THE DISTRIBUTION OF NONPRINCIPAL EIGENVALUES OF SINGULAR SECOND-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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We obtain the asymptotic distribution of the nonprincipal eigenvalues associated with the singular problem $x'' + \lambda q(t)x = 0$ on an infinite interval $[a, +\infty)$. Similar to the regular eigenvalue problem on compact intervals, we can prove a Weyl-type expansion of the eigenvalue counting function, and we derive the asymptotic behavior of the eigenvalues.

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1. Introduction

In this work we study the second-order linear ordinary differential equation

$$x'' + \lambda q(t)x = 0, \quad t \geq a, \quad (1.1)$$

with the boundary conditions

$$x(a, \lambda) = 0, \quad \lim_{t \rightarrow \infty} [x(t, \lambda) - t] = 0, \quad \lim_{t \rightarrow \infty} t[x'(t, \lambda) - 1] = 0, \quad (1.2)$$

where λ is a real parameter and $q(t)$ is a positive continuous function on $[a, \infty)$ satisfying

$$\int_a^\infty t^2 q(t) dt < \infty. \quad (1.3)$$

A nonoscillatory solution $x_0(t, \lambda)$ of (1.1) satisfying the boundary conditions (1.2) is called a nonprincipal eigenfunction if

$$\int_a^\infty \frac{dt}{(x_1(t, \lambda))^2} < \infty, \quad (1.4)$$

and the corresponding value of λ is called a nonprincipal eigenvalue.

2 Distribution of eigenvalues

Concerning the existence and uniqueness of nonprincipal eigenvalues, the main result is due to Elbert et al. [2]. There exists a sequence of positive constants $\{\lambda_k\}_k$, $0 \leq \lambda_0 < \lambda_1 < \dots < \lambda_k < \dots \nearrow \infty$ such that, for each $\lambda = \lambda_k$, (1.1) possesses a solution $x_k(t, \lambda_k)$ satisfying the boundary condition (1.2) and having exactly k zeros in (a, ∞) , $k = 0, 1, 2, \dots$, imposing the integrability condition (1.3) on $q(t)$.

We are interested in the distribution and asymptotic behavior of eigenvalues $\{\lambda_k\}_k$. To this end, we study the spectral counting function

$$N(\lambda) = \#\{k : \lambda_k \leq \lambda\}. \quad (1.5)$$

It is well known that the eigenvalue problem in a closed interval $[a, b]$ has the asymptotic distribution (see [1]):

$$N(\lambda) \sim \frac{\lambda^{1/2}}{\pi} \int_a^b q^{1/2}(t) dt \quad (1.6)$$

as $\lambda \rightarrow \infty$, generalizing the Weyl formula. Here, $f \sim g$ means that $f/g \rightarrow 1$.

Our main result is the following theorem.

THEOREM 1.1. *Let $\{\lambda_k\}$ be the sequence of nonprincipal eigenvalues of problem (1.1)-(1.2), and let $q(t)$ be a positive, continuous, and nonincreasing function satisfying (1.3). Then, the asymptotic expansion of $N(\lambda)$ is given by*

$$N(\lambda) = \frac{\lambda^{1/2}}{\pi} \int_a^\infty q^{1/2}(t) dt + o(\lambda^{1/2}) \quad (1.7)$$

as $\lambda \rightarrow \infty$. Also, the k th-eigenvalue has the following asymptotic behavior:

$$\lambda_{k-1} = \left(\frac{\pi k}{\int_a^\infty q^{1/2}(t) dt} \right)^2 + o(k^2) \quad (1.8)$$

as $k \rightarrow \infty$.

The paper is organized as follows. In Section 2 we prove some auxiliary results, and the proof of Theorem 1.1 is given in Section 3.

2. Sturm-Liouville bracketing of eigenvalues

Let us observe that problem (1.1)-(1.2) is not a variational one, since $x'(t) \sim 1$ as $t \rightarrow +\infty$ and $x'(t) \notin L^2(0, +\infty)$. Hence, we need the following generalization of the Dirichlet-Neumann bracketing of Courant (see [1]) in order to prove Theorem 1.1.

THEOREM 2.1. *Let $N(\lambda, I)$ be the spectral counting function on $I = (a, b)$ of the problem*

$$-x'' = \lambda q(t)x, \quad x(a) = 0 = x(b). \quad (2.1)$$

Let $c \in (a, b)$. Then,

$$N(\lambda, I) \sim N(\lambda, I_1) + N(\lambda, I_2) \quad (2.2)$$

as $\lambda \rightarrow \infty$, where $I_1 = (a, c)$ and $I_2 = (c, b)$.

Remark 2.2. For simplicity, we deal only with the Dirichlet boundary condition on a bounded interval. With minor modifications of the proof, the result is valid for different boundary conditions, including the case $b = +\infty$ and the boundary condition (1.2), since the proof is based on the Sturm-Liouville oscillation theory.

Let us sketch the proof of the Dirichlet Neumann bracketing for a second-order differential operator L with variational structure in an interval I . The eigenvalues of L are obtained minimizing a quadratic functional in a convenient subspace $H \subset H^1(I)$. We have

$$H_0^1(I_1) \oplus H_0^1(I_2) \subset H_0^1(I) \subset H \subset H^1(I) \subset H^1(I_1) \oplus H^1(I_2) \quad (2.3)$$

and we obtain the Dirichlet eigenvalues of L in I_1 and I_2 as an upper bound of the eigenvalues of L in I , and the Neumann eigenvalues of I_1 and I_2 as a lower bound.

In problem (1.1)-(1.2), the solutions and eigenvalues are obtained by a fixed point argument, instead of a minimization procedure, and we need a different argument to relate the eigenvalue of two intervals and those of the union of them. Since the eigenfunction x_k has exactly k zeros in (a, b) , it is possible to obtain the asymptotic distribution of eigenvalues from the asymptotic number of zeros of solutions, an idea which goes back at least to Hartman (see [3]). For the sake of self-completeness, we prove Theorem 2.1 here.

Proof of Theorem 2.1. Let us consider the following eigenvalue problems in I_1 and I_2 , with the original boundary conditions in a and b , and a Neumann boundary condition at c :

$$\begin{aligned} -u'' &= \mu q(t)u, & t \in (a, c), \\ u(a) &= 0, & u'(c) = 0, \end{aligned} \quad (2.4)$$

$$\begin{aligned} -v'' &= \nu q(t)v, & t \in (c, b), \\ v'(c) &= 0, & v(b) = 0. \end{aligned} \quad (2.5)$$

For each problem there exists a sequence of simple eigenvalues $\{\mu_k\}_k, \{\nu_k\}_k$ tending to infinity, and the k th eigenfunction u_k corresponding to μ_k (resp., ν_k, ν_k) has exactly k zeros.

Let λ be fixed. Let λ_n be the greater eigenvalue of problem (2.1) lower or equal than λ and $x_n(t)$ the corresponding eigenfunction, which has n zeros in (a, b) . Let k be the number of zeros of x_n in (a, c) , and let $n - k$ be the number of zeros in (c, b) .

Let μ_j be the greater eigenvalue of problem (2.4) lower or equal than λ , and let u_j be the corresponding eigenfunction. We will show that j , the number of zeros of u_j , satisfies

$$k - 1 \leq j \leq k + 2. \quad (2.6)$$

Let us suppose first that u_j has $k + 3$ zeros. Then, the Sturmian theory gives $\mu_j > \lambda_n$. Let $x_{\mu_j}(t)$ be the unique solution of (2.1) satisfying

$$\begin{aligned} x_{\mu_j}(c) &= u_j(c), \\ x'_{\mu_j}(c) &= u'_j(c). \end{aligned} \quad (2.7)$$

4 Distribution of eigenvalues

Hence, $x_{\mu_j} \equiv u_j$ in (a, c) , and $x_{\mu_j}(t)$ has at least $n - k - 1$ zeros in (c, b) (let us note that one of the original zeros of $x_n(t)$ could cross the point c to the left). Thus, the solution $x_{\mu_j}(t)$ has at least $n + 2$ zeros in (a, b) .

However, the eigenfunction $x_{n+1}(t)$ of problem (2.1) corresponding to the eigenvalue λ_{n+1} has $n + 1$ zeros and satisfy $\lambda_{n+1} < \mu_j$. Hence,

$$\lambda_{n+1} < \mu_j \leq \lambda, \quad (2.8)$$

which contradicts our assumption.

On the other hand, let us suppose that u_j has $k - 2$ zeros. Clearly, $\mu_j < \lambda_n < \lambda$. Let u_{j+1} be the eigenfunction of problem (2.4) with $k - 1$ zeros in (a, c) , and let μ_{j+1} be the corresponding eigenvalue. By using the Sturm-Liouville theory,

$$\mu_{j+1} < \lambda_n < \lambda, \quad (2.9)$$

because $x_n(t)$ has k zeros in (a, c) , which contradicts the fact that μ_j is the greater eigenvalue of problem (2.4) lower or equal than λ .

Let us consider now problem (2.5). Let v_h be the greater eigenvalue of problem (2.5) lower or equal than λ , and let v_h be the corresponding eigenfunction. In much the same way, fixing the boundary condition at $t = b$, we can show that h , the number of zeros of v_h , satisfy

$$n - k - 2 \leq h \leq n - k + 1. \quad (2.10)$$

Then, from inequalities (2.6) and (2.10),

$$N(\lambda, I_1) + N(\lambda, I_2) - 3 \leq N(\lambda, I) \leq N(\lambda, I_1) + N(\lambda, I_2) + 3 \quad (2.11)$$

and the proof is finished. $\square$

3. Asymptotic of nonprincipal eigenvalues

In this section we prove Theorem 1.1. First, we need the following lemma.

LEMMA 3.1. *Let $q(t)$ be a positive continuous function satisfying*

$$\int_a^\infty t^2 q(t) dt < \infty. \quad (3.1)$$

Then,

$$\int_a^\infty q^{1/2}(t) dt < \infty. \quad (3.2)$$

Proof. It follows from Holder's inequality:

$$\int_a^\infty q^{1/2}(t) dt < \left(\int_a^\infty t^2 q(t) dt \right)^{1/2} \left(\int_a^\infty t^{-2} dt \right)^{1/2} < \infty. \quad (3.3)$$

$\square$

We divide the proof of Theorem 1.1 in three parts. We obtain an optimal lower bound for $N(\lambda)$; then we obtain an upper bound for $N(\lambda)$; and finally, we improve the upper bound.

PROPOSITION 3.2. *Let $N(\lambda)$ be the eigenvalue counting function of Theorem 1.1. The following inequality holds:*

$$\frac{\lambda^{1/2}}{\pi} \int_a^{+\infty} q^{1/2}(t) dt + o(\lambda^{1/2}) \leq N(\lambda). \quad (3.4)$$

Proof. Let $\varepsilon > 0$ be fixed, there exist T_ε such that

$$\frac{1}{\pi} \int_{T_\varepsilon}^{\infty} q^{1/2}(t) dt \leq \frac{\varepsilon}{2}. \quad (3.5)$$

Let us consider the Dirichlet eigenvalue problem on $[a, T_\varepsilon]$:

$$-y''(t) = \mu q(t)y(t), \quad (3.6)$$

$$y(a) = 0 = y(T_\varepsilon). \quad (3.7)$$

It is well known that there exists a sequence of eigenvalues $\{\mu_k\}_{k \geq 0}$, with associated eigenfunctions $\{y_k\}_{k \geq 0}$. Each eigenvalue is isolated and y_k has exactly k zeros in the open interval (a, T_ε) .

The spectral counting function $N_D(\lambda, [a, T_\varepsilon])$ of problem (3.6) has the following asymptotic expansion:

$$N_D(\lambda, [a, T_\varepsilon]) = \frac{\lambda^{1/2}}{\pi} \int_a^{T_\varepsilon} q^{1/2}(t) dt + o(\lambda^{1/2}). \quad (3.8)$$

Therefore, for the same $\varepsilon > 0$, there exists $\lambda(\varepsilon)$ such that

$$\left| \frac{N_D(\lambda, [a, T_\varepsilon])}{\lambda^{1/2}} - \frac{1}{\pi} \int_a^{T_\varepsilon} q^{1/2}(t) dt \right| \leq \frac{\varepsilon}{2} \quad (3.9)$$

for every $\lambda \geq \lambda(\varepsilon)$.

By the Sturmian comparison theorem, we have the inequality $\lambda_k \leq \mu_k$, which gives the lower bound for $N(\lambda)$:

$$N_D(\lambda, [a, T_\varepsilon]) \leq N(\lambda). \quad (3.10)$$

Hence,

$$\frac{N(\lambda)}{\lambda^{1/2}} \geq \frac{N_D(\lambda, [a, T_\varepsilon])}{\lambda^{1/2}} \geq \frac{1}{\pi} \int_a^{T_\varepsilon} q^{1/2}(t) dt - \frac{\varepsilon}{2} \geq \frac{1}{\pi} \int_a^{\infty} q^{1/2}(t) dt - \varepsilon \quad (3.11)$$

for every $\lambda \geq \lambda(\varepsilon)$, and the proof is finished. $\square$

Remark 3.3. Let us note that Proposition 3.2 is valid whenever $\int_a^{\infty} q^{1/2}(t) dt < +\infty$, which is guaranteed by Lemma 3.1, without any monotonicity assumption.

6 Distribution of eigenvalues

PROPOSITION 3.4. *Let $N(\lambda)$ be the eigenvalue counting function of Theorem 1.1. The following inequality holds:*

$$\frac{4\lambda^{1/2}}{\pi} \int_a^{+\infty} q^{1/2}(t)dt + o(\lambda^{1/2}) \geq N(\lambda). \quad (3.12)$$

Proof. We need a lower bound for eigenvalues due to Nehari [4]. Let $q(t)$ be a monotonic function, and μ_k the k th Dirichlet eigenvalue of (1.1) in (a, b) . Then,

$$\mu_k \left(\int_a^b q^{1/2}(t)dt \right)^2 \geq \frac{\pi^2 k^2}{4}. \quad (3.13)$$

Let $\{\lambda_k\}_{k \geq 0}$ be the nonprincipal eigenvalues of problem (1.1)-(1.2), and let t_k be the k th zero of the associated eigenfunction $x_k(t)$. Clearly, λ_k coincides with the k th Dirichlet eigenvalue in (a, t_k) .

Hence,

$$\lambda_k \geq \frac{\pi^2 k^2}{4 \left(\int_a^{t_k} q^{1/2}(t)dt \right)^2} \geq \frac{\pi^2 k^2}{4 \left(\int_a^\infty q^{1/2}(t)dt \right)^2}. \quad (3.14)$$

We obtain

$$\begin{aligned} N(\lambda) &= \#\{k : \lambda_k \leq \lambda\} \\ &\leq \#\left\{k : \frac{\pi^2 k^2}{4 \left(\int_a^\infty q^{1/2}(t)dt \right)^2} \leq \lambda\right\} \\ &= \#\left\{k : k \leq \frac{2\lambda^{1/2}}{\pi} \int_a^\infty q^{1/2}(t)dt\right\} \\ &\leq \frac{2\lambda^{1/2}}{\pi} \int_a^\infty q^{1/2}(t)dt + O(1), \end{aligned} \quad (3.15)$$

and the proof is finished. $\square$

Now we prove Theorem 1.1.

Proof of Theorem 1.1. Let be T_ε such that

$$\int_{T_\varepsilon}^{+\infty} q^{1/2}(t)dt < \varepsilon. \quad (3.16)$$

Applying Theorem 2.1 we obtain

$$N(\lambda) \sim N(\lambda, (a, T_\varepsilon)) + N(\lambda, (T_\varepsilon, \infty)). \quad (3.17)$$

The asymptotic behavior of $N(\lambda, (a, T_\varepsilon))$ is obtained from the classical theory,

$$N(\lambda, (a, T_\varepsilon)) \sim \frac{\lambda^{1/2}}{\pi} \int_a^{T_\varepsilon} q^{1/2}(t)dt. \quad (3.18)$$

Hence, for $\lambda \geq \lambda(\varepsilon)$, we have

$$N(\lambda, (a, T_\varepsilon)) \leq \frac{\lambda^{1/2}}{\pi} \int_a^{T_\varepsilon} q^{1/2}(t) dt + \varepsilon \lambda^{1/2} \leq \frac{\lambda^{1/2}}{\pi} \int_a^{+\infty} q^{1/2}(t) dt + \varepsilon \lambda^{1/2}. \quad (3.19)$$

Now, $N(\lambda, (T_\varepsilon, \infty))$ can be bounded by using Proposition 3.4:

$$N(\lambda, (T_\varepsilon, \infty)) \leq \frac{2\lambda^{1/2}}{\pi} \int_{T_\varepsilon}^{+\infty} q^{1/2}(t) dt \leq \varepsilon \frac{2\lambda^{1/2}}{\pi}. \quad (3.20)$$

Hence,

$$N(\lambda) \leq \frac{\lambda^{1/2}}{\pi} \int_a^{+\infty} q^{1/2}(t) dt + \varepsilon \lambda^{1/2} + \varepsilon \frac{2\lambda^{1/2}}{\pi}. \quad (3.21)$$

Since ε is arbitrarily small, and by using Proposition 3.2, we have the asymptotic expansion

$$N(\lambda) \sim \frac{\lambda^{1/2}}{\pi} \int_a^{+\infty} q^{1/2}(t) dt. \quad (3.22)$$

Finally, from (3.22), we have

$$k = N(\lambda_{k-1}) \sim \frac{\lambda_k^{1/2}}{\pi} \int_a^{+\infty} q^{1/2}(t) dt, \quad (3.23)$$

which gives the asymptotic behavior of the k th-eigenvalue,

$$\lambda_k = \left(\frac{\pi k}{\int_a^{+\infty} q^{1/2}(t) dt} \right)^2 + o(k^2). \quad (3.24)$$

This completes the proof. □

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