sup\{p(t)(1-1/r(t))-1\}} \\
& \times \left[1 - \left(\frac{t}{b} \right)^{\sup(1/(1-1/r(t))q(t))} \right] f^{p(t)}(t) dt \\
& \leq \alpha^M \left\{ \inf \left\{ \left(1 - \frac{1}{r(t)} \right) q(t) \right\} \right\}^N \left[1 - \left(\frac{a}{b} \right)^{\sup(1/(1-1/r(t))q(t))} \right]^N \int_a^b f^{p(t)}(t) dt.
\end{aligned} \tag{3.6}$$

This completes the proof of our theorem. $\square$

THEOREM 3.4. *Let $a > 0$, for all $x \in (a, \infty)$, $p(x) > 1$, $p(x)^{-1} + q(x)^{-1} = 1 - r(x)^{-1}$, $f \geq 0$, $r(x) > 1$, and $0 < \int_a^\infty f^{p(t)}(t) dt < \infty$. Then*

$$\begin{aligned}
& \int_a^\infty \left(\frac{1}{x^{(1-1/r(x))}} \int_a^x f(t) dt \right)^{p(x)} dx \\
& < \alpha^M \left\{ \inf \left\{ \left(1 - \frac{1}{r(t)} \right) q(t) \right\} \right\}^N \int_a^\infty [1 - \theta_N(t)] f^{p(t)}(t) dt,
\end{aligned} \tag{3.7}$$

where $\theta_N(t) = (1/N)[\sum_{k=1}^\infty (\Gamma(N+1)/\Gamma(k+2)\Gamma(N-k))(-1)^{k-1}(a/t)^{k/(1-1/r(t))\inf q(t)}]$, for $t > a$, $M = \sup p(t)$, $\alpha = \sup p(t)^{-1} + \sup q(t)^{-1} + \sup r(t)^{-1}$, and $N = \sup p(t)(1-1/r(t))$.

Proof. In view of inequalities (3.3) and (3.5), we find

$$\begin{aligned}
& \int_a^\infty \left(\frac{1}{x^{1-1/r(x)}} \int_a^x f(t) dt \right)^{p(x)} dx \\
& < \alpha^{\sup p(x)} \left\{ \inf \left(1 - \frac{1}{r(x)} \right) q(x) \right\}^{\sup\{p(x)/q(x)\}} \\
& \times \int_a^\infty x^{-(1-1/r(x))p(x)} \left(x^{\sup 1/(1-1/r(x))q(x)} - a^{\sup(1/(1-1/r(x))q(x))} \right)^{p(x)/q(x)} \\
& \times \int_a^\infty t^{1/(1-1/r(t))q(t)} f^{p(t)}(t) dt dx \\
& \leq \alpha^{\sup p(t)} \left\{ \inf \left(1 - \frac{1}{r(t)} \right) q(t) \right\}^{\sup\{p(t)-p(t)/r(t)-1\}}
\end{aligned}$$

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$$\begin{aligned}
& \times \int_a^\infty \left\{ \int_t^\infty x^{-(1-1/r(x))p(x)+p(x)/(1-1/r(x))q(x)^2} \right. \\
& \quad \times \left(1 - \left(\frac{a}{x} \right)^{\sup_{1/(1-1/r(x))q(x)}} \right)^{p(x)/q(x)} dx \Big\} t^{1/(1-1/r(t))q(t)} f^{p(t)}(t) dt \\
& \leq \alpha^M \left\{ \inf \left(1 - \frac{1}{r(t)} \right) q(t) \right\}^{\sup\{p(t)(1-1/r(t))\}} \\
& \quad \times \int_a^\infty \left[\int_t^\infty \left(1 - \left(\frac{a}{x} \right)^{\sup_{1/(1-1/r(x))q(x)}} \right)^{\sup p(x)/q(x)} \right. \\
& \quad \times d \left(1 - \left(\frac{a}{x} \right)^{\sup_{1/(1-1/r(x))q(x)}} \right) \Big] \left(\frac{t}{a} \right)^{\sup_{1/(1-1/r(t))q(t)}} f^{p(t)}(t) dt \\
& \leq \alpha^M \left\{ \inf \left(1 - \frac{1}{r(t)} \right) q(t) \right\}^{\sup\{p(t)(1-1/r(t))\}} \\
& \quad \times \int_a^\infty \frac{1}{\sup(1-1/r(t))p(t)} \left\{ 1 - \left[1 - \left(\frac{a}{t} \right)^{\sup_{1/(1-1/r(t))q(t)}} \right]^{\sup(1-1/r(t))p(t)} \right\} \\
& \quad \times \left(\frac{t}{a} \right)^{\sup_{1/(1-1/r(t))q(t)}} f^{p(t)}(t) dt \\
& = \alpha^M \left\{ \inf \left(1 - \frac{1}{r(t)} \right) q(t) \right\}^N \int_a^\infty [1 - \theta_N(t)] f^{p(t)}(t) dt,
\end{aligned} \tag{3.8}$$

where $N = \sup 1/(1-1/r(t))p(t) > 1$,

$$\begin{aligned}
\theta_N(t) &= 1 - \frac{1}{N} \left\{ 1 - \left[1 - \left(\frac{a}{t} \right)^{\sup_{1/(1-1/r(t))q(t)}} \right]^N \right\} \left(\frac{t}{a} \right)^{\sup_{1/(1-1/r(t))q(t)}}, \quad t > a > 0, \\
\theta_N(a) &= \sup \frac{1}{(1-1/r(t))q(t)},
\end{aligned} \tag{3.9}$$

since

$$\begin{aligned}
\left[1 - \left(\frac{a}{t} \right)^{\sup_{1/(1-1/r(t))q(t)}} \right]^N &= \sum_{k=0}^\infty \frac{\Gamma(N+1)}{\Gamma(k+1)\Gamma(N-k+1)} (-1)^k \left(\frac{a}{t} \right)^{k/\inf_{1/(1-1/r(t))q(t)}}, \\
\theta_N(t) &= \frac{1}{N} \sum_{k=0}^\infty \frac{\Gamma(N+1)}{\Gamma(k+2)\Gamma(N-k)} (-1)^{k-1} \left(\frac{a}{t} \right)^{k/\inf_{1/(1-1/r(t))q(t)}}, \quad t > a > 0.
\end{aligned} \tag{3.10}$$

This completes the proof. $\square$

Note. When $t > a > 0$, by Bernoulli's inequality (see [2, Chapter 2.4]), we obtain

$$1 - N \left(\frac{a}{t} \right)^{\sup 1/(1-1/r(t))q(t)} < \left[1 - \left(\frac{a}{t} \right)^{\sup 1/(1-1/r(t))q(t)} \right]^N, \quad (3.11)$$

$$\theta_N(t) > 1 - \frac{1}{N} \left[1 - \left\{ 1 - N \left(\frac{a}{t} \right)^{\sup 1/(1-1/r(t))q(t)} \right\} \right] \left(\frac{t}{a} \right)^{\sup 1/(1-1/r(t))q(t)} = 0.$$

Applications

THEOREM 3.5. Let $0 < b \leq \infty$, for all $x \in (0, \infty)$, $s \geq N > 1$, $p(x)^{-1} + q(x)^{-1} = 1 - r(x)^{-1}$, $f \geq 0$, $r(x) > 1$, and $0 < \int_0^b x^{-s+N} f^{p(x)}(x) < \infty$. Then
(i) for $b \in (0, \infty)$,

$$\begin{aligned} & \int_0^b x^{-s} \left(\int_0^x f(t) dt \right)^{p(x)} dx \\ & < \frac{\alpha^M \{ \inf (1 - 1/r(t))q(t) \}^N}{\{ \inf (1 - 1/r(t))q(t) \} (s - N) + 1} \int_0^b \left[1 - \left(\frac{t}{b} \right)^{\{s-N+\inf 1/(1-1/r(t))q(t)\}} \right] \\ & \quad \times t^{-s+(1-1/r(t))p(t)} f^{p(t)}(t) dt, \end{aligned} \quad (3.12)$$

(ii) for $b = \infty$,

$$\int_0^\infty x^{-s} \left(\int_0^x f(t) dt \right)^{p(x)} dx < \frac{\alpha^M \{ \inf (1 - 1/r(t))q(t) \}^N}{\{ \inf (1 - 1/r(t))q(t) \} (s - N) + 1} \int_0^\infty t^{-s+N} f^{p(t)}(t) dt,$$

$$M = \sup p(t), \quad \alpha = \sup p(t)^{-1} + \sup q(t)^{-1} + \sup r(t), \quad N = \sup p(t) \left(1 - \frac{1}{r(t)} \right). \quad (3.13)$$

Proof. For case (i), $b \in (0, \infty)$, we use (3.1) to obtain

$$\begin{aligned} & \int_0^b x^{-r} \left(\int_0^x f(t) dt \right)^{p(x)} dx \\ & < \alpha^M \left\{ \inf \left(1 - \frac{1}{r(x)} \right) q(x) \right\}^{\sup p(x)/q(x)} \int_0^b x^{-s+\sup \{1/(1-1/r(x))q(x)2\}} \\ & \quad \times \int_0^x t^{1/(1-1/r(t))q(t)} f^{p(t)}(t) dt dx \\ & \leq \alpha^M \left\{ \inf \left(1 - \frac{1}{r(t)} \right) q(t) \right\}^{\sup \{p(t)/q(t)\}} \\ & \quad \times \int_0^b \left(\int_t^b x^{-s+\sup \{1/(1-1/r(x))p(x)-1/(1-1/r(x))q(x)-1\}} dx \right) t^{1/(1-1/r(t))q(t)} f^{p(t)}(t) dt \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{\alpha^M \{ \inf (1 - 1/r(t)) q(t) \}^{\sup(1-1/r(t))p(t)-1}}{-s + \sup \{ (1 - 1/r(t)) p(t) - 1/(1 - 1/r(t)) q(t) \}} \\
 &\quad \times \int_0^b (b^{-s+\sup \{ (1-1/r(t))p(t)-1/(1-1/r(t))q(t) \}} - t^{-s+\sup \{ (1-1/r(t))p(t)-1/(1-1/r(t))q(t) \}}) \\
 &\quad \times t^{1/(1-1/r(t))q(t)} f^{p(t)}(t) dt \\
 &\leq \frac{\alpha^M \{ \inf (1 - 1/r(t)) q(t) \}^N}{\{ \inf (1 - 1/r(t)) q(t) \} (s - N) + 1} \int_0^b \left[1 - \left(\frac{t}{b} \right)^{s-N+\inf \{ (1-1/r(t))q(t) \}} \right] \\
 &\quad \times t^{-s+\sup(1-1/r(t))p(t)} f^{p(t)}(t) dt.
 \end{aligned} \tag{3.14}$$

For case (ii), $b = \infty$, we use (3.1) to find

$$\begin{aligned}
 &\int_0^\infty x^{-s} \left(\int_0^x f(t) dt \right)^{p(x)} dx \\
 &\quad < \alpha^M \left\{ \inf \left(1 - \frac{1}{r(x)} \right) q(x) \right\}^{\sup p(x)/q(x)} \int_0^\infty x^{-s+\sup \{ 1/(1-1/r(x))q(x)^2 \}} \\
 &\quad \times \int_0^x t^{1/(1-1/r(t))q(t)} f^{p(t)}(t) dt dx \\
 &= \alpha^M \left\{ \inf \left(1 - \frac{1}{r(t)} \right) q(t) \right\}^{\sup p(t)/q(t)} \\
 &\quad \times \int_0^\infty \left(\int_t^\infty x^{-s+\sup \{ 1/(1-1/r(x))p(x)-1/(1-1/r(x))q(x)-1 \}} dx \right) \\
 &\quad \times t^{1/(1-1/r(t))q(t)} f^{p(t)}(t) dt \\
 &= \frac{\alpha^M \{ \inf (1 - 1/r(t)) q(t) \}^N}{\{ \inf (1 - 1/r(t)) q(t) \} (s - N) + 1} \int_0^\infty t^{-s+N} f^{p(t)}(t) dt.
 \end{aligned} \tag{3.15}$$

□

Remark 3.6. (a) If $p(x)$, $q(x)$, and $r(x)$ are constants in Lemmas 3.1 and 3.2 and Theorems 3.3 and 3.4, then our results reduce to the corresponding Lemmas 2.1 and 2.2 and Theorems 2.4 and 2.5 obtained in [3].

(b) If we take $r(x) \rightarrow \infty$ in Theorems 3.3 and 3.4, then it reduces to corresponding Theorems 2.4 and 2.5.

(c) In the limits $a \rightarrow 0$, $b \rightarrow \infty$, $r(x) \rightarrow \infty$, $p(x)$ and $q(x)$ constants, (3.5) reduces to (1.1). Hence (3.5) is the generalization of (1.1).

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