MIXED JACOBI-LIKE FORMS OF SEVERAL VARIABLES

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Received 4 November 2005; Accepted 26 March 2006

We study mixed Jacobi-like forms of several variables associated to equivariant maps of the Poincaré upper half-plane in connection with usual Jacobi-like forms, Hilbert modular forms, and mixed automorphic forms. We also construct a lifting of a mixed automorphic form to such a mixed Jacobi-like form.

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1. Introduction

Jacobi-like forms of one variable are formal power series with holomorphic coefficients satisfying a certain transformation formula with respect to the action of a discrete subgroup Γ of $SL(2, \mathbb{R})$, and they are related to modular forms for Γ , which of course play a major role in number theory. Indeed, by using this transformation formula, it can be shown that there is a one-to-one correspondence between Jacobi-like forms whose coefficients are holomorphic functions on the Poincaré upper half-plane and certain sequences of modular forms of various weights (cf. [1, 12]). More precisely, each coefficient of such a Jacobi-like form can be expressed in terms of derivatives of a finite number of modular forms in the corresponding sequence. Jacobi-like forms are also closely linked to pseudodifferential operators, which are formal Laurent series for the formal inverse ∂^{-1} of the differentiation operator ∂ with respect to the given variable (see, e.g., [1]). In addition to their natural connections with number theory and pseudodifferential operators, Jacobi-like forms have also been found to be related to conformal field theory in mathematical physics in recent years (see [2, 10]).

The generalization of Jacobi-like forms to the case of several variables was studied in [8] in connection with Hilbert modular forms, which are essentially modular forms of several variables. As it is expected, Jacobi-like forms of several variables correspond to sequences of Hilbert modular forms. Another type of generalization can be provided by considering mixed Jacobi-like forms of one variable for a discrete subgroup $\Gamma \subset SL(2, \mathbb{R})$, which are associated to a holomorphic map of the Poincaré upper half-plane that is equivariant with respect to a homomorphism of Γ into $SL(2, \mathbb{R})$ (cf. [7, 9]). Mixed Jacobi-like

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forms are related to mixed automorphic forms, and examples of mixed automorphic forms include holomorphic forms of the highest degree on the fiber product of elliptic surfaces (see [6]).

In this paper, we study mixed Jacobi-like forms of several variables associated to equivariant maps of the Poincaré upper half-plane in connection with usual Jacobi-like forms, Hilbert modular forms, and mixed automorphic forms. We also construct a lifting of a mixed automorphic form to such a mixed Jacobi-like form.

2. Jacobi-like forms

In this section, we review Jacobi-like forms of several variables and describe some of their properties. We also describe Hilbert modular forms, which are closely linked to such Jacobi-like forms.

Throughout this paper, we fix a positive integer n . Let $(z_1, \dots, z_n)$ be the standard coordinate system for $\mathbb{C}^n$, and denote the associated partial differentiation operators by

$$\partial_1 = \frac{\partial}{\partial z_1}, \dots, \partial_n = \frac{\partial}{\partial z_n}. \quad (2.1)$$

We will often use the multi-index notation. Thus, given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ and $u = (u_1, \dots, u_n) \in \mathbb{C}^n$, we have

$$\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}, \quad u^\alpha = u_1^{\alpha_1} \dots u_n^{\alpha_n}, \quad (2.2)$$

and for $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$, we write $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for each $i = 1, \dots, n$. Furthermore, we also write $\mathbf{c} = (c, \dots, c) \in \mathbb{Z}^n$ if $c \in \mathbb{Z}$, and denote by $\mathbb{Z}_+$ the set of nonnegative integers. Given $\alpha \in \mathbb{Z}^n$ and $\beta \in \mathbb{Z}_+^n$, we write $\beta! = \beta_1! \dots \beta_n!$ and

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}, \quad (2.3)$$

where for $1 \leq i \leq n$, we have $\binom{\alpha_i}{0} = 1$ and

$$\binom{\alpha_i}{\beta_i} = \frac{\alpha_i(\alpha_i - 1) \cdots (\alpha_i - \beta_i + 1)}{\beta_i!} \quad (2.4)$$

for $\beta_i > 0$.

Let $\mathcal{H} \subset \mathbb{C}$ be the Poincaré upper half-plane. Then the usual action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathcal{H}$ by linear fractional transformations induces an action of $\mathrm{SL}(2, \mathbb{R})^n$ on the product $\mathcal{H}^n$ of n copies of $\mathcal{H}$. Thus, if $\gamma \in \mathrm{SL}(2, \mathbb{R})^n$ and $z = (z_1, \dots, z_n) \in \mathcal{H}^n$ with

$$\gamma = (\gamma_1, \dots, \gamma_n), \quad \gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}) \quad (1 \leq i \leq n), \quad (2.5)$$

then we have

$$\gamma z = (\gamma_1 z_1, \dots, \gamma_n z_n) = \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \dots, \frac{a_n z_n + b_n}{c_n z_n + d_n} \right) \in \mathcal{H}^n. \quad (2.6)$$

For such γ and z , we set

$$J(\gamma, z) = (j(\gamma_1, z_1), \dots, j(\gamma_n, z_n)) \in \mathbb{C}^n, \quad j(\gamma_i, z_i) = c_i z_i + d_i \quad (2.7)$$

for $1 \leq i \leq n$. We denote by $\tilde{J}(\gamma, z)$ the diagonal matrix with diagonal entries $j(\gamma_i, z_i)$ with $1 \leq i \leq n$, that is,

$$\tilde{J}(\gamma, z) = \text{diag}(j(\gamma_1, z_1), \dots, j(\gamma_n, z_n)). \quad (2.8)$$

Then the map $(\gamma, z) \mapsto \tilde{J}(\gamma, z)$ satisfies the cocycle condition

$$\tilde{J}(\gamma\gamma', z) = \tilde{J}(\gamma, \gamma'z)\tilde{J}(\gamma', z) \quad (2.9)$$

for all $\gamma, \gamma' \in \text{SL}(2, \mathbb{R})^n$ and $z \in \mathcal{H}^n$. Given an element $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{Z}^n$ and a map $f : \mathcal{H}^n \rightarrow \mathbb{C}$, we set

$$(f|_\eta \gamma)(z) = J(\gamma, z)^{-\eta} f(\gamma z) \quad (2.10)$$

for all $z \in \mathcal{H}^n$ and $\gamma \in \text{SL}(2, \mathbb{R})^n$. Let Γ be a discrete subgroup of $\text{SL}(2, \mathbb{R})^n$.

Definition 2.1. Given $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{Z}_+^n$, a *Hilbert modular form of weight η* for Γ is a holomorphic function $f : \mathcal{H}^n \rightarrow \mathbb{C}$ such that

$$f|_\eta \gamma = f \quad (2.11)$$

for all $\gamma \in \Gamma$, where $f|_\eta \gamma$ is as in (2.10). Denote by $\mathcal{M}_\eta(\Gamma)$ the space of all Hilbert modular forms of weight η for Γ .

Remark 2.2. The usual definition of Hilbert modular forms also includes the regularity condition at the cusps, which is satisfied automatically for $n > 1$ according to Koecher's principle (cf. [3, 4]).

We denote by R the ring of holomorphic functions $f(z_1, \dots, z_n)$ on $\mathcal{H}^n$ and by $R[[X]] = R[[X_1, \dots, X_n]]$ the set of all formal power series in $X_1, \dots, X_n$ with coefficients in R . Thus, using the multi-index notation, an element of $R[[X]]$ can be written in the form

$$\Phi(z, X) = \sum_{\alpha \geq 0} f_\alpha(z) X^\alpha \quad (2.12)$$

with $z = (z_1, \dots, z_n) \in \mathcal{H}^n$ and $X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$.

Let $\mathbb{C}^\times = \mathbb{C} - \{0\}$ be the set of nonzero complex numbers. Given $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^\times)^n$, we denote by $\tilde{\lambda} = \text{diag}(\lambda_1, \dots, \lambda_n)$ the associated $n \times n$ diagonal matrix, and set

$$\mathbb{C}^\times X = \{X\tilde{\lambda} \mid \lambda \in (\mathbb{C}^\times)^n\} = \{(\lambda_1 X_1, \dots, \lambda_n X_n) \mid \lambda_1, \dots, \lambda_n \in \mathbb{C}^\times\}, \quad (2.13)$$

where $X = (X_1, \dots, X_n)$ is regarded as a row vector. Using (2.9), we see that $\text{SL}(2, \mathbb{R})^n$ acts on $\mathcal{H}^n \times \mathbb{C}^\times X$ by

$$\gamma \cdot (z, X\tilde{\lambda}) = (\gamma z, X\tilde{J}(\gamma, z)^{-2}\tilde{\lambda}) \quad (2.14)$$

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for all $z \in \mathcal{H}^n$, $\lambda \in (\mathbb{C}^\times)^n$, and $\gamma \in \mathrm{SL}(2, \mathbb{R})^n$, where $\tilde{J}(\gamma, z)$ is as in (2.8) so that

$$X\tilde{J}(\gamma, z)^{-2}\tilde{\lambda} = (j(\gamma_1, z_1)^{-2}\lambda_1 X_1, \dots, j(\gamma_n, z_n)^{-2}\lambda_n X_n). \quad (2.15)$$

We now set

$$K_{\xi, \eta}(\gamma, (z, X\tilde{\lambda})) = J(\gamma, z)^\xi \exp\left(\sum_{i=1}^n c_i \eta_i j(\gamma_i, z_i)^{-1} \lambda_i X_i\right) \quad (2.16)$$

for $z \in \mathcal{H}^n$, γ as in (2.5), and $\lambda \in (\mathbb{C}^\times)^n$. Then it can be shown that

$$K_{\xi, \eta}(\gamma\gamma', (z, X\tilde{\lambda})) = K_{\xi, \eta}(\gamma, \gamma' \cdot (z, X\tilde{\lambda})) K_{\xi, \eta}(\gamma', (z, X\tilde{\lambda})) \quad (2.17)$$

for all $\gamma, \gamma' \in \mathrm{SL}(2, \mathbb{R})^n$, where $\gamma' \cdot (z, X\tilde{\lambda})$ is as in (2.14).

Definition 2.3. Given $\xi, \eta \in \mathbb{Z}^n$, a *Jacobi-like form* for Γ of n variables of weight ξ , and index η is an element,

$$\Phi(z, X) = \Phi(z, X_1, \dots, X_n) \quad (2.18)$$

of $R[[X]]$ satisfying

$$\Phi(\gamma z, X\tilde{J}(\gamma, z)^{-2}) = K_{\xi, \eta}(\gamma, (z, X)) \Phi(z, X) \quad (2.19)$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{H}^n$. Denote by $\mathcal{J}_{\xi, \eta}(\Gamma)$ the space of all Jacobi-like forms of n variables for Γ of weight ξ and index η .

Remark 2.4. Jacobi-like forms of several variables in $\mathcal{J}_{\xi, \eta}(\Gamma)$ with $\xi = \mathbf{0}$ and $\eta = \mathbf{1}$ were considered in [8], while Jacobi-like forms of one variable with index 0 were studied in [12].

PROPOSITION 2.5. *Given $\varepsilon \in \mathbb{Z}_+^n$, consider a formal power series*

$$\Phi(z, X) = \sum_{\alpha \geq \varepsilon} \phi_\alpha(z) X^\alpha \in R[[X]]. \quad (2.20)$$

Then the following conditions are equivalent.

- (i) *The power series $\Phi(z, X)$ is a Jacobi-like form belonging to $\mathcal{J}_{\xi, \eta}(\Gamma)$.*
- (ii) *The coefficient functions $\phi_\alpha : \mathcal{H} \rightarrow \mathbb{C}$ satisfy*

$$(\phi_\alpha|_{2\alpha+\xi}\gamma)(z) = \sum_{\delta=0}^{\alpha-\varepsilon} \frac{1}{\delta!} \frac{c^\delta \eta^\delta}{J(\gamma, z)^\delta} \phi_{\alpha-\delta}(z) \quad (2.21)$$

for all $z \in \mathcal{H}^n$ and $\alpha \geq \varepsilon$, where $\gamma \in \Gamma$ is as in (2.5) with $c = (c_1, \dots, c_n)$.

- (iii) *There exist modular forms $f_\nu \in \mathcal{M}_{2\nu+\xi}(\Gamma)$ for $\nu \geq \varepsilon$ such that*

$$\phi_\alpha(z) = \sum_{\beta=0}^{\alpha-\varepsilon} \frac{\eta^\beta}{\beta!(2\alpha+\xi-\beta-\varepsilon)!} \partial^\beta f_{\alpha-\beta}(z) \quad (2.22)$$

for all $\alpha \geq \varepsilon$.

Proof. The proposition can be proved by slightly modifying the proofs of [8, Lemma 4.2 and Theorem 4.4]. $\square$

If $\Phi(z, X) = \sum_{\alpha \geq \varepsilon} \phi_\alpha(z) X^\alpha \in \mathcal{F}_{\xi, \eta}(\Gamma)$, then (2.21) implies that

$$\phi_\varepsilon|_{2\varepsilon+\xi}\gamma = \phi_\varepsilon \quad (2.23)$$

for all $\gamma \in \Gamma$; hence the initial coefficient $\phi_\varepsilon(z)$ of the formal power series $\Phi(z, X)$ is a Hilbert modular form of weight $2\varepsilon + \xi$ for Γ . We set

$$\mathcal{F}_{\xi, \eta}(\Gamma)_\varepsilon = X^\varepsilon \mathcal{F}_{\xi, \eta}(\Gamma), \quad (2.24)$$

which is a subspace of $\mathcal{F}_{\xi, \eta}(\Gamma)$ consisting of the elements of the form $\sum_{\alpha \geq \varepsilon} \phi_\alpha(z) X^\alpha$.

Then we see that there is a linear map

$$\mathfrak{F} : \mathcal{F}_{\xi, \eta}(\Gamma)_\varepsilon \longrightarrow \mathcal{M}_{2\varepsilon+\xi}(\Gamma) \quad (2.25)$$

sending an element of $\mathcal{F}_{\xi, \eta}(\Gamma)_\varepsilon$ to its coefficient of X^ε .

3. Mixed Jacobi-like forms

In this section, we discuss Jacobi-like forms of several variables associated to holomorphic maps of the Poincaré upper half-plane $\mathcal{H}$ that are equivariant with respect to a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$. Such Jacobi-like forms are related to mixed automorphic forms.

Let Γ be a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$, and for each $k \in \{1, \dots, n\}$, let $\omega_k : \mathcal{H} \rightarrow \mathcal{H}$ and $\chi_k : \Gamma \rightarrow \mathrm{SL}(2, \mathbb{R})$ be a holomorphic map and a group homomorphism, respectively, satisfying

$$\omega_k(\gamma\zeta) = \chi_k(\gamma)\omega_k(\zeta) \quad (3.1)$$

for all $\zeta \in \mathcal{H}$ and $\gamma \in \Gamma$. By setting

$$\omega = (\omega_1, \dots, \omega_n), \quad \chi = (\chi_1, \dots, \chi_n), \quad (3.2)$$

we obtain a holomorphic map $\omega : \mathcal{H} \rightarrow \mathbb{C}^n$ and a homomorphism $\chi : \Gamma \rightarrow \mathrm{SL}(2, \mathbb{R})^n$. Given $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{Z}^n$, we define the map $J_{\omega, \chi} : \mathrm{SL}(2, \mathbb{R}) \times \mathcal{H} \rightarrow \mathbb{C}^n$ by

$$J_{\omega, \chi}(\gamma, \zeta) = (j(\chi_1(\gamma), \omega_1(\zeta)), \dots, j(\chi_n(\gamma), \omega_n(\zeta))) \quad (3.3)$$

for all $\gamma \in \mathrm{SL}(2, \mathbb{R})$ and $\zeta \in \mathcal{H}$, where $j : \mathrm{SL}(2, \mathbb{R}) \times \mathcal{H} \rightarrow \mathbb{C}$ is as in (2.7).

Definition 3.1. Given $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{Z}^n$, a *mixed automorphic form of type ξ associated to Γ , ω , and χ* is a holomorphic map $f : \mathcal{H} \rightarrow \mathbb{C}$ satisfying

$$f(\gamma\zeta) = J_{\omega, \chi}(\gamma, \zeta)^\xi f(\zeta) = j(\chi_1(\gamma), \omega_1(\zeta))^{\xi_1} \cdots j(\chi_n(\gamma), \omega_n(\zeta))^{\xi_n} f(\zeta) \quad (3.4)$$

for all $\zeta \in \mathcal{H}$ and $\gamma \in \Gamma$. Denote by $\mathcal{M}_\xi(\Gamma, \omega, \chi)$ the space of mixed automorphic forms of type ξ associated to Γ , ω , and χ .

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Definition 3.2. Let $\mathcal{F}$ be the set of holomorphic functions on $\mathcal{H}$, and let $\mathcal{F}[[X]]$ be the space of formal power series in $X = (X_1, \dots, X_n)$. Given $\xi = (\xi_1, \dots, \xi_n), \eta = (\eta_1, \dots, \eta_n) \in \mathbb{Z}^n$, a formal power series $F(\zeta, X) \in \mathcal{F}[[X]]$ is a *mixed Jacobi-like form of weight ξ and index η associated to Γ, ω , and χ* if it satisfies

$$F(\gamma\zeta, X\tilde{J}_{\omega,\chi}(\gamma, \zeta)^{-2}) = J_{\omega,\chi}(\gamma, \zeta)^\xi \exp\left(\sum_{k=1}^n \frac{c_{\chi,k}\eta_k X_k}{j(\chi_k(\gamma), \omega_k(\zeta))}\right) F(\zeta, X) \quad (3.5)$$

for all $\zeta \in \mathcal{H}$ and $\gamma \in \Gamma$, where $\tilde{J}_{\omega,\chi}(\gamma, \zeta)$ denotes the diagonal matrix

$$\text{diag}(j(\chi_1(\gamma), \omega_1(\zeta)), \dots, j(\chi_n(\gamma), \omega_n(\zeta))) \quad (3.6)$$

and $c_{\chi,k}$ is the $(2, 1)$ -entry of the matrix $\chi_k(\gamma) \in \text{SL}(2, \mathbb{R})$. Denote by $\mathcal{F}_{\xi,\eta}(\Gamma, \omega, \chi)$ the space of mixed Jacobi-like forms of weight ξ and index η associated to Γ, ω , and χ .

Given $\mu \in \mathbb{Z}^n$ and a function $h: \mathcal{H} \rightarrow \mathbb{C}$, set

$$(h|_\mu^{\omega,\chi}\gamma)(\zeta) = h(\gamma\zeta)J_{\omega,\chi}(\gamma, \zeta)^{-\mu} \quad (3.7)$$

for all $\zeta \in \mathcal{H}$ and $\gamma \in \Gamma$.

LEMMA 3.3. *A formal power series $F(\zeta, X) = \sum_{\alpha \geq \varepsilon} f_\alpha(\zeta)X^\alpha \in \mathcal{F}[[X]]$ with $\varepsilon \in \mathbb{Z}_+^n$ is an element of $\mathcal{F}_{\xi,\eta}(\Gamma, \omega, \chi)$ if and only if*

$$(f_\alpha|_{2\alpha+\xi}^{\omega,\chi}\gamma)(\zeta) = \sum_{\delta=0}^{\alpha-\varepsilon} \frac{1}{\delta!} \frac{c_\chi^\delta \eta^\delta}{J_{\omega,\chi}(\gamma, \zeta)^\delta} f_{\alpha-\delta}(\zeta) \quad (3.8)$$

for all $\gamma \in \Gamma$ with $c_\chi = (c_{\chi,1}, \dots, c_{\chi,n})$, $\zeta \in \mathcal{H}^n$, and $\alpha \geq \varepsilon$, where $c_{\chi,j}$ denotes the $(2, 1)$ -entry of the matrix $\chi_j(\gamma) \in \text{SL}(2, \mathbb{R})$ for $1 \leq j \leq n$. In particular, the initial coefficient $f_\varepsilon(\zeta)$ of $F(\zeta, X)$ is an element of $\mathcal{M}_{2\varepsilon+\xi}(\Gamma, \omega, \chi)$ if $F(\zeta, X) \in \mathcal{F}_{\xi,\eta}(\Gamma, \omega, \chi)$.

Proof. Given $\gamma \in \Gamma$ as described by (3.4) and (3.5), the formal power series $F(\zeta, X) = \sum_{\alpha \geq \varepsilon} f_\alpha(\zeta)X^\alpha$ is an element of $\mathcal{F}_{\xi,\eta}(\Gamma, \omega, \chi)$ if and only if

$$\begin{aligned} \sum_{\alpha \geq \varepsilon} f_\alpha(\gamma\zeta)J_{\omega,\chi}(\gamma, \zeta)^{-2\alpha-\xi} X^\alpha &= \prod_{i=1}^n \left(\sum_{\mu_i=0}^{\infty} \frac{1}{\mu_i!} \frac{c_{\chi,i}^{\mu_i} \eta_i^{\mu_i} X_i^{\mu_i}}{j(\chi_i(\gamma), \omega_i(z))^{\mu_i}} \right) \cdot \sum_{\gamma \geq \varepsilon} f_\gamma(\zeta) X^\gamma \\ &= \sum_{\mu \geq 0} \sum_{\gamma \geq \varepsilon} \frac{1}{\mu!} \frac{c^\mu \eta^\mu}{J_{\omega,\chi}(\gamma, \zeta)^\mu} f_\gamma(\zeta) X^{\mu+\gamma} \end{aligned} \quad (3.9)$$

for all $\zeta \in \mathcal{H}$. Thus by comparing the coefficients of X^α , we obtain

$$f_\alpha(\gamma\zeta)J_{\omega,\chi}(\gamma, \zeta)^{-2\alpha-\xi} = \sum_{\delta=0}^{\alpha-\varepsilon} \frac{1}{\delta!} \frac{c^\delta \eta^\delta}{J_{\omega,\chi}(\gamma, \zeta)^\delta} f_{\alpha-\delta}(\zeta), \quad (3.10)$$

and therefore the lemma follows. $\square$

For each $\varepsilon \in \mathbb{Z}^n$ with $\varepsilon \geq \mathbf{0}$, we set

$$\mathcal{F}_{\xi,\eta}(\Gamma, \omega, \chi)_\varepsilon = X^\varepsilon \mathcal{F}_{\xi,\eta}(\Gamma, \omega, \chi). \quad (3.11)$$

Then by Lemma 3.3, we see that there is a linear map

$$\mathcal{F}_{\omega,\chi} : \mathcal{F}_{\xi,\eta}(\Gamma, \omega, \chi)_\varepsilon \longrightarrow \mathcal{M}_{2\varepsilon+\xi}(\Gamma, \omega, \chi) \quad (3.12)$$

sending an element $\sum_{\alpha \geq \varepsilon} f_\alpha(\zeta) X^\alpha$ of $\mathcal{F}_{\xi,\eta}(\Gamma, \omega, \chi)$ to its initial coefficient $f_\varepsilon(\zeta)$.

If R is the set of holomorphic functions on $\mathcal{H}^n$ as in Section 2, we define the maps

$$\Delta^\omega : R \longrightarrow \mathcal{F}, \quad \Delta_X^\omega : R[[X]] \longrightarrow \mathcal{F}[[X]] \quad (3.13)$$

associated to the map $\omega : \mathcal{H} \rightarrow \mathcal{H}^n$ as in (3.2) by

$$(\Delta^\omega h)(\zeta) = h(\omega(\zeta)), \quad (\Delta_X^\omega F)(\zeta, X) = F(\omega(\zeta), X) \quad (3.14)$$

for all $\zeta \in \mathcal{H}$, $h \in R$, and $F \in R[[X]]$. Given a discrete subgroup Γ of $\mathrm{SL}(2, \mathbb{R})$, let $\tilde{\Gamma}_\chi$ be a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})^n$ such that

$$\chi(\Gamma) = \chi_1(\Gamma) \times \cdots \times \chi_n(\Gamma) \subset \tilde{\Gamma}_\chi, \quad (3.15)$$

where $\chi = (\chi_1, \dots, \chi_n)$ is as in (3.2).

THEOREM 3.4. (i) If $\Delta^\omega : R \rightarrow \mathcal{F}$ and $\Delta_X^\omega : R[[X]] \rightarrow \mathcal{F}[[X]]$ are as in (3.14), then

$$\Delta^\omega(\mathcal{M}_\xi(\tilde{\Gamma}_\chi)) \subset \mathcal{M}_\xi(\Gamma, \omega, \chi), \quad \Delta_X^\omega(\mathcal{F}_{\xi,\eta}(\tilde{\Gamma}_\chi)_\varepsilon) \subset \mathcal{F}_{\xi,\eta}(\Gamma, \omega, \chi)_\varepsilon \quad (3.16)$$

for all $\xi, \eta \in \mathbb{Z}^n$.

(ii) If $\mathcal{F}$ and $\mathcal{F}_{\omega,\chi}$ are the linear maps in (2.25) and (3.12), respectively, then the diagram

$$\begin{array}{ccc} \mathcal{F}_{\xi,\eta}(\tilde{\Gamma}_\chi)_\varepsilon & \xrightarrow{\mathcal{F}} & \mathcal{M}_{2\varepsilon+\xi}(\tilde{\Gamma}_\chi) \\ \Delta_X^\omega \Big\downarrow & & \Big\downarrow \Delta^\omega \\ \mathcal{F}_{\xi,\eta}(\Gamma, \omega, \chi)_\varepsilon & \xrightarrow{\mathcal{F}_{\omega,\chi}} & \mathcal{M}_{2\varepsilon+\xi}(\Gamma, \omega, \chi) \end{array} \quad (3.17)$$

is commutative.

Proof. If $f : \mathcal{H}^n \rightarrow \mathbb{C}$ is an element of $\mathcal{M}_\xi(\tilde{\Gamma}_\chi)$, then by (3.14) we have

$$\begin{aligned} (\Delta^\omega f)(\gamma\zeta) &= f(\omega(\gamma\zeta)) = f(\chi_1(\gamma)\omega_1(\zeta), \dots, \chi_n(\gamma)\omega_n(\zeta)) \\ &= J(\chi(\gamma), \omega(\zeta))^\xi f(\omega(\zeta)) = J(\chi(\gamma), \omega(\zeta))^\xi (\Delta^\omega f)(\zeta) \end{aligned} \quad (3.18)$$

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for all $\zeta \in \mathcal{H}$ and $\gamma \in \Gamma$; hence $\Delta^\omega f$ is an element of $\mathcal{M}_\xi(\tilde{\Gamma}_\chi, \omega, \chi)$. On the other hand, if F is an element of $\mathcal{F}_{\xi, \eta}(\tilde{\Gamma}_\chi)$ by (3.5) and (3.14), we see that

$$\begin{aligned} (\Delta_X^\omega(F))(\gamma\zeta, X\tilde{J}_{\omega, \chi}(\gamma, \zeta)^{-2}) &= F(\chi_1(\gamma)\omega_1(\zeta), \dots, \chi_n(\gamma)\omega_n(\zeta), X\tilde{J}_{\omega, \chi}(\gamma, \zeta)^{-2}) \\ &= J(\chi(\gamma), \omega(\zeta))^\xi \exp\left(\sum_{k=1}^n \frac{c_{\chi, k}\eta_k X_k}{j(\chi_k(\gamma), \omega_k(\zeta))}\right) F(\omega(\zeta), X) \end{aligned} \quad (3.19)$$

for all $\zeta \in \mathcal{H}$ and $\gamma \in \Gamma$. Thus $\Delta_X^\omega F$ is an element of $\mathcal{F}_{\xi, \eta}(\Gamma, \omega, \chi)$, and $\Delta_X^\omega F \in \mathcal{F}_{\xi, \eta}(\Gamma, \omega, \chi)_\varepsilon$ if $F \in \mathcal{F}_{\xi, \eta}(\tilde{\Gamma}_\chi)_\varepsilon$, which proves (i). In order to verify (ii), consider an element $\Phi(\zeta, X) = \sum_{\alpha \geq \varepsilon} \phi_\alpha(\zeta) X^\alpha \in \mathcal{F}_{\xi, \eta}(\tilde{\Gamma}_\chi)_\varepsilon$. Then we have

$$((\Delta^\omega \circ \mathcal{F})(\Phi))(\zeta) = (\Delta^\omega \phi_\varepsilon)(\zeta) = \phi_\varepsilon(\omega_1(\zeta), \dots, \omega_n(\zeta)) \quad (3.20)$$

for $\zeta \in \mathcal{H}$. On the other hand, we have

$$\begin{aligned} (\Delta_X^\omega \Phi)(\zeta, X) &= \Phi(\omega(\zeta), X) = \Phi(\omega_1(\zeta), \dots, \omega_n(\zeta), X) \\ &= \sum_{\alpha \geq \varepsilon} \phi_\alpha(\omega_1(\zeta), \dots, \omega_n(\zeta)) X^\alpha. \end{aligned} \quad (3.21)$$

Thus we see that

$$((\mathcal{F}_{\omega, \chi} \circ \Delta_X^\omega)(\Phi))(\zeta) = \phi_\varepsilon(\omega_1(\zeta), \dots, \omega_n(\zeta)) = ((\Delta^\omega \circ \mathcal{F})(\Phi))(\zeta), \quad (3.22)$$

which implies (ii); hence the proof of the theorem is complete. $\square$

4. Examples

In this section, we discuss two examples related to mixed Jacobi-like forms. The first one involves a fiber bundle over a Riemann surface whose generic fiber is the product of elliptic curves, and the second one is linked to solutions of linear ordinary differential equations.

Example 4.1. Let E be an elliptic surface (cf. [5]). Thus E is a compact surface over $\mathbb{C}$ that is the total space of an elliptic fibration $\pi : E \rightarrow X$ over a Riemann surface X . Let E_0 be the union of the regular fibers of π , and let $\Gamma \subset \text{PSL}(2, \mathbb{R})$ be the fundamental group of $X_0 = \pi(E_0)$. Then the universal covering space of X_0 may be identified with the Poincaré upper half-plane $\mathcal{H}$, and we have $X_0 = \Gamma \backslash \mathcal{H}$, where Γ is regarded as a subgroup of $\text{SL}(2, \mathbb{R})$ and the quotient is taken with respect to the action given by linear fractional transformations. Given $z \in \mathcal{H}_0$, let Φ be a holomorphic 1-form on $E_z = \pi^{-1}(z)$, and choose an ordered basis $\{\alpha_1(z), \alpha_2(z)\}$ for $H_1(E_z, \mathbb{Z})$ which depends on the parameter z in a continuous manner. If we set

$$\omega_1(z) = \int_{\alpha_1(z)} \Phi, \quad \omega_2(z) = \int_{\alpha_2(z)} \Phi, \quad (4.1)$$

then ω_1/ω_2 is a many-valued function from X_0 to $\mathcal{H}$ which can be lifted to a single-valued function $\omega : \mathcal{H} \rightarrow \mathcal{H}$ on the universal cover $\mathcal{H}$ of X_0 . Then it can be shown that there is a group homomorphism $\chi : \Gamma \rightarrow \mathrm{SL}(2, \mathbb{R})$, called the monodromy representation for the elliptic surface E , such that

$$\omega(\gamma z) = \chi(\gamma)\omega(z) \quad (4.2)$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$. Thus the maps χ and ω form an equivariant pair.

Let (χ_j, ω_j) be an equivariant pair associated to an elliptic surface E of the type described above for each $j \in \{1, \dots, p\}$, and set

$$\tilde{\chi} = (1, \chi_1, \dots, \chi_p), \quad \tilde{\omega} = (1, \omega_1, \dots, \omega_p). \quad (4.3)$$

Then, given a positive integer p and an element $\mathbf{m} = (m_1, \dots, m_p) \in \mathbb{Z}^q$ with $m_1, \dots, m_p > 0$, the semidirect product $\Gamma \ltimes_{\tilde{\chi}} (\mathbb{Z}^2)^{|\mathbf{m}|p}$ with $|\mathbf{m}| = m_1 + \dots + m_p$ associated to $\tilde{\chi}$ acts on $\mathcal{H} \times \mathbb{C}^{|\mathbf{m}|p}$ by

$$(\gamma, \ell_1, \dots, \ell_p) \cdot (z, \zeta_1, \dots, \zeta_p) = (\gamma z, \hat{\zeta}_1, \dots, \hat{\zeta}_p) \quad (4.4)$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$, where

$$\begin{aligned} \ell_j &= ((\mu_{1,j}, \nu_{1,j}), \dots, (\mu_{m_j,j}, \nu_{m_j,j})) \in (\mathbb{Z}^2)^{m_j}, \\ \zeta_j &= (\zeta_{1,j}, \dots, \zeta_{m_j,j}), \hat{\zeta}_j = (\hat{\zeta}_{1,j}, \dots, \hat{\zeta}_{m_j,j}) \in \mathbb{C}^{m_j} \end{aligned} \quad (4.5)$$

for $1 \leq j \leq p$ with

$$\hat{\zeta}_{r,j} = \frac{\zeta_{r,j} + \omega_j(z)\mu_{r,j} + \nu_{r,j}}{c_{\chi_j}\omega_j(z) + d_{\chi_j}} \quad (4.6)$$

for each $r \in \{1, \dots, m_j\}$ if

$$\chi_j(\gamma) = \begin{pmatrix} a_{\chi_j} & b_{\chi_j} \\ c_{\chi_j} & d_{\chi_j} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}). \quad (4.7)$$

We denote by $E_0^{|\mathbf{m}|p}$ the associated quotient space, that is,

$$E_0^{|\mathbf{m}|p} = \Gamma \times (\mathbb{Z}^2)^{|\mathbf{m}|p} \backslash \mathcal{H} \times \mathbb{C}^{|\mathbf{m}|p}. \quad (4.8)$$

Given $\varepsilon \in \mathbb{Z}^{p+1}$, we set $\xi = (2, m_1, \dots, m_p) - 2\varepsilon$, and let $F(z, X) \in \mathcal{F}_{\xi, \eta}(\Gamma, \omega, \chi)_\varepsilon$. Then by Lemma 3.3, we see that $\mathcal{F}_{\omega, \chi}(F(z, X))$ is an element of $\mathcal{M}_{(2, m_1, \dots, m_p)}(\Gamma, \omega, \chi)$, and it can be shown that the associated holomorphic form

$$\omega_F(\mathbf{z}) = \mathcal{F}_{\omega, \chi}(F(z, X)) dz \wedge d\zeta_1 \wedge \dots \wedge d\zeta_p \quad (4.9)$$

on $\mathcal{H} \times \mathbb{C}^{|\mathbf{m}|p}$ with $\mathbf{z} = (z, \zeta_1, \dots, \zeta_p) \in \mathcal{H} \times \mathbb{C}^{|\mathbf{m}|p}$ is invariant under the action of $\Gamma \times (\mathbb{Z}^2)^p$. Hence $\omega_F(\mathbf{z})$ can be regarded as a holomorphic $(|\mathbf{m}|p + 1)$ -form on $E_0^{|\mathbf{m}|p}$, and

therefore we obtain a canonical map

$$\mathcal{J}_{\xi,\eta}(\Gamma, \omega, \chi)_\varepsilon \longrightarrow \Omega^{p+1}(E_0^{|\mathbf{m}|p}) \quad (4.10)$$

from $\mathcal{J}_{\xi,\eta}(\Gamma, \omega, \chi)_\varepsilon$ to the space $\Omega^{p+1}(E_0^{|\mathbf{m}|p})$ of holomorphic $(|\mathbf{m}|p+1)$ -forms on $E_0^{|\mathbf{m}|p}$.

Example 4.2. Let Γ be a Fuchsian group of the first kind, and let $K(X)$ be the function field of the smooth complex algebraic curve $X = \Gamma \backslash \mathcal{H} \cup \{\text{cusps}\}$. Consider a second-order linear differential equation

$$\left(\frac{d^2}{dx^2} + \tilde{P}(x) \frac{d}{dx} + \tilde{Q}(x) \right) \tilde{f} = 0 \quad (4.11)$$

for $x \in X$ and $\tilde{P}(x), \tilde{Q}(x) \in K(X)$ with regular singular points, whose singular points are contained in $\Gamma \backslash \{\text{cusps}\} \subset X$. Let

$$\Lambda f = \left(\frac{d^2}{dz^2} + P(z) \frac{d}{dz} + Q(z) \right) f = 0, \quad (4.12)$$

for $z \in \mathcal{H}$, be the differential equation obtained by pulling back (4.11) via the natural projection $\mathcal{H} \rightarrow \Gamma \backslash \mathcal{H} \subset X$. Let σ_1 and σ_2 be linearly independent solutions of (4.12), and let $S^m(\Lambda)$ be the linear ordinary differential operator of order $m+1$ such that the $m+1$ functions

$$\sigma_1^m, \sigma_1^{m-1} \sigma_2, \dots, \sigma_1 \sigma_2^{m-1}, \sigma_2^m \quad (4.13)$$

are linearly independent solutions of the corresponding linear homogeneous equation $S^m(\Lambda)f = 0$. Let $\chi: \Gamma \rightarrow \text{SL}(2, \mathbb{R})$ be the monodromy representation of Γ for the second-order equation $\Lambda f = 0$. Then the period map $\omega: \mathcal{H} \rightarrow \mathcal{H}$ defined by $\omega(z) = \sigma_1(z)/\sigma_2(z)$ for all $z \in \mathcal{H}$ is equivariant with respect to χ . Let $\psi: \mathcal{H} \rightarrow \mathbb{C}$ be a function corresponding to an element of $K(X)$ satisfying the *parabolic residue condition* in the sense of [11, Definition 3.20], and let f^ψ be a solution of the nonhomogeneous equation $S^m(\Lambda)f = \psi$. Then the function

$$\frac{d^{m+1}}{d\omega(z)^{m+1}} \left(\frac{f^\psi(z)}{\sigma_2(z)^m} \right) \quad (4.14)$$

is a mixed automorphic form of type $(0, m+2)$ associated to Γ , ω , and χ (cf. [11, page 32]).

Given a positive integer p and $\mathbf{m} = (m_1, \dots, m_p) \in \mathbb{Z}^p$ with $m_1, \dots, m_p > 0$, we consider a system of ordinary differential equations

$$S^{m_j}(\Lambda_j) f_j(z_j) = \psi_j(z_j), \quad 1 \leq j \leq p, \quad (4.15)$$

of the type described above and for each $j \in \{1, \dots, p\}$, choose a solution $f_j^{\psi_j}(z_j)$ for the j th equation. For $1 \leq j \leq p$, let $\chi_j: \Gamma_j \rightarrow \text{SL}(2, \mathbb{R})$ and $\omega_j: \mathcal{H} \rightarrow \mathcal{H}$ be the monodromy representation and the period map, respectively, associated to the operator $S^{m_j}(\Lambda_j)$, and

set

$$\tilde{\chi} = (\chi_1, \dots, \chi_p), \quad \tilde{\omega} = (\omega_1, \dots, \omega_p), \quad \Gamma = \Gamma_1 \cap \dots \cap \Gamma_p. \quad (4.16)$$

Then we see that the function $\hat{f} : \mathcal{H} \rightarrow \mathbb{C}$ defined by

$$\hat{f}(z) = f_1(z) \cdots f_p(z) \quad (4.17)$$

for all $z \in \mathcal{H}$ is a mixed automorphic form belonging to $\mathcal{M}_{\mathbf{m}}(\Gamma, \tilde{\omega}, \tilde{\chi})$.

5. Liftings of mixed automorphic forms

Let $\omega = (\omega_1, \dots, \omega_n)$ and $\chi = (\chi_1, \dots, \chi_n)$ be as in Section 3. Thus $\omega_i : \mathcal{H} \rightarrow \mathcal{H}$ is a holomorphic map equivariant with respect to the homomorphism $\chi_i : \Gamma \rightarrow \mathrm{SL}(2, \mathbb{R})$ for each $i \in \{1, \dots, n\}$, where Γ is a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$. In this section, we construct liftings of mixed automorphic forms associated to Γ , ω , and χ of certain types to mixed Jacobi-like forms associated to Γ , ω , and χ .

We first consider discrete subgroups $\Gamma_1, \dots, \Gamma_n$ of $\mathrm{SL}(2, \mathbb{R})$ satisfying

$$\chi_i(\Gamma) \subset \Gamma_i \quad (5.1)$$

for all $i \in \{1, \dots, n\}$. Given $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{Z}^n$ and $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}_+^n$, let $M_{2\mu+\xi_i}(\Gamma_i)$ denote the space of automorphic forms of one variable for Γ_i of weight $2\mu_i + \xi_i$. If Δ^{ω_i} is the map in (3.14) associated to $\omega_i : \mathcal{H} \rightarrow \mathcal{H}$ in the case of $n = 1$, then we see that

$$\Delta^{\omega_i}(M_{2\mu+\xi_i}(\Gamma_i)) = \{h \circ \omega_i \mid h \in M_{2\mu+\xi_i}(\Gamma_i)\} \quad (5.2)$$

for $1 \leq i \leq n$. We denote the tensor product of these spaces by

$$\mathcal{M}_{2\mu+\xi}^0(\Gamma, \omega, \chi) = \bigotimes_{i=1}^n \Delta^{\omega_i}(M_{2\mu+\xi_i}(\Gamma_i)), \quad (5.3)$$

and consider an element of the form

$$\mathfrak{h} = \sum_{k=1}^p C_k \bigotimes_{i=1}^n (h_{i,k} \circ \omega_i) \in \mathcal{M}_{2\mu+\xi}^0(\Gamma, \omega, \chi) \quad (5.4)$$

with $C_k \in \mathbb{C}$ and $h_{i,k} \in M_{2\mu+\xi_i}(\Gamma_i)$ for $1 \leq i \leq n$ and $1 \leq k \leq p$. Then we have

$$\begin{aligned} \mathfrak{h}(\gamma z) &= \sum_{k=1}^p C_k \bigotimes_{i=1}^n (h_{i,k}(\omega_i(\gamma z))) = \sum_{k=1}^p C_k \bigotimes_{i=1}^n (h_{i,k}(\chi_i(\gamma)\omega_i(z))) \\ &= \sum_{k=1}^p C_k \bigotimes_{i=1}^n (j(\chi_i(\gamma), \omega_i(z))^{2\mu_i+\xi_i} h_{i,k}(\chi_i(\gamma)\omega_i(z))) \\ &= \left(\prod_{i=1}^n j(\chi_i(\gamma), \omega_i(z))^{2\mu_i+\xi_i} \right) \mathfrak{h}(z) \end{aligned} \quad (5.5)$$

12 Mixed Jacobi-like forms

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$; hence $\mathfrak{h}$ is a mixed automorphic form belonging to $\mathcal{M}_{2\mu+\xi}(\Gamma, \omega, \chi)$. Thus we see that $\mathcal{M}_{2\mu+\xi}^0(\Gamma, \omega, \chi)$ is a subspace of $\mathcal{M}_{2\mu+\xi}(\Gamma, \omega, \chi)$.

We now discuss a lifting of an element of $\mathcal{M}_{2\mu+\xi}^0(\Gamma, \omega, \chi)$ to a Jacobi-like form belonging to $\mathcal{J}_{\xi, \eta}(\Gamma, \omega, \chi)_\varepsilon$ with $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{Z}_+^n$ and $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{Z}^n$. Given $i \in \{1, \dots, n\}$ and $k \in \{1, \dots, p\}$, assuming that $\mu \geq \varepsilon$, we set

$$\hat{h}_{i,k,\ell} = \frac{\eta_i^{\ell-\mu_i} h_{i,k}^{(\ell-\mu_i)}}{(\ell-\mu_i)!(\ell+\xi_i+\mu_i-\varepsilon_i)!} \quad (5.6)$$

for $\ell \geq \mu_i$ and

$$\hat{h}_{k,\alpha}^\omega(z) = (\hat{h}_{1,k,\alpha_1}(\omega_1(z)), \dots, \hat{h}_{n,k,\alpha_n}(\omega_n(z))) \quad (5.7)$$

for all $z \in \mathcal{H}$ and $\alpha = (\alpha_1, \dots, \alpha_n) \geq \mu$. We define the formal power series $\Phi_{\mathfrak{h}}(z, X) \in R[[X]]$ associated to $\mathfrak{h}$ by

$$\Phi_{\mathfrak{h}}(z, X) = \sum_{k=1}^p C_k \sum_{\alpha \geq \mu} (\hat{h}_{k,\alpha}^\omega(z))^1 X^\alpha, \quad (5.8)$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^n$ so that

$$(\hat{h}_{k,\alpha}^\omega(z))^{\mathbf{1}} = \hat{h}_{1,k,\alpha_1}(\omega_1(z)) \cdots \hat{h}_{n,k,\alpha_n}(\omega_n(z)) \quad (5.9)$$

for $\alpha = (\alpha_1, \dots, \alpha_n)$.

THEOREM 5.1. *The map $\mathfrak{h} \mapsto \Phi_{\mathfrak{h}}$ determines a lifting of an element of $\mathcal{M}_{2\mu+\xi}^0(\Gamma, \omega, \chi)$ to a Jacobi-like form belonging to $\mathcal{J}_{\xi, \eta}(\Gamma, \omega, \chi)_\mu \subset \mathcal{J}_{\xi, \eta}(\Gamma, \omega, \chi)_\varepsilon$ such that*

$$\mathcal{F}_{\omega, \chi}(\Phi_{\mathfrak{h}}) = \frac{\mathfrak{h}}{(2\mu + \xi - \varepsilon)!} \quad (5.10)$$

for all $\mathfrak{h} \in \mathcal{M}_{2\mu+\xi}^0(\Gamma, \omega, \chi)$, where $\mathcal{F}_{\omega, \chi}$ is the map sending $\Phi_{\mathfrak{h}}(z, X)$ to the coefficient of X^μ as in (3.12).

Proof. For $1 \leq i \leq n$, applying Proposition 2.5 to the case of $n = 1$, we see that the formal power series

$$\Phi_i(z, X_i) = \sum_{\ell \geq \varepsilon_i} \phi_\ell(z) X_i^\ell \quad (5.11)$$

in X_i is a Jacobi-like form of one variable belonging to $\mathcal{J}_{\xi_i, \eta_i}(\Gamma_i)_{\varepsilon_i}$ if and only if there is a sequence of modular forms $\{f_r\}_{r \geq 0}$ with $f_r \in M_{2r+\xi_i}(\Gamma_i)$ satisfying

$$\phi_\ell = \sum_{j=0}^{\ell-\varepsilon_i} \frac{\eta_i^j f_{\ell-j}^{(j)}}{j!(2\ell + \xi_i - j - \varepsilon_i)!} \quad (5.12)$$

for all $\ell \geq \varepsilon_i$. We now consider an element $\mathfrak{h} \in \mathcal{M}_\varepsilon^0(\Gamma, \omega, \chi)$ given by (5.4). Given i and k , let $\{f_r\}_{r \geq 0}$ be the sequence of functions on $\mathcal{H}$ defined by

$$f_r = \begin{cases} h_{i,k} & \text{if } r = \mu_i, \\ 0 & \text{otherwise.} \end{cases} \quad (5.13)$$

Then clearly $f_r \in M_{2r+\xi_i}(\Gamma_i)$ for each $r \geq \varepsilon_i$. If $\Phi_{i,k}(z, X_i) = \sum_{\ell \geq \varepsilon_i} \phi_{i,k,\ell}(z) X_i^\ell$ is the corresponding Jacobi-like form belonging to $\mathcal{F}_{\xi_i, \eta_i}(\Gamma_i)$, then by (5.12) the coefficient function $\phi_{i,k,\ell}$ coincides with $\hat{h}_{i,k,\ell}$ in (5.6). Thus for each k , we see that the product

$$\begin{aligned} \Phi_k^\omega(z, X) &= \Phi_{1,k}(\omega_1(z), X_1) \cdots \Phi_{n,k}(\omega_n(z), X_n) \\ &= \sum_{\alpha_1 \geq \mu_1} \cdots \sum_{\alpha_n \geq \mu_n} \hat{h}_{1,k,\alpha_1}(\omega_1(z)) \cdots \hat{h}_{n,k,\alpha_n}(\omega_n(z)) X_1^{\alpha_1} \cdots X_n^{\alpha_n} \\ &= \sum_{\alpha \geq \mu} (\hat{h}_{k,\alpha}^\omega(z))^1 X^\alpha \end{aligned} \quad (5.14)$$

is a Jacobi-like form belonging to $\mathcal{F}_{\xi,\eta}(\Gamma, \omega, \chi)_\mu \subset \mathcal{F}_{\xi,\eta}(\Gamma, \omega, \chi)_\varepsilon$. From this and the fact that the formal power series in (5.8) can be written in the form

$$\Phi_{\mathfrak{h}}(z, X) = \sum_{k=1}^p C_k \Phi_k^\omega(z, X), \quad (5.15)$$

we see that $\Phi_{\mathfrak{h}}(z, X)$ is a Jacobi-like form belonging to $\mathcal{F}_{\xi,\eta}(\Gamma, \omega, \chi)_\mu$. On the other hand, from (5.6) we have

$$\hat{h}_{i,k,\mu_i} = \frac{h_{i,k}}{(2\mu_i + \xi_i - \varepsilon_i)!} \quad (5.16)$$

for $1 \leq i \leq n$ and $1 \leq k \leq p$, which implies that

$$\begin{aligned} \mathcal{F}_{\omega,\chi}(\Phi_k^\omega(z, X)) &= \frac{h_{1,k}(\omega_1(z)) \cdots h_{n,k}(\omega_n(z))}{(2\mu_1 + \xi_1 - \varepsilon_1)! \cdots (2\mu_n + \xi_n - \varepsilon_n)!} \\ &= \frac{h_{1,k}(\omega_1(z)) \cdots h_{n,k}(\omega_n(z))}{(2\mu + \xi - \varepsilon)!}. \end{aligned} \quad (5.17)$$

Combining this with (5.15), we obtain

$$\mathcal{F}_{\omega,\chi}(\Phi_{\mathfrak{h}}(z, X)) = \sum_{k=1}^p C_k \left(\frac{h_{1,k}(\omega_1(z)) \cdots h_{n,k}(\omega_n(z))}{(2\mu + \xi - \varepsilon)!} \right) = \frac{\mathfrak{h}(z)}{(2\mu + \xi - \varepsilon)!}, \quad (5.18)$$

where we identified the tensor product with the usual product in $\mathbb{C}$; hence the proof of the theorem is complete. $\square$

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