A DENSITY RESULT IN VECTOR OPTIMIZATION

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We study a class of vector minimization problems on a complete metric space such that all its bounded closed subsets are compact. We show that a subclass of minimization problems with a nonclosed set of minimal values is dense in the whole class of minimization problems.

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1. Introduction and the main result

The study of vector optimization problems has recently been a rapidly growing area of research. See, for example, [1–5] and the references mentioned therein. In this paper we study a class of vector minimization problems on a complete metric space such that all its bounded closed subsets are compact. This class of problems is associated with a complete metric space of continuous vector functions $\mathcal{A}$ defined below. For each F from $\mathcal{A}$ we denote by $\nu(F)$ the set of all minimal elements of the image $F(X) = \{F(x) : x \in X\}$. In this paper we will study the sets $\nu(F)$ with $F \in \mathcal{A}$. It is clear that for a minimization problem with only one criteria the set of minimal values is a singleton. In the present paper we will show that the subspace of all $F \in \mathcal{A}$ with nonclosed sets $\nu(F)$ is dense in $\mathcal{A}$. Therefore in general the sets $\nu(F)$, $F \in \mathcal{A}$ can be rather complicated.

In this paper we use the convention that $\infty/\infty = 1$ and denote by $\text{Card}(E)$ the cardinality of the set E .

Let $\mathbb{R}$ be the set of real numbers and let n be a natural number. Consider the finite-dimensional space R^n with the norm

$$\|x\| = \|(x_1, \dots, x_n)\| = \max \{|x_i| : i = 1, \dots, n\}, \quad x = (x_1, \dots, x_n) \in R^n. \quad (1.1)$$

Let $\{e_1, \dots, e_n\}$ be the standard basis in R^n :

$$e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1). \quad (1.2)$$

2 A density result in vector optimization

We equip the space R^n with the natural order. Let $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in R^n$. We say that

$$\begin{aligned} x &\geq y && \text{if } x_i \geq y_i \quad \forall i \in \{1, \dots, n\}, \\ x &> y && \text{if } x \geq y, x \neq y, \\ x &\gg y && \text{if } x_i > y_i \quad \forall i \in \{1, \dots, n\}. \end{aligned} \tag{1.3}$$

We say that $x \ll y$ (resp., $x < y, x \leq y$) if $y \gg x$ (resp., $y > x, y \geq x$).

Let (X, ρ) be a complete metric space such that each of its bounded closed subsets is compact. Fix $\theta \in X$.

Denote by $\mathcal{A}$ the set of all continuous mappings $F = (f_1, \dots, f_n) : X \rightarrow R^n$ such that for $i = 1, \dots, n$

$$\lim_{\rho(x, \theta) \rightarrow \infty} f_i(x) = \infty. \tag{1.4}$$

For each $F = (f_1, \dots, f_n), G = (g_1, \dots, g_n) \in \mathcal{A}$ set

$$\begin{aligned} \tilde{d}(F, G) &= \sup \{ |f_i(x) - g_i(x)| : x \in X, i = 1, \dots, n \}, \\ d(F, G) &= \tilde{d}(F, G)(1 + \tilde{d}(F, G))^{-1}. \end{aligned} \tag{1.5}$$

Clearly the metric space $(\mathcal{A}, d)$ is complete.

Let $A \subset R^n$ be a nonempty set. An element $x \in A$ is called a minimal element of A if there is no $y \in A$ for which $y < x$.

Let $F \in \mathcal{A}$. A point $x \in X$ is called a point of minimum of F if $F(x)$ is a minimal element of $F(X)$. If $x \in X$ is a point of minimum of F , then $F(x)$ is called a minimal value of F . Denote by $M(F)$ the set of all points of minimum of F and put $v(F) = F(M(F))$.

The following proposition is proved in [6].

PROPOSITION 1.1. *Let $F = (f_1, \dots, f_n) \in \mathcal{A}$. Then $M(F)$ is a nonempty bounded subset of (X, ρ) and for each $z \in F(X)$ there is $y \in v(F)$ such that $y \leq z$.*

In the sequel we assume that $n \geq 2$ and that the space (X, ρ) has no isolated points.

The following theorem is our main result. It will be proved in Section 2.

THEOREM 1.2. *Suppose that the space (X, ρ) is connected. Let $F = (f_1, \dots, f_n) \in \mathcal{A}$ and let $\epsilon > 0$. Then there exists $G \in \mathcal{A}$ such that $\tilde{d}(F, G) \leq \epsilon$ and the set $v(G)$ is not closed.*

2. Proof of Theorem 1.2

By Proposition 1.1 there exists $x_* \in X$ such that

$$F(x_*) \in v(F). \tag{2.1}$$

There exists $\delta \in (0, 1/4)$ such that

$$\|F(x) - F(x_*)\| \leq \frac{\epsilon}{8} \quad \text{for each } x \in X \text{ such that } \rho(x, x_*) \leq 2\delta, \quad (2.2)$$

$$\{z \in X : \rho(z, x_*) = 8\delta\} \neq \emptyset. \quad (2.3)$$

Since the metric space X is connected for each $t \in (0, 8\delta]$, there is $z \in X$ such that $\rho(z, x_*) = t$.

It is clear that there exists a continuous function $\psi : X \rightarrow [0, 1]$ such that

$$\begin{aligned} \psi(x) &= 1 \quad \text{for each } x \in X \text{ satisfying } \rho(x, x_*) \leq \delta, \\ \psi(x) &= 0 \quad \text{for each } x \in X \text{ satisfying } \rho(x, x_*) \geq 2\delta. \end{aligned} \quad (2.4)$$

For $x \in X$ and $i = 1, \dots, n$ define

$$f_i^{(1)}(x) = \psi(x)f_i(x_*) + (1 - \psi(x))f_i(x), \quad (2.5)$$

$$F^{(1)} = (f_1^{(1)}, \dots, f_n^{(1)}). \quad (2.6)$$

Clearly, $F^{(1)} \in \mathcal{A}$ and

$$F^{(1)}(x) = F(x_*) \quad \forall x \in X \text{ satisfying } \rho(x, x_*) \leq \delta. \quad (2.7)$$

We will show that $\tilde{d}(F, F^{(1)}) \leq \epsilon/8$. Let $x \in X$. If $\rho(x, x_*) \geq 2\delta$, then $\psi(x) = 0$ and $F^{(1)}(x) = F(x)$. If $\rho(x, x_*) \leq 2\delta$, then by (2.2)

$$\|F(x) - F(x_*)\| \leq \frac{\epsilon}{8}. \quad (2.8)$$

Combined with (2.5) the inequality above implies that

$$\|F^{(1)}(x) - F(x)\| = \|\psi(x)(F(x_*) - F(x))\| \leq \|F(x_*) - F(x)\| \leq \frac{\epsilon}{8}. \quad (2.9)$$

Therefore

$$\tilde{d}(F, F^{(1)}) \leq \frac{\epsilon}{8}. \quad (2.10)$$

We will show that $F^{(1)}(x_*) \in \nu(F^{(1)})$. Assume that $x \in X$ and that

$$F^{(1)}(x) \leq F^{(1)}(x_*) = F(x_*). \quad (2.11)$$

By (2.11), (2.5), and (2.6)

$$F(x_*) \geq F^{(1)}(x) = \psi(x)F(x_*) + (1 - \psi(x))F(x). \quad (2.12)$$

If $\psi(x) = 1$, then $F^{(1)}(x) = F(x_*)$. If $\psi(x) < 1$, then by (2.12) and (2.1) $F(x_*) \geq F(x)$ and $F(x_*) = F(x)$ and in view of (2.5) and (2.6) $F^{(1)}(x) = F(x_*)$. Therefore $F(x_*) \in \nu(F^{(1)})$.

4 A density result in vector optimization

For $x \in X$ and $i \in \{1, \dots, n\}$ set

$$f_i^{(2)}(x) = f_i^{(1)}(x) + \min \left\{ 1, \max \left\{ \rho(x, x_*) - \frac{\delta}{2}, 0 \right\} \right\} \left(\frac{\epsilon}{8} \right), \quad (2.13)$$

$$F^{(2)} = (f_1^{(2)}, \dots, f_n^{(2)}). \quad (2.14)$$

Clearly

$$F^{(2)} \in \mathcal{A}, \quad F^{(2)}(x) \geq F^{(1)}(x) \quad \forall x \in X. \quad (2.15)$$

By (2.13), and (2.7)

$$F^{(2)}(x) = F^{(1)}(x) = F(x_*) \quad \forall x \in X \text{ satisfying } \rho(x, x_*) \leq \frac{\delta}{2}. \quad (2.16)$$

By (2.13) and (2.7) for each $x \in X$ satisfying $\rho(x, x_*) \in [\delta/2, \delta]$,

$$F^{(2)}(x) = F^{(1)}(x_*) + \left(\frac{\epsilon}{8} \right) \left[\rho(x, x_*) - \frac{\delta}{2} \right] (1, 1, \dots, 1). \quad (2.17)$$

By (2.13)

$$\tilde{d}(F^{(2)}, F^{(1)}) \leq \frac{\epsilon}{8}. \quad (2.18)$$

This inequality and (2.10) imply that

$$\tilde{d}(F, F^{(2)}) \leq \frac{\epsilon}{4}. \quad (2.19)$$

It is clear that the inclusion $F(x_*) \in \nu(F^{(1)})$, (2.15), and (2.16) imply that

$$F(x_*) \in \nu(F^{(2)}). \quad (2.20)$$

We will show that

$$\text{if } x \in X \text{ satisfies } F^{(2)}(x) = F(x_*), \text{ then } \rho(x, x_*) \leq \frac{\delta}{2}. \quad (2.21)$$

Assume that

$$x \in X, \quad F^{(2)}(x) = F(x_*). \quad (2.22)$$

In view of (2.15) and (2.22) $F^{(1)}(x) \leq F^{(2)}(x) = F(x_*)$. Together with the inclusion $F(x_*) \in \nu(F^{(1)})$ the inequality above implies that $F^{(1)}(x) = F(x_*) \geq F^{(2)}(x)$. Combined with (2.13) this relation implies that

$$\rho(x, x_*) \leq \frac{\delta}{2}. \quad (2.23)$$

Thus (2.21) is proved.

Since $F(x_*) \in \nu(F^{(1)})$, it follows from (2.13) that the following property holds.

(P1) For each $x \in X$ satisfying $\rho(x, x_*) > \delta/2$ there is $i \in \{1, \dots, n\}$ such that

$$f_i^{(1)}(x) \geq f_i^{(1)}(x_*) = f_i(x_*) \quad (2.24)$$

and that

$$\begin{aligned} f_i^{(2)}(x) &= f_i^{(1)}(x) + \left(\frac{\epsilon}{8}\right) \min \left\{ 1, \rho(x, x_*) - \frac{\delta}{2} \right\} \\ &\geq f_i^{(1)}(x_*) + \left(\frac{\epsilon}{8}\right) \min \left\{ 1, \rho(x, x_*) - \frac{\delta}{2} \right\} \\ &= f_i(x_*) + \left(\frac{\epsilon}{8}\right) \min \left\{ 1, \rho(x, x_*) - \frac{\delta}{2} \right\}. \end{aligned} \quad (2.25)$$

Choose

$$\delta_0 \in \left(0, \frac{\delta}{8}\right), \quad \lambda_0 \in \left(0, \frac{\epsilon}{16}\right). \quad (2.26)$$

Define functions $\phi_1, \phi_2 : [0, \infty) \rightarrow R$ as follows:

$$\begin{aligned} \phi_1(x) &= x, \quad x \in [0, 1], \quad \phi_1(x) = 1, \quad x \in (1, 2], \\ \phi_1(x) &= x - 1, \quad x \in (2, 8], \quad \phi_1(x) = 15 - x, \quad x \in (8, 14], \\ \phi_1(x) &= 1, \quad x \in (14, 15], \quad \phi_1(x) = 16 - x, \quad x \in (15, 16], \quad \phi_1(x) = 0, \quad x \in (16, \infty), \end{aligned} \quad (2.27)$$

$$\begin{aligned} \phi_2(x) &= -x, \quad x \in [0, 2], \quad \phi_2(x) = x - 4, \quad x \in (2, 8], \\ \phi_2(x) &= 12 - x, \quad x \in (8, 14], \quad \phi_2(x) = -16 + x, \quad x \in (14, 16], \\ \phi_2(x) &= 0, \quad x \in (16, \infty). \end{aligned} \quad (2.28)$$

It is clear that ϕ_1, ϕ_2 are continuous functions and that

$$\sup \{ |\phi_i(x)| : x \in R, i = 1, 2 \} \leq 7. \quad (2.29)$$

Define a function $G = (g_1, \dots, g_n) : X \rightarrow R^n$ as follows:

$$g_1(x) = f_1^{(2)}(x) + \lambda_0 \phi_1(16\rho(x, x_*)\delta_0^{-1}), \quad x \in X; \quad (2.30)$$

for $i \in \{2, \dots, n\}$

$$g_i(x) = f_i^{(2)}(x) + \lambda_0 \phi_2(16\rho(x, x_*)\delta_0^{-1}), \quad x \in X. \quad (2.31)$$

Clearly, $G \in \mathcal{A}$. By (2.31), (2.30), (2.29), and (2.26) $\tilde{d}(G, F^{(2)}) \leq \epsilon/2$. Together with (2.19) this implies that

$$\tilde{d}(G, F) < \epsilon. \quad (2.32)$$

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Relations (2.30), (2.31), and (2.28) imply that for each $x \in X$ satisfying $\rho(x, x_*) \geq \delta_0$

$$G(x) = F^{(2)}(x). \quad (2.33)$$

In view of (2.33) and (2.16) for each $x \in X$ satisfying $\rho(x, x_*) \in [\delta_0, \delta/2]$ we have

$$G(x) = F(x_*). \quad (2.34)$$

It follows from (2.33) and (2.17) that for each $x \in X$ satisfying $\rho(x, x_*) \in [\delta/2, \delta]$

$$G(x) = F(x_*) + \left(\frac{\epsilon}{8}\right) \left[\rho(x, x_*) - \frac{\delta}{2} \right] (1, 1, \dots, 1). \quad (2.35)$$

By (2.30), (2.31), (2.16), (2.27), and (2.28) for each $x \in X$ satisfying $\rho(x, x_*) \leq \delta_0/16$ we have

$$g_1(x) = f_1^{(2)}(x) + \lambda_0 \rho(x, x_*) \delta_0^{-1} 16 = f_1(x_*) + \lambda_0 \rho(x, x_*) \delta_0^{-1} 16; \quad (2.36)$$

for $i = 2, \dots, n$

$$g_i(x) = f_i^{(2)}(x_*) - \lambda_0 \rho(x, x_*) \delta_0^{-1} 16 = f_i(x_*) - \lambda_0 \rho(x, x_*) \delta_0^{-1} 16, \quad (2.37)$$

$$G(x) = F(x_*) + \lambda_0 \rho(x, x_*) \delta_0^{-1} 16 (1, -1, \dots, -1). \quad (2.38)$$

Relations (2.27), (2.30), (2.31), (2.16), and (2.28) imply that for each $x \in X$ satisfying $\rho(x, x_*) = \delta_0/8$ we have

$$g_1(x) = f_1^{(2)}(x) + \lambda_0 \phi_1(2) = f_1(x_*) + \lambda_0; \quad (2.39)$$

for $i = 2, \dots, n$

$$g_i(x) = f_i^{(2)}(x) + \lambda_0 \phi_2(2) = f_i(x_*) - 2\lambda_0, \quad (2.40)$$

$$G(x) = F(x_*) + \lambda_0 (1, -2, -2, \dots, -2). \quad (2.41)$$

We will show that for each $x \in X$ satisfying $\rho(x, x_*) \leq \delta_0$ the following property holds.

(P2) There is $z \in X$ such that $\rho(z, x_*) \in [0, \delta_0/16] \cup \{\delta_0/8\}$ and $G(z) \leq G(x)$.

Let $x \in X$ satisfy

$$\rho(x, x_*) \leq \delta_0. \quad (2.42)$$

Clearly, if $\rho(x, x_*) \leq \delta_0/16$, then (P2) holds with $z = x$. We consider the following cases:

$$\rho(x, x_*) \in \left[\frac{\delta_0}{16}, \frac{\delta_0}{8} \right]; \quad (2.43)$$

$$\rho(x, x_*) \in \left(\frac{\delta_0}{8}, \frac{\delta_0}{2} \right]; \quad (2.44)$$

$$\rho(x, x_*) \in \left(\frac{\delta_0}{2}, \frac{7}{8}\delta_0 \right]; \quad (2.45)$$

$$\rho(x, x_*) \in \left(\frac{7}{8}\delta_0, \frac{1}{16}\delta_0 \right]; \quad (2.46)$$

$$\rho(x, x_*) \in \left(\frac{15}{16}\delta_0, \delta_0 \right]. \quad (2.47)$$

Let (2.43) hold. Then by (2.43), (2.30), (2.31), (2.27), (2.28), and (2.16) we have

$$g_1(x) = f_1^{(2)}(x) + \lambda_0 \phi_1(\rho(x, x_*)\delta_0^{-1}16) = f_1(x_*) + \lambda_0; \quad (2.48)$$

for $i = 2, \dots, n$

$$g_i(x) = f_i^{(2)}(x) + \lambda_0 \phi_2(\rho(x, x_*)\delta_0^{-1}16) = f_i(x_*) - \lambda_0(\rho(x, x_*)\delta_0^{-1}16) \geq f_i(x_*) - 2\lambda_0. \quad (2.49)$$

Together with (2.41) these relations imply that

$$G(x) \geq G(z) \quad \text{if } z \in X \text{ satisfies } \rho(z, x_*) = \frac{\delta_0}{8}. \quad (2.50)$$

Thus property (P2) holds if (2.43) is valid.

Assume that (2.44) is true. By (2.44), (2.30), (2.31), (2.27), (2.28), and (2.16) we have

$$g_1(x) = f_1(x_*) + \lambda_0(\rho(x, x_*)\delta_0^{-1}16 - 1) \geq f_1(x_*) + \lambda_0; \quad (2.51)$$

for $i = 2, \dots, n$

$$g_i(x) = f_i(x_*) + \lambda_0(\rho(x, x_*)\delta_0^{-1}16 - 4) \geq f_i(x_*) + \lambda_0(-2). \quad (2.52)$$

Together with (2.41) these relations imply that for each $z \in X$ satisfying $\rho(z, x_*) = \delta_0/8$ we have $G(z) \leq G(x)$. Thus property (P2) holds if (2.44) is valid.

Assume that (2.45) holds. By (2.45), (2.30), (2.31), (2.27), (2.28), and (2.16) we have

$$g_1(x) = f_1(x_*) + \lambda_0(15 - \rho(x, x_*)\delta_0^{-1}16) \geq f_1(x_*) + \lambda_0; \quad (2.53)$$

for $i = 2, \dots, n$

$$g_i(x) = f_i(x_*) + \lambda_0(12 - \rho(x, x_*)\delta_0^{-1}16) \geq f_i(x_*) - 2\lambda_0. \quad (2.54)$$

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Together with (2.41) these relations imply that for each $z \in X$ satisfying $\rho(z, x_*) = \delta_0/8$ we have $G(z) \leq G(x)$. Thus property (P2) holds if (2.45) is valid.

Assume that (2.46) holds. By (2.46), (2.30), (2.31), (2.27), (2.28), and (2.16) we have

$$g_1(x) = f_1(x_*) + \lambda_0; \quad (2.55)$$

for $i = 2, \dots, n$

$$g_i(x) = f_i(x_*) + \lambda_0(-16 + \rho(x, x_*)\delta_0^{-1}16) \geq f_i(x_*) - 2\lambda_0. \quad (2.56)$$

Together with (2.41) these relations imply that for each $z \in X$ satisfying $\rho(z, x_*) = \delta_0/8$ we have $G(z) \leq G(x)$. Thus property (P2) holds if (2.46) is valid.

Assume that (2.47) holds. By (2.47), (2.30), (2.31), (2.27), (2.28), and (2.16) we have

$$g_1(x) = f_1(x_*) + \lambda_0(16 - \rho(x, x_*)\delta_0^{-1}16); \quad (2.57)$$

for $i = 2, \dots, n$

$$g_i(x) = f_i(x_*) + \lambda_0(\rho(x, x_*)\delta_0^{-1}16 - 16). \quad (2.58)$$

Since the space X is connected, it follows from (2.3) and (2.47) that there is $z \in X$ such that

$$\rho(z, x_*) = \delta_0 - \rho(x, x_*) \in \left[0, \frac{\delta_0}{16}\right]. \quad (2.59)$$

In view of (2.59), (2.38), (2.57), and (2.58)

$$\begin{aligned} G(z) &= F(x_*) + \lambda_0\rho(z, x_*)\delta_0^{-1}16(1, -1, \dots, -1) \\ &= F(x_*) + \lambda_0(\delta_0 - \rho(x, x_*))\delta_0^{-1}16(1, -1, -1, \dots, -1) = G(x). \end{aligned} \quad (2.60)$$

Thus property (P2) holds if (2.47) is valid.

We have shown that (P2) holds in all the cases. We have also shown that the following property holds.

(P3) For each $x \in X$ satisfying $\rho(x, x_*) \leq \delta_0$ there is $z \in X$ such that

$$\rho(z, x_*) \in \left[0, \frac{\delta_0}{16}\right] \cup \left\{\frac{\delta_0}{8}\right\}, \quad G(z) \leq G(x). \quad (2.61)$$

We will show that the following property holds.

(P4) If $x \in X$ satisfies $\rho(x, x_*) \geq \delta/2$ and $z \in X$ satisfies $\rho(z, x_*) \in \{\delta_0/8\} \cup [0, \delta_0/16]$, then the inequality $G(x) \leq G(z)$ does not hold.

Assume that

$$x \in X, \quad z \in X, \quad \rho(x, x_*) > \frac{\delta}{2}, \quad \rho(z, x_*) \in \left\{\frac{\delta_0}{8}\right\} \cup \left[0, \frac{\delta_0}{16}\right]. \quad (2.62)$$

By property (P1) and (2.62) there is $j \in \{1, \dots, n\}$ such that

$$f_j^{(2)}(x) \geq f_j(x_*) + \left(\frac{\epsilon}{8}\right) \min\left\{1, \rho(x, x_*) - \frac{\delta}{2}\right\}. \quad (2.63)$$

By (2.13) and (2.62)

$$f_i^{(2)}(x) > f_i(x_*), \quad i = 1, \dots, n. \quad (2.64)$$

Together with (2.33) and (2.26) this implies that

$$g_i(x) > f_i(x_*), \quad i = 1, \dots, n. \quad (2.65)$$

It follows from (2.62) that

$$\begin{aligned} G(z) \in \{ & F(x_*) + \lambda_0(1, -1, -1, \dots, -1)t : t \in [0, 1] \} \\ & \cup \{ F(x_*) + \lambda_0(1, -2, -2, \dots, -2) \}. \end{aligned} \quad (2.66)$$

It follows from this inclusion and (2.65) that $G(x) \leq G(z)$ does not hold. Therefore property (P4) holds.

Let $t \in [0, 1]$. We show that $F(x_*) + \lambda_0 t(1, -1, -1, \dots, -1) \in \nu(G)$. Since the space X is connected, it follows from (2.3) that there is $z \in X$ such that

$$\rho(z, x_*) = \frac{\delta_0}{16} t. \quad (2.67)$$

By (2.67) and (2.38)

$$\begin{aligned} G(z) &= F(x_*) + \lambda_0 \rho(z, x_*) \delta_0^{-1} 16(1, -1, -1, \dots, -1) \\ &= F(x_*) + t \lambda_0(1, -1, -1, \dots, -1) \in G(X). \end{aligned} \quad (2.68)$$

Assume that

$$x \in X, \quad G(x) \leq G(z). \quad (2.69)$$

We will show that $G(x) = G(z)$. If $\rho(x, x_*) > \delta/2$, then by (2.13), (2.33), and (2.26) the relation (2.65) is true and together with (2.69) this implies that $G(z) \gg F(x_*)$. This contradicts (2.68). Therefore

$$\rho(x, x_*) \leq \frac{\delta}{2}. \quad (2.70)$$

If $\rho(x, x_*) \in [\delta_0, \delta/2]$, then by (2.34), (2.69), (2.68), and (2.67)

$$G(x) = F(x_*), \quad t = 0, \quad z = x_*, \quad G(z) = G(x). \quad (2.71)$$

Assume that

$$\rho(x, x_*) \leq \delta_0. \quad (2.72)$$

By (2.72) and property (P3) there is $y \in X$ such that

$$G(y) \leq G(x), \quad \rho(y, x_*) \in \left[0, \frac{\delta_0}{16}\right] \cup \left\{\frac{\delta_0}{8}\right\}. \quad (2.73)$$

In view of (2.73), (2.69), and (2.68)

$$G(y) \leq G(x) \leq G(z) = F(x_*) + \lambda_0 t(1, -1, -1, \dots, -1). \quad (2.74)$$

If $\rho(y, x_*) = \delta_0/8$, then by (2.41)

$$G(y) = F(x_*) + \lambda_0(1, -2, -2, \dots, -2) \quad (2.75)$$

and since $t \in [0, 1]$, the equality above contradicts (2.74). Therefore in view of (2.73)

$$\rho(y, x_*) \in \left[0, \frac{\delta_0}{16}\right]. \quad (2.76)$$

By (2.76) and (2.38)

$$G(y) = F(x_*) + \lambda_0 \rho(x, x_*) \delta_0^{-1}(16)(1, -1, -1, \dots, -1). \quad (2.77)$$

Together with (2.74) this equality implies that $t = \rho(x, x_*) \delta_0^{-1} 16$ and $G(y) = G(x) = G(z)$. Thus we have shown that (2.69) implies that $G(x) = G(z)$. Therefore

$$F(x_*) + \lambda_0 t(1, -1, -1, \dots, -1) \in v(G) \quad \forall t \in [0, 1]. \quad (2.78)$$

Let $x \in X$ satisfy $\rho(x, x_*) = \delta_0/8$. (Note that by (2.3) such an x exists.) In view of (2.41)

$$G(x) = F(x_*) + \lambda_0(1, -2, -2, \dots, -2) < F(x_*) + \lambda_0(1, -1, -1, \dots, -1). \quad (2.79)$$

Thus

$$F(x_*) + \lambda_0(1, -1, -1, \dots, -1) \notin v(G). \quad (2.80)$$

Together with (2.78) this implies that $v(G)$ is not closed. Theorem 1.2 is proved.

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