We give a different proof of a lemma by Phelps (1960) which asserts, roughly speaking, that if two norm-one functionals $f$ and $g$ have their hyperplanes $f^{-1}(0)$ and $g^{-1}(0)$ sufficiently close together, then either $\|f - g\|$ or $\|f + g\|$ must be small. We also extend this result to a complex Banach space.

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In 1960 in [2], Phelps proved the following lemma.

**Lemma 1.** Suppose that $E$ is a real normed linear space and that $\epsilon > 0$. If $f, g \in S^*$ are such that $f^{-1}(0) \cap U \subset g^{-1}[-\epsilon/2, \epsilon/2]$, then either $\|f - g\| \leq \epsilon$ or $\|f + g\| \leq \epsilon$. (Here, $U$ represents the unit ball of $E$ and $S^*$ is the unit sphere of $E^*$.)

This lemma was then used the following year as a crucial step in the proof of the well-known Bishop-Phelps theorem [1] that every Banach space is subreflexive; in other words, every functional on a Banach space $E$ can be approximated by a norm-attaining functional on the same space. The original proof of this lemma uses the Hahn-Banach theorem and is therefore fairly abstract.

In this note, we present an alternate proof for Lemma 1 stated above. This proof gives a geometric argument while extending the lemma to a complex Banach space. Lemma 1 is shown to be a special case when the bound of $\epsilon$ on $\|f \pm g\|$ is replaced by $5\epsilon$. This replacement does not affect the fundamental conclusion of Lemma 1.

We now state the extended lemma.

**Lemma 2.** Let $X$ be a complex Banach space and let $\epsilon$ be such that $0 < \epsilon < 1/2$. Let $\varphi, \psi \in X^*$, $\|\varphi\| = \|\psi\| = 1$. Suppose that for all $x \in X$ with $\|x\| \leq 1$ and $\varphi(x) = 0$, it holds that $|\psi(x)| \leq \epsilon$. Then there is some complex number $\alpha$ such that $|\alpha| = 1$ and $\|\varphi - \alpha \psi\| \leq 5\epsilon$.

It will be shown that if $\varphi$ and $\psi$ are real-valued functionals on a real Banach space $X$, then $\alpha$ will in fact be either 1 or $-1$, thus proving the amended original result.

We now prove Lemma 2.
A new proof of a lemma by Phelps

Proof. Let \( e \in X \) be such that \( \|e\| = 1 \) and \( |\varphi(e)| \geq 1 - \epsilon/4 \). We will first show that \( |\psi(e)| \geq 1 - (5/2)\epsilon \). To see this, let \( f \in X \) such that \( \|f\| = 1 \) and \( |\psi(f)| \geq 1 - \epsilon/4 \). Let \( k = 1 - \epsilon/4 \) and let \( t = \varphi(f)/\varphi(e) \). Then \( 0 \leq |t| \leq 1/(1 - \epsilon/4) = 1/k \leq 8/7 \) and if we take \( w = (k/(k + 1))(f - te) \), then \( \|w\| \leq (k/(k + 1))(\|f\| + |t|\|e\|) \leq (k/(k + 1))(1 + 1/k) = 1 \).

Moreover,

\[
\varphi(w) = \frac{k}{k + 1}(\varphi(f) - \frac{\varphi(f)}{\varphi(e)}\varphi(e)) = 0
\]

so we have

\[
\epsilon \geq |\psi(w)| = \frac{k}{k + 1}|\varphi(f) - t\psi(e)|
\]

\[
\geq \frac{k}{k + 1}| |\varphi(f)| - |t||\psi(e)| |
\]

\[
\geq \frac{k}{k + 1}( |\varphi(f)| - |t||\psi(e)| ).
\]

Thus

\[
\frac{1}{k} |\psi(e)| \geq |t||\psi(e)| \geq |\varphi(f)| - \frac{k + 1}{k}\epsilon \geq \left(1 - \frac{\epsilon}{4}\right) - \frac{k + 1}{k}\epsilon = k - \frac{k + 1}{k}\epsilon.
\]

This gives

\[
|\psi(e)| \geq k^2 - (k + 1)e = \left(1 - \frac{\epsilon}{4}\right)^2 - \left(2 - \frac{\epsilon}{4}\right)e = 1 - \epsilon + \frac{\epsilon^2}{16} - 2\epsilon + \frac{\epsilon^2}{4} \geq 1 - \frac{5}{2}\epsilon
\]

as required. Notice that, if \( \varphi \) and \( \psi \) are real valued, the above still holds.

Now, there exist \( \beta, \gamma \in \mathbb{C} \) such that \( |\beta| = |\gamma| = 1 \), \( \beta\varphi(e) \in [1 - \epsilon/4, 1] \subset \mathbb{R} \), and

\[
\gamma\psi(e) \in [1 - 5\epsilon/2, 1] \subset \mathbb{R};
\]

and so \( |\beta\varphi(e) - \gamma\psi(e)| \leq 5\epsilon/2 \).

Let \( x \in X \) be such that \( \|x\| \leq 1 \) and write \( x = \lambda e + y \), where \( \lambda = \varphi(x)/\varphi(e) \) and \( y = x - \lambda e \). Then \( |\lambda| \leq |\varphi(x)|/|\varphi(e)| \leq 1/(1 - \epsilon/4) \leq 8/7 \), \( \|y\| \leq \|x\| + |\lambda||e\| \leq 15/7 \), and \( \varphi(y) = \varphi(x) - (\varphi(x)/\varphi(e))\varphi(e) = 0 \). So, by hypothesis, \( |\psi((7/15)y)| \leq \epsilon \), that is, \( |\psi(y)| \leq (15/7)e \). Then, if we take \( \alpha = \gamma/\beta \), we have \( |\alpha| = 1 \) and

\[
|\varphi(x) - \alpha\psi(x)| = \frac{1}{|\beta|} |\beta\varphi(x) - \gamma\psi(x)| = |\beta\lambda\varphi(e) + \beta\varphi(y) - \gamma\lambda\psi(e) - \gamma\psi(y)|
\]

\[
\leq |\lambda||\beta\varphi(e) - \gamma\psi(e)| + |\gamma||\psi(y)| \leq \frac{8}{7} \cdot \frac{5}{2}\epsilon + 1 \cdot \frac{15}{7}\epsilon = 5\epsilon.
\]

But \( x \) was an arbitrary element of the unit ball of \( X \), so we have \( \|\varphi - \alpha\psi\| \leq 5\epsilon \).

Notice that if \( X \) is a real Banach space, the above argument still holds. Also, if \( \varphi \) and \( \psi \) are real valued, we can choose \( e \in X \) such that \( \varphi(e) \geq 1 - \epsilon/4 \), giving \( \beta = 1 \), and then from the claim, either \( \gamma = 1 \) or \( \gamma = -1 \). So either \( \alpha = 1 \) or \( \alpha = -1 \), yielding Phelps’ result, up to a constant. \( \square \)
References


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