Denote the $2 \times 2$ upper triangular matrix rings over $\mathbb{Z}$ and $\mathbb{Z}_p$ by $\text{UTM}_2(\mathbb{Z})$ and $\text{UTM}_2(\mathbb{Z}_p)$, respectively. We prove that if a ring $R$ is a p.p.-ring, then $R$ is reduced if and only if $R$ does not contain any subrings isomorphic to $\text{UTM}_2(\mathbb{Z})$ or $\text{UTM}_2(\mathbb{Z}_p)$. Other conditions for a p.p.-ring to be reduced are also given. Our results strengthen and extend the results of Fraser and Nicholson on r.p.p.-rings.
results strengthen and extend the results obtained by Fraser and Nicholson in [3]. Also, some of our results can be applied to r.p.p.-monoids with zero.

2. Definitions and basic results

The following crucial lemma of p.p.-rings was given by Fraser and Nicholson in [3].

**Lemma 2.1** [3]. Let $R$ be a ring and $a \in R$. Then $R$ is an l.p.p.-ring if and only if $\ell(a) = Re$ for some idempotent $e \in E(R)$.

By using Lemma 2.1, we can give some properties of a p.p.-ring which is reduced.

**Theorem 2.2**. Let $R$ be a p.p.-ring and $E(R)$ the set of all idempotents of $R$. Then the following statements are equivalent:

(i) $R$ is reduced;

(ii) $ef = fe$ for all $e, f \in E(R)$;

(iii) $E(R)$ is a subsemigroup of the semigroup $(R, \cdot)$;

(iv) $ef = 0$ if and only if $fe = 0$ for all $e, f \in E(R)$;

(v) $eR = Re$ for all $e \in E(R)$.

**Proof.** (i)⇒(ii)⇒(iii) are trivial.

(iii)⇒(iv). Let $e, f \in E(R)$. Suppose that $ef = 0$. Then by (iii), we have $fe \in E(R)$ and so $fe = (fe)^2 = f(ef)e = 0$. Similarly, we can show that if $fe = 0$, then $ef = 0$. This proves (iv).

(iv)⇒(v). Let $x \in r(e)$. Then $ex = 0$ and so $e \in \ell(x)$. Since $R$ is a p.p.-ring, by Lemma 2.1, we have $\ell(x) = Rf$, for some $f \in E(R)$. Now, by Pierce decomposition, we have $R = R(1 - f) \oplus Rf$ and hence $\ell(1 - f) = Rf$. Consequently $e \in \ell(1 - f) = \ell(x)$ and thereby $e(1 - f) = 0$ since $ex = 0$. Because $(1 - f) \in E(R)$, by (iv), we have $(1 - f)e = 0$. It is now easy to check that $e + xe \in E(R)$. Since $(e + xe)(1 - f) = 0$, we have, by (iv), $0 = (1 - f)(e + xe) = (1 - f)xe$. However, by $\ell(x) = Rf$ and $1 \in R$, we have $fx = 0$ so that $fxe = 0$. This leads to $xe = (1 - f)xe + fxe = 0$, and thereby $x \in \ell(e)$. Thus $r(e) \subseteq \ell(e)$. Dually, we can show that $\ell(e) \subseteq r(e)$. Therefore $r(e) = \ell(e)$. Thus, for all $e \in R$, $r(1 - e) = \ell(1 - e)$, that is, $eR = Re$. This proves (v).

(v)⇒(i). Since (v) easily yields that the idempotents of $R$ are central, so (v)⇒(i) by [3].

The following example illustrates that there exists a p.p.-ring which is not reduced.

**Example 2.3.** Let $UTM_2(\mathbb{R})$ be the subring of the matrix ring $M_2(\mathbb{R})$ consisting of all $2 \times 2$ upper triangular matrices over the field $\mathbb{R}$. We claim that $UTM_2(\mathbb{R})$ is a p.p.-ring. In order to establish our claim, let

\[
A = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}, \quad B = \begin{pmatrix} x & z \\ 0 & y \end{pmatrix}
\]

be elements of $UTM_2(\mathbb{R})$. Then we see immediately that $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ if and only if $ax = 0$, $by = 0$ and $az + cy = 0$. The following cases now arise.
(i) $x \neq 0$ and $y \neq 0$. In this case, we have $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ if and only if $a = b = c = 0$. Hence, we have

$$\ell(B) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} = UTM_2(\mathbb{R}) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (2.2)

(ii) $x \neq 0$ and $y = 0$. In this case, we have $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ if and only if $a = 0$. This leads to

$$\ell(B) = \left\{ \begin{pmatrix} 0 & c \\ 0 & b \end{pmatrix} : b, c \in \mathbb{R} \right\} = UTM_2(\mathbb{R}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (2.3)

(iii) $x = 0$ and $y \neq 0$. In this case, we have $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ if and only if $b = 0$ and $c = az^{-1}$. This leads to

$$\ell(B) = \left\{ \begin{pmatrix} a & az^{-1} \\ 0 & 0 \end{pmatrix} : a \in R \right\} = UTM_2(\mathbb{R}) \begin{pmatrix} 1 & zy^{-1} \\ 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (2.4)

Summing up the above cases, we can easily see that $\ell(B)$ of $UTM_2(\mathbb{R})$ is generated by an idempotent. Clearly, $UTM_2(\mathbb{R})$ is not reduced.

3. Main theorem

In proving the main theorem of this paper, we first denote by $o(r)$ the (additive) order of $r \in R$, that is, the smallest positive integer $n$ such that $nr = 0$. If $r$ is of infinite order, then we simply write $o(r) = \infty$.

We now prove a useful lemma for p.p.-rings.

**Lemma 3.1.** Let $R$ be a p.p.-ring with 1 such that $ef = 0$ but $fe \neq 0$ for some $e, f \in E(R)$. Then, $o(e) = o(f) = o(fe)$, and if $o(e) < \infty$, then there exist $u, v \in E(R)$ and a prime $p$ such that $o(u) = o(v) = o(vu) = p$ with $uv = 0$ but $vu \neq 0$.

**Proof.** Since $R$ is a p.p.-ring, by Theorem 2.2, $R$ is clearly not reduced. Also, since $1 \in R$, by Lemma 2.1, there exists some $g, h \in E(R)$ such that $\ell(fe) = R(1 - g)$ and $r(fe) = (1 - h)R$. These lead to $\ell(f) = \ell(g)$ and $r(f) = r(h)$. Since $1 - f \in \ell(fe)$, we have $(1 - f)g = 0$ and so $g = fg$. Since $g = fg$, we see that $gf \in E(R)$ and $\ell(g) = \ell(gf)$. Thus, $(1 - gf)fe = 0$ since $(1 - gf)g = 0$ and $\ell(g) = \ell(fe)$, that is, $fe - gf = 0$. Thereby, we have $gf = fe$. Similarly, we can prove that there exists $h \in E(R)$ such that $h = he, eh \in E(R), r(ef) = r(fe)$, and $fe = feh$. Hence, $fe = gf = (gf)(eh)$. On the other hand, we have $(eh)(gf) = e(he)(fg)f = 0$. Because $\ell(f) = \ell(gf)$ and $r(f) = r(ef)$, we can easily see that $o(gf) = o(ef) = o(fe)$.

Now two cases arise.

(i) $o(gf) = \infty$. In this case, there is nothing to prove.

(ii) $o(gf) < \infty$. Without loss of generality, let $o(gf) = pk$, where $p$ is a prime number. Then, we can easily check that $o(kfe) = p$. By using similar arguments as above, we also have $u, v \in E(R)$ such that $o(u) = o(v) = o(kfe)$ with $uv = 0$ but $vu \neq 0$. Hence, $u$ and $v$ are the required idempotents in $R$. The proof is completed. \hfill $\blacksquare$
We now formulate the following main theorem.

**Theorem 3.2.** Let $R$ be a p.p.-ring. Then $R$ is reduced if and only if $R$ has no subrings which are isomorphic either to $\text{UTM}_2(\Bbb{Z})$ or to $\text{UTM}_2(\Bbb{Z}_p)$, where $p$ is a prime.

**Proof.** The necessity part of the theorem follows from Theorem 2.2 since $\text{UTM}_2(\Bbb{Z})$ and $\text{UTM}_2(\Bbb{Z}_p)$ both contain some noncommuting idempotents.

To prove the sufficiency part of the theorem, we suppose that $R$ is not reduced. Then we can let $i, j \in E(R)$ such that $ij = 0$, $ji \neq 0$, and $o(i) = o(ji) = o(ji)$; and $o(i) = o(j) = o(ji) = p$ if $o(i) < \infty$, where $p$ is a prime. Consider the subring of $R$ generated by $i$ and $j$. Clearly, $\{0, i, j, ji\}$ forms a subsemigroup of $R$ under ring multiplication and so $S = \{ai + bji + ci : a, b, c \in \Bbb{Z}\}$ forms a subring of $R$, under the ring multiplication and addition.

Now, we define a mapping $\theta : \text{UTM}_2(\Bbb{Z}) \rightarrow S$ by

$$
\begin{pmatrix}
  a & b \\
  0 & c
\end{pmatrix} \mapsto a + (b - c)ji + ci. \tag{3.1}
$$

Then, we can easily verify that $\theta$ is a surjective homomorphism of $\text{UTM}_2(\Bbb{Z})$ onto $S$.

We now consider the kernel of $\theta$. Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \ker \theta$. Then we have $a + (b - c)ji + ci = 0$. Multiplying $i$ on the left gives $ci = 0$, and multiplying $j$ on the right gives $aj = 0$. Hence, we have $(b - c)ji = 0$.

The following cases arise.

(i) $o(i) = o(ji) = \infty$. Then $a = 0$, $c = 0$, and $(b - c) = 0$. Thus $a = b = c = 0$ and thereby $A = 0$. Hence $\ker \theta = \{0\}$ and $\theta$ is an isomorphism.

(ii) $o(i) = o(ji) = p$. In this case, we have $p | a$, $p | c$, and $p | (b - c)$. Hence $p | a$, $p | c$, and $p | b$. Consequently $\ker \theta = \{(a \ b) : p | a, p | b, p | c\}$. Observing that $\text{UTM}_2(\Bbb{Z})/\ker \theta \cong \text{UTM}_2(\Bbb{Z}_p)$, we have $S \cong \text{UTM}_2(\Bbb{Z}_p)$. This contradicts our assumption and therefore our proof is completed.

As an application of our main theorem, we give a new criterion for a p.p.-ring to be reduced.

**Theorem 3.3.** Let $R$ be a p.p.-ring having no subrings isomorphic to $\text{UTM}_2(\Bbb{Z}_p)$ for prime $p$. If $o(e) < \infty$ for all $e \in E(R)$, then $R$ is reduced.

In fact, Theorem 3.3 follows from the following lemma.

**Lemma 3.4.** Let $R$ be a p.p.-ring having no subring isomorphic to $\text{UTM}_2(\Bbb{Z}_p)$. Suppose that at least one of the idempotents $e, f \in E(R)$ has a prime order $p$. Then $ef = 0$ if and only if $fe = 0$.

**Proof.** Suppose that $ef = 0$ but $fe \neq 0$. Also, suppose that $e$ or $f$ has a prime order $p$. Then, $fe$ must have an order $p$. Now, by using the arguments in the proof of Lemma 3.1, we can construct some idempotents $g, h \in R$ and that $o(g) = o(h) = o(hg) = p$ such that $hg = fe$ but $gh = 0$. By using the arguments in the proof of Theorem 3.2, we can show similarly that the subring $S = \langle g, h \rangle$ of the ring $R$ (the subring of $R$ generated by $f$ and $g$) is isomorphic to $\text{UTM}_2(\Bbb{Z}_p)$. However, this is clearly a contradiction. Thus, we have $fe = 0$. This proves Lemma 3.4.
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