COVARIANCE OF THE NUMBER OF REAL ZEROS OF A RANDOM TRIGONOMETRIC POLYNOMIAL

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For random coefficients a_j and b_j we consider a random trigonometric polynomial defined as $T_n(\theta) = \sum_{j=0}^{n}\{a_j \cos j\theta + b_j \sin j\theta\}$. The expected number of real zeros of $T_n(\theta)$ in the interval $(0, 2\pi)$ can be easily obtained. In this note we show that this number is in fact $n/\sqrt{3}$. However the variance of the above number is not known. This note presents a method which leads to the asymptotic value for the covariance of the number of real zeros of the above polynomial in intervals $(0, \pi)$ and $(\pi, 2\pi)$. It can be seen that our method in fact remains valid to obtain the result for any two disjoint intervals. The applicability of our method to the classical random trigonometric polynomial, defined as $P_n(\theta) = \sum_{j=0}^{n} a_j(\omega) \cos j\theta$, is also discussed. $T_n(\theta)$ has the advantage on $P_n(\theta)$ of being stationary, with respect to θ, for which, therefore, a more advanced method developed could be used to yield the results.

1. Introduction

Let $(\Omega, \Pr, \mathscr{A})$ be a fixed probability space and for $\omega \in \Omega$ let $\{a_j(\omega)\}_{j=0}^{n}$ and $\{b_j(\omega)\}_{j=0}^{n}$ be sequences of independent, identically and normally distributed random variables, both with means zero and variances one. Denote by $N_n(\alpha, \beta)$ the number of real zeros of random trigonometric polynomial

$$T_n(\theta, \omega) \equiv T_n(\theta) = \sum_{j=0}^{n} \{a_j(\omega) \cos j\theta + b_j(\omega) \sin j\theta\} \tag{1.1}$$

in the interval (α, β) and by $EN_n(\alpha, \beta)$ its expected value. Indeed the above definition of random trigonometric polynomials differs from the classical case of

$$P_n(\theta, \omega) \equiv P_n(\theta) = \sum_{j=0}^{n} a_j(\omega) \cos j\theta \tag{1.2}$$

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which has been extensively studied. The literature includes the original work of Dunnage [3] which was later extended by Das [2] and Wilkins [8] and was reviewed by Bharucha-Reid and Sambandham [1] and recently by Farahmand [7]. They generally show that for all sufficiently large n and for different classes of distributions of the coefficients or in different cases, for example, the level crossing case instead of zero crossings, $EN_n(0,2\pi)$ is asymptotic to $2n/\sqrt{3}$. In particular the above work of Wilkins is of interest as it shows that the error term involved in the asymptotic estimate is small and in fact is $O(1)$. However, finding the variance of the number of real zeros involves a different level of difficulties. There have been several attempts, for instance, see [4] or [6], to obtain the asymptotic value for the variance of $N_n(0,2\pi)$ for $P_n(\theta)$. So far, the results are only in the form of upper bounds. As far as the expected number of zeros is concerned the asymptotic value of $EN_n(0,2\pi)$ for $T_n(\theta)$ and $P_n(\theta)$ is the same. Therefore, we conjecture that their variances are also the same. In addition, with the above assumptions of independence of the coefficients $a_j(\omega)$ and $b_j(\omega)$ the inner term of $T_n(\theta)$ given in (1.1) has the property of being stationary with respect to θ. This can be seen by evaluating its covariance function as

$$\begin{aligned} &E\{[a_j(\omega)\cos j\theta + b_j(\omega)\sin j\theta][a_j(\omega)\cos j(\theta+\tau) + b_j \sin j(\theta+\tau)]\} \\ &\quad = \cos j\theta \cos j(\theta+\tau) + \sin j\theta \sin j(\theta+\tau) = \cos j\tau. \end{aligned} \tag{1.3}$$

Therefore it is natural to seek to evaluate the variance of number of zeros of $T_n(\theta)$ which possess the above stationary property instead of $P_n(\theta)$ given in (1.2). We, however, are unable to make any substantial progress in this direction. Instead, we obtain the covariance of the number of real zeros in the intervals $(0,\pi)$ and $(\pi,2\pi)$. As our main aim remains to estimate the variance of $N_n(0,2\pi)$ we will present our results and discussions in such a way that they could be used to be generalized for variance. Although we are considering two intervals $(0,\pi)$ and $(\pi,2\pi)$ our proof is also valid for any two disjoint intervals. A small modification and some generalization to our analysis should lead to an asymptotic value for the variance. Looking at our proof it suggests that our estimate for the covariance will remain the same as for the variance. Furthermore, although we are considering the polynomial $T_n(\theta)$ given in (1.1) as far as the results for the covariance, and, therefore, the variance are concerned, it should remain invariant also for $P_n(\theta)$. For random trigonometric polynomial $T_n(\theta)$ given in (1.1) we prove the following.

Theorem 1.1. *With the above assumption of independent and Gaussian distribution of the coefficients $\{a_j(\omega)\}_{j=0}^n$ and $\{b_j(\omega)\}_{j=0}^n$ the covariance of the number of real zeros of $T_n(\theta)$ is*

$$\operatorname{cov}\{N_n(0,\pi), N_n(\pi,2\pi)\} = 4n + O(1). \tag{1.4}$$

2. Covariance of the number of real zeros

For any two intervals (α,β) and (δ,γ) it is known that

$$E\{N_n(\alpha,\beta)N_n(\delta,\beta)\} = \int_\alpha^\beta \int_\delta^\beta \iint_{-\infty}^{\infty} |xy|\, p_{\theta_1,\theta_2}(0,0,x,y)\,dx\,dx\,d\theta_1\,d\theta_2, \tag{2.1}$$

where $p_{\theta_1,\theta_2}(z_1,z_2,x,y)$ denotes the four-dimensional joint probability density function of $T_n(\theta_1)$, $T_n(\theta_2)$, $T_n'(\theta_1)$, and $T_n'(\theta_2)$. For our purpose and using the above formula to obtain the result for The covariance case, the two intervals (α,β) and (δ,γ) are disjoint. However the above formula and the following discussions remain valid for any two intervals, whether or not they are overlapping. Let Π be the 4×4 variance-covariance matrix of random variables $T_n(\theta_1)$, $T_n(\theta_2)$, $T_n'(\theta_1)$, and $T_n'(\theta_2)$ with cofactor Π_{ij} of ijth element. Then using the Gaussian assumption for the coefficients we can calculate the above-required joint density function as

$$P_{\theta_1,\theta_2}(0,0,x,y) = \frac{1}{4\pi^2\sqrt{|\Pi|}}\exp\left\{-\frac{\Pi_{33}x^2+\Pi_{44}y^2+(\Pi_{34}+\Pi_{43})xy}{2|\Pi|}\right\}. \tag{2.2}$$

In order to evaluate (2.1) further in (2.2) we let $q = x\sqrt{\Pi_{33}/|\Pi|}$ and $s = y\sqrt{\Pi_{44}/|\Pi|}$. As we will see later Π_{33} and Π_{44} are positive and therefore q and s are real. Hence from (2.2) we obtain

$$\begin{aligned}
&\iint_{-\infty}^{\infty}|xy|p_{\theta_1,\theta_2}(0,0,x,y)\,dx\,dx \\
&\quad= \frac{|\Pi|^{3/2}}{4\pi^2\Pi_{33}\Pi_{44}}\iint_{-\infty}^{\infty}|qs|\exp\left\{-\frac{q^2+s^2}{2}-\left(\frac{\Pi_{34}+\Pi_{43}}{\sqrt{\Pi_{33}\Pi_{44}}}\right)qs\right\}dq\,ds \\
&\quad= \frac{|\Pi|^{3/2}}{4\pi^2\Pi_{33}\Pi_{44}}\iint_{-\infty}^{\infty}|qs|\exp\left(-\frac{q^2+s^2+2\rho qs}{2}\right)dq\,ds,
\end{aligned} \tag{2.3}$$

where $\rho = (\Pi_{34}+\Pi_{43})/2\sqrt{\Pi_{33}\Pi_{44}}$. Now let $\rho = \cos\phi$, then from [7, page 97] the integral appear in (2.3) can be evaluated as $4\{1+(\pi/2-\phi)\cot\phi\}/\cos^2\phi$. Therefore for Gaussian assumption the required formula in (2.1) is simplified as

$$EN_n(\alpha,\beta)N_n(\gamma,\delta) = \frac{1}{\pi^2}\int_\alpha^\beta\int_\gamma^\delta \frac{|\Pi|^{3/2}\{1+(\pi/2-\phi)\cot\phi\}}{\Pi_{33}\Pi_{44}\cos^2\phi}d\theta_1\,d\theta_2. \tag{2.4}$$

Now we let

$$\begin{gathered}
A_n(\theta_1,\theta_2) = \operatorname{cov}\{T_n(\theta_1),T_n(\theta_2)\}, \qquad C_n(\theta_1,\theta_2) = \operatorname{cov}\{T_n(\theta_1),T_n'(\theta_2)\}, \\
B_n(\theta_1,\theta_2) = \operatorname{cov}\{T_n'(\theta_1),T_n'(\theta_2)\},
\end{gathered} \tag{2.5}$$

where $T_n'(\theta)$ is the derivative of $T_n(\theta)$ with respect to θ. It is easy to show that the $\operatorname{cov}\{T_n(\theta),T_n(\theta)\} = 0$, also $A_n(\theta,\theta) = \operatorname{var}\{T_n(\theta)\} = n$ and $B_n(\theta,\theta) = \operatorname{var}\{T_n'(\theta)\} = n(n+1)(2n+1)/6$ are independent of θ. Therefore we can obtain the variance-covariance matrix of random variables $T_n(\theta_1)$, $T_n(\theta_2)$, $T_n'(\theta_1)$, and $T_n'(\theta_2)$ as

$$\Pi = \begin{pmatrix} n & A_n(\theta_1,\theta_2) & 0 & C_n(\theta_1,\theta_2) \\ A_n(\theta_1,\theta_2) & n & -C_n(\theta_1,\theta_2) & 0 \\ 0 & -C_n(\theta_1,\theta_2) & \dfrac{n(n+1)(2n+1)}{6} & B_n(\theta_1,\theta_2) \\ C_n(\theta_1,\theta_2) & 0 & B_n(\theta_1,\theta_2) & \dfrac{n(n+1)(2n+1)}{6} \end{pmatrix}. \tag{2.6}$$

Let

$$S_n(\theta_1,\theta_2) = \frac{\sin\{(2n+1)(\theta_1-\theta_2)/2\}}{\sin\{(\theta_1-\theta_2)/2\}},$$
$$Z_n(\theta_1,\theta_2) = \frac{\cos\{(2n+1)(\theta_1-\theta_2)/2\}}{\sin\{(\theta_1-\theta_2)/2\}}. \tag{2.7}$$

Then we can obtain the remaining elements of the above matrix as

$$\begin{aligned}
A_n(\theta_1,\theta_2) &\equiv A_n(\theta_2,\theta_1) = \sum_{j=0}^{n}\cos j(\theta_1,\theta_2) = \frac{S_n(\theta_1,\theta_2)-1}{2},\\
C_n(\theta_1,\theta_2) &\equiv -C_n(\theta_2,\theta_1) = \sum_{j=0}^{n} j\sin j(\theta_1-\theta_2)\\
&= -\frac{(2n+1)Z_n(\theta_1,\theta_2)}{4} + \frac{S_n(\theta_1,\theta_2)\cot\{(\theta_1-\theta_2)/2\}}{4},\\
B_n(\theta_1,\theta_2) &\equiv B_n(\theta_2,\theta_1) = \sum_{j=0}^{n} j^2\cos j(\theta_1-\theta_2)\\
&= -\frac{(2n+1)^2}{8}S_n(\theta_1,\theta_2) - \frac{S_n(\theta_1,\theta_2)}{8} + \frac{(2n+1)}{4}Z_n(\theta_1,\theta_2)\cot\left\{\frac{(\theta_1-\theta_2)}{2}\right\}\\
&\quad - \frac{S_n(\theta_1,\theta_2)}{4}\cot^2\left\{\frac{(\theta_1-\theta_2)}{2}\right\}.
\end{aligned} \tag{2.8}$$

Now we are in the position to proceed with the proof of our theorem.

3. Proof of the theorem

From (2.6) we can obtain the determinate of Π as

$$\begin{aligned}
|\Pi| &= n^2\left\{\frac{n(n+1)(2n+1)}{6}\right\}^2 - n^2B_n^2(\theta_1,\theta_2)\\
&\quad - 2nC_n^2(\theta_1,\theta_2)\frac{n(n+1)(2n+1)}{6} - A_n^2(\theta_1,\theta_2)\left\{\frac{n(n+1)(2n+1)}{6}\right\}^2\\
&\quad + A_n^2(\theta_1,\theta_2)B_n^2(\theta_1,\theta_2) + 2A_n(\theta_1,\theta_2)C^2(\theta_1,\theta_2)B_n(\theta_1,\theta_2) + C_n^4(\theta_1,\theta_2).
\end{aligned} \tag{3.1}$$

Also the required cofactors are

$$\Pi_{43} = \Pi_{34} = n^2B_n(\theta_1,\theta_2) - A^2(\theta_1,\theta_2)B_n(\theta_1,\theta_2) - A(\theta_1,\theta_2)C^2(\theta_1,\theta_2), \tag{3.2}$$

$$\Pi_{33} = \Pi_{44} = \frac{n^3(n+1)(2n+1)}{6} - n^2C^2(\theta_1,\theta_2) + \frac{n(n+1)(2n+1)}{6}A_n^2(\theta_1,\theta_2). \tag{3.3}$$

Now we use the advantage that θ_1 and θ_2 are disjoint and therefore $S_n(\theta_1,\theta_2)$, $Z_n(\theta_1,\theta_2)$ and $\cot(\theta_1-\theta_2)$ are bounded. Therefore from (3.1)–(3.3) we obtain

$$\begin{aligned} |\Pi| &\sim n^2\left\{\frac{n(n+1)(2n+1)}{6}\right\}^2, \\ |\Pi_{33}| &\sim \frac{n^3(n+1)(2n+1)}{6}, \\ |\Pi_{34}| \sim \Pi_{43} &\sim n^2 B_n(\theta_1,\theta_2) = O(n^4 S_n(\theta_1,\theta_2)). \end{aligned} \tag{3.4}$$

In order to evaluate the integral that appears in (2.4) we note that from (3.4)

$$\rho = \frac{\Pi_{34}+\Pi_{43}}{2\sqrt{\Pi_{33}\Pi_{44}}} \longrightarrow 0 \tag{3.5}$$

as $n\to\infty$ and therefore for sufficiently large n,

$$\phi = \arccos\rho \longrightarrow \frac{\pi}{2}. \tag{3.6}$$

This summarizes the value of (2.4) to

$$E\{N_n(\alpha,\beta)N_n(\delta,\gamma)\} = \int_\alpha^\beta\int_\delta^\gamma \frac{|\Pi|^{3/2}}{\pi^2\Pi_{33}\Pi_{44}}d\theta_1 d\theta_2. \tag{3.7}$$

With our assumptions of our theorem it turns out that $|\Pi|$, Π_{33}, and Π_{44} are independent of θ_1 and θ_2. Also for all sufficiently large n,

$$\begin{aligned} |\Pi| &\sim \frac{n^4(n+1)^2(2n+1)^2}{36} \sim \frac{n^8}{9}+\frac{n^7}{3}, \\ \Pi_{44} \sim \Pi_{33} &\sim \frac{n^3(n+1)(2n+1)}{6} \sim \frac{n^5}{3}. \end{aligned} \tag{3.8}$$

Therefore

$$E\{N_n(0,\pi)N_n(\pi,2\pi)\} \sim \frac{|\Pi|^{3/2}}{\Pi_{33}\Pi_{44}} \sim \frac{\{n^8/9+n^7/3\}^{3/2}}{(n^5/3)^2} \sim \frac{n^2}{3}+\frac{9n}{2}. \tag{3.9}$$

In order to proceed we need to find $EN_n(0,\pi)$ and $EN_n(\pi,2\pi)$. To this end we use the Kac-Rice formula and because of the stationary property of $T_n(\theta)$ mentioned above, we are able to obtain an estimate with small error easily. Using a same method as [5] and since $\operatorname{cov}(T_n(\theta),T'(\theta))=0$, we have

$$EN_n(0,\pi) = \frac{1}{\pi}\int_0^\pi \frac{B}{\pi A}d\theta, \tag{3.10}$$

where

$$\begin{aligned} A^2 &= \operatorname{var}\{T_n(\theta)\} = n, \\ B^2 &= \operatorname{var}\{T_n(\theta)\} = \frac{n(n+1)(2n+1)}{6}. \end{aligned} \tag{3.11}$$

Therefore

$$EN_n(0,\pi) = \frac{\sqrt{n(n+1)(2n+1)}}{\sqrt{6n}} = \sqrt{\frac{n^2}{3} + \frac{n}{2} + \frac{1}{6}} = \frac{n}{\sqrt{3}} + \frac{\sqrt{3}}{4} + O\left(\frac{1}{n}\right). \tag{3.12}$$

Hence by (3.9), (3.12) and since a similar result to (3.12) can be obtained for $EN_n(\pi, 2\pi)$, we can obtain

$$\operatorname{cov}\{N_n(0,\pi), N_n(\pi,2\pi)\} \sim \frac{n^2}{3} + \frac{9n}{2} - \left\{\frac{n}{\sqrt{3}} + \frac{\sqrt{3}}{4} + O(1)\right\}^2 \sim 4n + O(1). \tag{3.13}$$

This completes the proof of the theorem.

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