A ring $R$ is called a right Ikeda-Nakayama (for short IN-ring) if the left annihilator of the intersection of any two right ideals is the sum of the left annihilators, that is, if $\ell(I \cap J) = \ell(I) + \ell(J)$ for all right ideals $I$ and $J$ of $R$. $R$ is called Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each $i, j$. In this paper, we show that if $R[x]$ is a right IN-ring, then $R$ is a right IN-ring in case $R$ is an Armendariz ring.
2 Ikeda-Nakayama modules

A module $M$ is called $\alpha$-Armendariz if
\begin{enumerate}[(i)]
\item for any $m \in M$ and $a \in R$, $\alpha(am) = 0$ if and only if $\alpha(a)m = 0$;
\item for any $m(x) = \sum_{i=0}^{n} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{s} a_j x^j \in R[x]$, $m(x)f(x) = 0$ implies $m_i a_j = 0$ for each $i, j$ (cf. [8, 9]).
\end{enumerate}

In [5, Proposition 3.1], Hirano showed that if $R$ is Armendariz ring if and only if $r\text{Ann}_R(2^R) \subseteq r\text{Ann}_R(R[x])$; $A \rightarrow AR[x]$ is bijective, where $r\text{Ann}_R(2^R) = \{ r_R(U) : U \subseteq R \}$. Using this proposition, in this paper, it is shown that if $R[x]$ is a right IN-ring, then $R$ is a right IN-ring, in case $R$ is an Armendariz ring.

2. Ikeda-Nakayama modules

Let $S[x]$ and $R[x]$ be the polynomial rings over rings $S$ and $R$ and, for a module $sM_R$, let $M[x]$ be the set of all formal polynomials in indeterminate $x$ with coefficients from $M$. Then $M[x]$ becomes an $(S[x], R[x])$-bimodule under usual addition and multiplication of polynomials. Extend the notion of an IN-ring to module such as the following.

Definition 2.1. Recall that $M[x]$ is called an Ikeda-Nakayama module (IN-module) if
\begin{equation}
\ell_{S[x]}(U \cap V) = \ell_{S[x]}(U) + \ell_{S[x]}(V)
\end{equation}
for any $R[x]$-submodules $U$ and $V$ of $M[x]_{R[x]}$. Such modules and rings were studied by many authors (cf. [4, 6, 10]). Professor Harmanci asked (private communication) for a description of modules $M$ (rings $R$) such that $M[x]$ are Ikeda-Nakayama modules (right Ikeda-Nakayama rings), respectively.

Note that there is a canonical ring homomorphism $\lambda : S[x] \rightarrow \text{End}(M[x]_{R[x]})$ given by $\lambda(s(x))(m(x)) = s(x)m(x)$ for $m(x) \in M[x]$ and $s(x) \in S[x]$.

Let $U$ and $V$ be $R[x]$-submodules of $M[x]$. Then, for any $t(x) \in \ell_{S[x]}(U \cap V)$, $\alpha_{t(x)} : U + V \rightarrow M[x]$, $u + v \rightarrow t(x)u$ is well defined, where $u \in U$ and $v \in V$.

Lemma 2.2. Let $S[x]M[x]_{R[x]}$-bimodule and $U$ and $V$ be $R[x]$-submodules of $M[x]_{R[x]}$. Then, for any $t(x) \in \ell_{S[x]}(U \cap V)$, every $U + V \xrightarrow{\alpha_{t(x)}} M[x]$ extends commutatively to $M[x]$ by $\lambda(s(x))$ for some $s(x) \in S[x]$ if and only if $M[x]$ is an IN-module.

In particular, if $U \cap V = 0$, then every $U + V \xrightarrow{\alpha_{t(x)}} M[x]$ extends commutatively to $M[x]$ by $\lambda(s(x))$ for some $s(x) \in S[x]$ if and only if $S[x] = \ell_{S[x]}(U) + \ell_{S[x]}(V)$.

Proof. Let $t(x) \in \ell_{S[x]}(U \cap V)$. Then $\alpha_{t(x)} : U + V \rightarrow M[x]$, $u + v \rightarrow t(x)u$ is a well-defined $R[x]$-module homomorphism, where $u \in U$ and $v \in V$. By assumption, there exists $s(x) \in S[x]$ such that $\lambda(s(x))$ extends to $\alpha_{t(x)}$. Thus, for all $u \in U$ and $v \in V$, $t(x)u = \alpha_{t(x)}(u + v) = \lambda(s(x))(u + v) = s(x)(u + v)$ and so $t(x) - s(x)u + (s(x))v = 0$. It follows that $t(x) - s(x) \in \ell_{S[x]}(U)$ and $-s(x) \in \ell_{S[x]}(V)$. Hence $t(x) = (t(x) - s(x)) + (s(x)) \in \ell_{S[x]}(U) + \ell_{S[x]}(V)$. The other inclusion is clear.

For converse, assume that $M[x]$ is an IN-module and, for any $t(x) \in \ell_{S[x]}(U \cap V)$, $\alpha_{t(x)} : U + V \rightarrow M[x]$ defined as above. For $a(x) \in \ell_{S[x]}(U)$ and $b(x) \in \ell_{S[x]}(V)$, write $t(x) = a(x) + b(x)$. Then, for all $u \in U$ and $v \in V$, $\alpha_{t(x)}(u + v) = t(x)u = (a(x) + b(x))u = a(x)u + b(x)u = 0 + b(x)u = b(x)u = b(x)u + b(x)v = b(x)v = \lambda(b(x)(u + v))$. □

As a result of Lemma 2.2, we have the following proposition.
**Proposition 2.3.** Let $R[x]$ be the set of all polynomials in indeterminate $x$ with coefficients from $R$. If $I$ and $J$ are right ideals of $R[x]$ such that every $R[x]$-linear map $I + J \to R[x]$ extends to $R[x]$, then

$$\ell_{R[x]}(I \cap J) = \ell_{R[x]}(I) + \ell_{R[x]}(J).$$

(2.2)

In particular, this holds if $I + J = R[x]$, in which case $\ell_{R[x]}(I \cap J) = \ell_{R[x]}(I) \oplus \ell_{R[x]}(J)$.

Let $N$ be an $R[x]$-submodule of $M[x]$ and $N_C = \{m_i \in M : \exists n \in N \text{ with } n = m_0 + m_1x + \cdots + m_ix^i\}$.

**Theorem 2.4.** Let $M$ be an Ikeda-Nakayama module and let $N$ and $K$ be $R[x]$-submodules of $M[x]$ such that $\ell_S(N \cap K)_C = \ell_S(N_C \cap K_C)$. Then $M[x]$ is an IN-module.

**Proof.** Let $U$ and $V$ be $R[x]$-submodules of $M[x]$. Let $t(x) \in \ell_{S[x]}(U \cap V)$. Then $\alpha_t(x) : U + V \to M[x], u + v \to t(x)u$ is a well defined $R[x]$-homomorphism, where $u \in U$ and $v \in V$. Similarly, for all $t \in \ell_S(U_C \cap V_C)$, the $\alpha_t : U_C + V_C \to M, u' + v' \to tu'$ is a well defined $R$-homomorphism, where $u' \in U_C$ and $v' \in V_C$. Since $M$ is an IN-module, we have $\ell_S(U \cap V)_C = \ell_S(N_C \cap K_C) = \ell_S(U_C) + \ell_S(V_C)$ by assumption and definition. Hence there exists a homomorphism $h_t : M \to M$ such that $h_t i = \alpha_t$, where $i : U_C + V_C \to M$ is the inclusion map by [10, Lemma 1]. We define $h' : M[x] \to M[x]$ such that $h'_i(k_0 + k_1x + \cdots + k_nx^n) = h_i(k_0) + h_i(k_1)x + \cdots + h_i(k_n)x^n$. It is clear that $h'_i$ is well defined. Let $t(x) = t_0 + t_1x + t_2x^2 + \cdots + t_nx^n \in \ell_{S[x]}(U \cap V)$. Then $t_0, t_1, \ldots, t_n \in \ell_S(U \cap V)_C = \ell_S(U_C) + \ell_S(V_C)$. For each $t_j, \alpha_t : U_C + V_C \to M, u' + v' \to tu'$ is a well defined $R$-homomorphism, and then we define a map $h_t : M \to M$ such that $h_t i = \alpha_t$, where $i : U_C + V_C \to M$ is the inclusion map. We extend it by defining $h'_j : M[x] \to M[x]$ such that, for $j = 0, 1, 2, \ldots, n$, $h'_j(k_0 + k_1x + \cdots + k_nx^n) = (h_i(k_0) + h_i(k_1)x + \cdots + h_i(k_n)x^n)x^j$.

To complete the proof, we show that $h_i = \alpha_{t_i(x)}$, where $i' : U + V \to M[x]$ is the inclusion map. Let $h = \sum_{j=0}^n h'_j$ and $u = u_0 + u_1x + \cdots + u_nx^n \in U$ and $v(x) = v_0 + v_1x + \cdots + v_nx^i \in V$. Then $u_0, u_1, \ldots, u_r \in U_C$ and $v_0, v_1, \ldots, v_r \in V_C$. So $h'_j(u + v) = (h_i(u_0) + h_i(u_1)x + \cdots + h_i(u_r)x^r)x^j = t_jx^j(u_0 + u_1x + \cdots + u_rx^r)$ and $(u + v) = \sum_{j=0}^n h'_j(u + v) = t(x)(u + v)$. Hence $M[x]$ is an IN-module by Lemma 2.2. \hfill \Box

Let $\alpha$ be an endomorphism of $R$, that is, $\alpha$ is a ring homomorphism from $R$ to $R$ with $\alpha(1) = 1$. Following [9], a module $M$ is called $\alpha$-Armendariz if

1. for any $m \in M$ and $a \in R$, $ma = 0$ if and only if $m\alpha(a) = 0$;
2. for any $m(x) = \sum_{i=0}^n m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^n a_j x^j \in R[x]$, $m(x)f(x) = 0$ implies $m_ia_j = 0$ for each $i, j$.

Note that 1-Armendariz module is called Armendariz module.

We denote $r\text{Ann}_R(2^M) = \{r_R(U) \mid U \subseteq M\}$ and $\ell\text{Ann}_R(2^M) = \{\ell_R(U) \mid U \subseteq M\}$. If $U$ is a subset of $M$, then $\ell_{R[x]}(U) = \ell_R(U)[x]$ and $r_{R[x]}(U) = r_R(U)[x]$. Hence we have the maps

$$\Phi : r\text{Ann}_R(2^M) \to r\text{Ann}_{R[x]}(2^{M[x]})$$

(2.3)
defined by $\Phi(r_R(U)) = r_R[x](U) = r_R(U)[x]$ for every $r_R(U) \in r\text{Ann}_R(2^M)$ and

$$\Phi' : \ell\text{Ann}_R(2^M) \rightarrow \ell\text{Ann}_{R[x]}(2^{M[x]})$$

(2.4)

defined by $\Phi'(\ell_R(U)) = \ell_R[x](U) = \ell_R(U)[x]$ for every $\ell_R(U) \in \ell\text{Ann}_R(2^M)$.

For a polynomial $m(x) \in M[x]$, $C_m(x)$ denotes the set of coefficients of $m(x)$ and for a subset $V$ of $M[x]$, $C_V$ denotes the set $\bigcup_{m(x) \in V} C_m(x)$. Then

$$r_{R[x]}(V) \cap R = r_R(V) = r_R(C_V), \quad \ell_{R[x]}(V) \cap R = \ell_R(V) = \ell_R(C_V).$$

(2.5)

Hence we also have the maps

$$\Psi : r\text{Ann}_{R[x]}(2^{M[x]}) \rightarrow r\text{Ann}_R(2^M)$$

(2.6)

defined by $\Psi(r_{R[x]}(V)) = r_{R[x]}(V) \cap R$ for every $r_{R[x]}(V) \in r\text{Ann}_{R[x]}(2^{M[x]})$ and

$$\Psi' : \ell\text{Ann}_{R[x]}(2^{M[x]}) \rightarrow \ell\text{Ann}_R(2^M)$$

(2.7)

defined by $\Psi'(\ell_{R[x]}(V)) = \ell_{R[x]}(V) \cap R$ for every $\ell_{R[x]}(V) \in \ell\text{Ann}_{R[x]}(2^{M[x]})$.

Obviously $\Phi$ (or $\Phi'$) is injective and $\Psi$ (or $\Psi'$) is surjective. Also, $\Phi$ (or $\Phi'$) is surjective if and only if $\Psi$ (or $\Psi'$) is injective and in this case $\Phi$ and $\Psi$ (or $\Phi$ and $\Psi'$) are the inverses of each other.

Proposition 2.5. Let $M_R$ be a module. Then the following are equivalent.

1. $M_R$ is an Armendariz module.
2. The map $\Phi : r\text{Ann}_R(2^M) \rightarrow r\text{Ann}_{R[x]}(2^{M[x]})$ defined by $\Phi(r_R(U)) = r_R[x](U) = r_R(U)[x]$, for every $r_R(U) \in r\text{Ann}_R(2^M)$, is bijective.
3. The map $\Phi' : \ell\text{Ann}_R(2^M) \rightarrow \ell\text{Ann}_{R[x]}(2^{M[x]})$ defined by $\Phi'(\ell_R(U)) = \ell_{R[x]}(U) = \ell_{R[U]}(V)$, for every $\ell_R(U) \in \ell\text{Ann}_R(2^M)$, is bijective.

Proof. (1) $\Rightarrow$ (2). Assume $M$ is an Armendariz module. Obviously $\Phi$ is injective. So it is enough to show $\Phi$ is surjective. Let $\ell_{R[x]}(V) \in \ell\text{Ann}_{R[x]}(2^{M[x]})$ for some $V \subseteq M[x]$. Then for $\ell_R(C_V) \in \ell\text{Ann}_R(2^M)$, $\Phi'(\ell_R(C_V)) = \ell_{R[x]}(C_V) = \ell_{R[x]}(V)$. In fact, let $f(x) \in \ell_{R[x]}(C_V)$, where $f(x) = a_0 + a_1x + \cdots + a_nx^n$. Then $f(x)C_V = 0$. Thus for all $m \in C_V$, $f(x)m = a_0m + a_1mx + \cdots + a_nmx^n = 0$ and hence $a_jm = 0$ for all $j$. Let $n(x) = n_0 + n_1x + \cdots + n_lx^l \in V$ be arbitrary. Then $f(x)n(x) = 0$ since $n_i \in C_V$ for all $i$. Hence $f(x) \in \ell_{R[x]}(V)$. Conversely, let $g(x) = b_0 + b_1x + \cdots + b_kx^k \in \ell_{R[x]}(V)$. Then for all $m(x) \in V$, $g(x)m(x) = 0$, where $m(x) = m_0 + m_1x + \cdots + m_lx^l \in V$. Since $M_R$ is Armendariz, $b_jm_i = 0$ for all $i$ and $j$. Hence $g(x)m_i = 0$ for all $i$. So $g(x) \in \ell_{R[x]}(C_V)$ since $m(x) \in V$ is arbitrary. Consequently for each $\ell_{R[x]}(V) \in \ell\text{Ann}_{R[x]}(2^{M[x]})$ for some $V \subseteq M[x]$ there exists $\ell_R(C_V) \in \ell\text{Ann}_R(2^M)$ such that $\Phi'(\ell_R(C_V)) = \ell_{R[x]}(V)$, and therefore $\Phi'$ is surjective.
(3) ⇒ (1). Conversely, assume \( f(x)m(x) = 0 \), where \( m(x) = m_0 + m_1x + \cdots + m_ix^i \in M[x] \) and \( f(x) = a_0 + a_1x + \cdots + a_kx^k \in R[x] \). By hypothesis, \( \ell_{R[x]}(m(x)) = \ell_R(U)[x] \) for some \( U \subseteq M \). Then \( f(x) \in \ell_R(U)[x] \) and hence \( a_j \in \ell_R(U) \) for all \( j \). So \( a_j \in \ell_R(U) \subseteq \ell_R(U)[x] = \ell_{R[x]}(m(x)) \) then \( a_jm(x) = 0 \). Consequently, \( a_jm_i = 0 \) for all \( i \) and \( j \). Therefore \( M_R \) is an Armendariz module.

By Proposition 2.5, we can obtain [5, Proposition 3.1].

**Proposition 2.6.** Let \( R \) be a ring. The following statements are equivalent.

1. \( R \) is Armendariz ring.
2. \( r\text{Ann}_R(2^R) \rightarrow r\text{Ann}_R(2^R[x]) \); \( A \rightarrow AR[x] \) is bijective, where \( r\text{Ann}_R(2^R) = \{r_R(U) : U \subseteq R \} \).
3. \( \ell\text{Ann}_R(2^R) \rightarrow \ell\text{Ann}_R(2^R[x]) \); \( B \rightarrow R[x]B \) is bijective, where \( \ell\text{Ann}_R(2^R) = \{\ell_R(U) : U \subseteq R \} \).

Now, we give the main result of this work.

**Theorem 2.7.** Let \( R[x] \) is a right IN-ring, then \( R \) is a right IN-ring.

**Proof.** Let \( I \) and \( J \) be right ideals of \( R \). Since \( R \) is an Armendariz ring, we have \( \ell_{R[x]}(I) = \ell_R(I)[x] \) by Proposition 2.6, for every right ideal \( I \) of \( R \). Note that \( \ell_{R[x]}(I) = \ell_{R[x]}(I[x]) \). By assumption, \( \ell_{R[x]}(I) + \ell_{R[x]}(J) = \ell_{R[x]}(I[x]) + \ell_{R[x]}(J[x]) = \ell_{R[x]}(I[x] \cap J[x]) = \ell_{R[x]}((I \cap J)[x]) \). Then \( \ell_R((I \cap J)[x]) = \ell_R(I[x]) + \ell_R(J[x]) = (\ell_R(I) + \ell_R(J))[x] \) implies that \( \ell_R(I \cap J) = \ell_R(I) + \ell_R(J) \). So \( R \) is a right IN-ring.

**Example 2.8.** (i) Since \( Z \) is an Armendariz ring, \( Z \) is a right IN-ring if and only if \( Z[x] \) is an IN-ring.

(ii) Let \( R \) be a trivial extension of \( Z \) and the \( Z \)-module \( Z_{2^n} \), that is, \( R = Z \oplus Z_{2^n} \) with the following addition and multiplication:

\[
(n,a) + (m,b) = (n + m, a + b),
\]

\[
(n,a)(m,b) = (nm, nb + ma).
\]

Also \( R \) is the subring \{ \((a, n) \) : \( a \in Z, n \in Z_{2^n} \) \}. \( R \) is an IN-ring by [10]. As Lee and Zhou pointed out [8, Corollary 2.7], \( R \) is an Armendariz ring. We consider the right ideals \( I \) and \( J \) of \( R[x] \):

\[
I = \left\{ \begin{pmatrix} px^2 & u(x) \\ 0 & px^2 \end{pmatrix} : u(x) \in Z_{2^n}, \ p \text{ is prime} \right\},
\]

\[
J = \left\{ \begin{pmatrix} qx + qx^2 & 0 \\ 0 & qx + qx^2 \end{pmatrix} : q \text{ is prime and } (p,q) = 1 \right\}.
\]

Clearly, \( \ell_{R[x]}(I \cap J) = R[x] \) since \( p \) and \( q \) are primes with \( (p,q) = 1 \) and so \( I \cap J = 0 \). But \( \ell_{R[x]}(I) \) and \( \ell_{R[x]}(J) \) do not contain constant. Therefore, \( \ell_{R[x]}(I) + \ell_{R[x]}(J) \neq \ell_{R[x]}(I \cap J) \). So \( R[x] \) is not a right IN-ring by Proposition 2.3.
Recall that, a ring $R$ is called \textit{reduced ring} if it has no nonzero nilpotent elements, a ring $R$ is called \textit{right p.p.-ring} for all $a \in R$, $r_R(a) = eR$, where $e^2 = e \in R$ and $R$ is called \textit{Baer ring}, for all $I \leq_R R$, $r_R(I) = eR$, where $e^2 = e \in R$.

As a result of Theorem 2.7, we can say the following corollary.

**Corollary 2.9.** Let $R[x]$ be a right IN-ring. Then $R$ is a right IN-ring in each of the following cases.

1. $R^2 = 0$.
2. $R$ is a reduced ring.
3. $R$ is an Abelian (if every idempotent of $R$ is central) and von Neumann regular ring.
4. $R$ is an Abelian right (left) p.p.-ring.
5. $R$ is an Abelian Baer ring.

**Proof.** Assume $R[x]$ is a right IN-ring.

1. By [1], if $R^2 = 0$, then $R$ is an Armendariz ring.
2. By [2], reduced rings are Armendariz.
3. Every Abelian von Neumann regular ring is a reduced ring.
4. By [1, Theorem 6] or [7, Lemma 7], if $R$ is an Abelian right (left) p.p.-ring, then $R$ is an Armendariz (a Reduced and so Armendariz) ring.
5. Every Abelian Baer ring is a reduced ring.

Hence $R$ is a right IN-ring by Theorem 2.7.

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