We introduce the notion of sensible fuzzy ideals of BCK-algebras with respect to a $t$-conorm and investigate some of their properties. We give the conditions for a sensible fuzzy subalgebra with respect to a $t$-conorm to be a sensible fuzzy ideal with respect to a $t$-conorm. Some properties of the direct product and $S$-product of fuzzy ideals of BCK-algebras with respect to a $t$-conorm are also discussed.

1. Introduction

Imai and Iséki [3] introduced the class of logical algebras: BCK-algebras. This notion is originated from two different ways: one of the motivations is based on set theory, another motivation is from classical and nonclassical propositional calculus.

The notion of fuzzy sets was first introduced by Zadeh [8]. On the other hand, Schweizer and Sklar [5, 6] introduced the notions of triangular norm ($t$-norm) and triangular conorm ($t$-conorm). Triangular norm ($t$-norm) and triangular conorm ($t$-conorm) are the most general families of binary operations that satisfy the requirement of the conjunction and disjunction operators, respectively. Thus, the $t$-norm generalizes the conjunctive (AND) operator and the $t$-conorm generalizes the disjunctive (OR) operator. In application, $t$-norm $T$ and $t$-conorm $S$ are two functions that map the unit square into the unit interval. Jun and Kim [4] introduced the notion of imaginable fuzzy ideals of BCK-algebras with respect to a $t$-norm. Cho et al. [1] have recently introduced the notion of sensible fuzzy subalgebras of BCK-algebras with respect to $s$-norm and studied some of their properties. In this paper, we introduce the notion of sensible fuzzy ideals of BCK-algebras with respect to a $t$-conorm and investigate some of their properties. We give conditions for a sensible fuzzy subalgebra with respect to a $t$-conorm to be a sensible fuzzy ideal with respect to a $t$-conorm. Some properties of the direct product and $S$-product of fuzzy ideals of BCK-algebras with respect to a $t$-conorm are also obtained.
2. Preliminaries

In this section, we review some definitions and results that will be used in the sequel.

An algebra \((X; *, 0)\) of type \((2, 0)\) is called a BCK-algebra if it satisfies the following conditions:

1. \((x * y) * (x * z)) * (z * y) = 0,
2. \(x * (x * y)) * y = 0,
3. \(x * x = 0,
4. x * y = 0, y * x = 0 \implies x = y,
5. 0 * x = 0

for all \(x, y, z \in X\). We can define a partial ordering relation \(\leq\) on \(X\) by letting \(x \leq y\) if and only if \(x * y = 0\). Let \(S\) be a nonempty subset of a BCK-algebra \(X\), then \(S\) is called a subalgebra of \(X\) if \(x * y \in S\) for all \(x, y \in S\). A mapping \(f : X \rightarrow Y\) of BCK-algebras is a homomorphism if \(f(x * y) = f(x) * f(y)\) for all \(x, y \in X\). A nonempty subset \(A\) of a BCK-algebra \(X\) is called an ideal of \(X\) if, for all \(x, y \in X\), it satisfies (I1) \(0 \in A\), (I2) \(x * y, y \in A \implies x \in A\). A mapping \(\mu : X \rightarrow [0, 1]\), where \(X\) is an arbitrary nonempty set, is called a fuzzy set in \(X\). For any fuzzy set \(\mu\) in \(X\) and any \(\alpha \in [0, 1]\), we define the set \(L(\mu; \alpha) = \{x \in X \mid \mu(x) \leq \alpha\}\), which is called lower level cut of \(\mu\).

Definition 2.1 [2]. A fuzzy set \(\mu\) in a BCK-algebra \(X\) is called an antifuzzy ideal of \(X\) if

\begin{align}
(\text{AF1}) & \mu(0) \leq \mu(x) \text{ for all } x \in X; \\
(\text{AF2}) & \mu(x) \leq \max(\mu(x * y), \mu(y)) \text{ for all } x, y \in X.
\end{align}

Definition 2.2 [7]. A triangular conorm (\(t\)-conorm \(S\)) is a mapping \(S : [0, 1] \times [0, 1] \rightarrow [0, 1]\) that satisfies the following conditions:

\begin{align}
(\text{S1}) & S(x, 0) = x, \\
(\text{S2}) & S(x, y) = S(y, x), \\
(\text{S3}) & S(x, S(y, z)) = S(S(x, y), z), \\
(\text{S4}) & S(x, y) \leq S(x, z) \text{ whenever } y \leq z
\end{align}

for all \(x, y, z \in [0, 1]\).

Replacing 0 by 1 in condition S1, we obtain the concept of \(t\)-norm \(T\).

Definition 2.3. Given a \(t\)-norm \(T\) and a \(t\)-conorm \(S\), \(T\) and \(S\) are dual (with respect to the negation ‘) if and only if \((T(x, y))' = S(x', y')\).

Proposition 2.4. Conjunctive (AND) operator is a \(t\)-norm \(T\) and disjunctive (OR) operator is its dual \(t\)-conorm \(S\).

Proposition 2.5 [5]. For a \(t\)-conorm \(T\), the following statement holds:

\begin{align}
S(x, y) \geq \max(x, y), \quad \forall x, y \in [0, 1].
\end{align}

Definition 2.6. Let \(S\) be a \(t\)-conorm. A fuzzy set \(\mu\) in \(X\) is called sensible with respect to \(S\) if \(\text{Im} \mu \subseteq \Delta_S\), where \(\Delta_S = \{\alpha \in [0, 1] \mid S(\alpha, \alpha) = \alpha\}\).

3. Fuzzy ideals with respect to a \(t\)-conorm

In what follows, let \(X\) denote a BCK-algebra unless otherwise specified.
Definition 3.1. Let $S$ be a $t$-conorm. A fuzzy set $\mu : X \to [0,1]$ is called a fuzzy ideal of $X$ with respect to $S$ if

(SF1) $\mu(0) \leq \mu(x)$,
(SF2) $\mu(x) \leq S(\mu(x \ast y), \mu(y))$

for all $x, y \in X$.

Example 3.2. Let $X = \{0,a,b,1\}$ be a BCK-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>1</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>a</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Define a fuzzy set $\mu : X \to [0,1]$ by $\mu(0) = 0$ if $x \in \{0,a\}$ and $\mu(x) = 1$ for all $x \notin \{0,a\}$ and let $S_m : [0,1] \times [0,1] \to [0,1]$ be a function defined by $S_m(x,y) = \min(x+y,1)$ which is a $t$-conorm for all $x, y \in [0,1]$. By routine calculations, it is easy to check that $\mu$ is a sensible fuzzy ideal of $X$ with respect to $S_m$.

Proposition 3.3. Let $S$ be a $t$-conorm. Then every sensible fuzzy ideal of $X$ with respect to $S$ is an antifuzzy ideal of $X$.

Proof. The proof is obtained dually by using the notion of $t$-conorm $S$ instead of $t$-norm $T$ in [4]. \qed

The converse of Proposition 3.3 is not true in general as seen in the following example.

Example 3.4. Let $X = \{0,1,2,3,4\}$ be a BCK-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
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<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
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<tr>
<td>4</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Define a fuzzy set $\mu : X \to [0,1]$ by $\mu(0) = 0.1, \mu(1) = \mu(2) = \mu(3) = 0.4$ and $\mu(4) = 0.7$ is an antifuzzy ideal of $X$. Let $\gamma \in (0,1)$ and define the binary operation $S_\gamma$ on $(0,1)$ as follows:

$$S_\gamma(\alpha, \beta) = \begin{cases} \max\{\alpha, \beta\} & \text{if } \min\{\alpha, \beta\} = 0, \\ 1 & \min\{\alpha, \beta\} > 0, \alpha + \beta \geq 1 + \gamma, \\ \gamma & \text{otherwise} \end{cases}$$ (3.1)

for all $\alpha, \beta \in [0,1]$. Then $S_\gamma$ is a $t$-conorm. Thus $S_\gamma(\mu(0), \mu(0)) = S_\gamma(0.1,0.1) = \gamma \neq \mu(0)$ whenever $\gamma < 0.8$. Hence $\mu$ is not a sensible fuzzy ideal of $X$ with respect to $S_\gamma$.

Theorem 3.5. Let $S$ be a $t$-conorm and $\mu$ a nonempty fuzzy set of $X$. Then $\mu$ is fuzzy ideal of $X$ with respect to $S$ if and only if each nonempty level subset $L(\mu; \alpha)$ of $\mu$ is an ideal of $X$. 
Proof. Suppose that $\mu$ is a fuzzy ideal of $X$ with respect to $S$. Since $L(\mu, \alpha)$ is nonempty, there exists $x \in L(\mu, \alpha)$. Now, from (SF1), $\mu(0) \leq \mu(x) \leq \alpha$, we have $0 \in L(\mu, \alpha)$. Let $x, y \in X$ be such that $x \ast y \in L(\mu, \alpha)$ and $y \in L(\mu, \alpha)$. Then we have $\mu(x) \leq S(\mu(x \ast y), \mu(y)) \leq S(\alpha, \alpha) = \alpha$, and so $x \in L(\mu, \alpha)$. This shows that the level set $L(\mu, \alpha)$ is an ideal of $X$.

Conversely, assume that every nonempty level subset $L(\mu, \alpha)$ of $\mu$ is an ideal of $X$. Then it can be easily checked that $\mu$ satisfies (SF1). If there exist $x, y \in X$ such that $\mu(x) > S(\mu(x \ast y), \mu(y))$, then by taking $t_0 := (1/2)\{\mu(x) + S(\mu(x \ast y), \mu(y))\}$, we have $x \ast y \in L(\mu; t_0)$ and $y \in L(\mu; t_0)$. Since $\mu$ is an ideal of $X$, $x \in L(\mu; t_0)$, we have $\mu(x) \leq t_0$, a contradiction. Hence $\mu$ is a fuzzy ideal of $X$ with respect to $S$. \hfill $\Box$

**Definition 3.6.** Let $X$ be a BCK-algebra and a family of fuzzy sets $\{\mu_i \mid i \in I\}$ in a BCK-algebra $X$. Then the union $\bigvee_{i \in I} \mu_i$ of $\{\mu_i \mid i \in I\}$ is defined by

$$\left( \bigvee_{i \in I} \mu_i \right)(x) = \sup \{\mu_i(x) \mid i \in I\}$$

(3.2)

for each $x \in X$.

**Theorem 3.7.** If $\{\mu_i \mid i \in I\}$ is a family of fuzzy ideals of a BCK-algebra $X$ with respect to $S$, then $\bigvee_{i \in I} \mu_i(x)$ is a fuzzy ideal of $X$ with respect to $S$.

**Proof.** Let $\{\mu_i \mid i \in I\}$ be a family of fuzzy ideals of $X$ with respect to $S$. It is easy to see that $\mu_i(0) \leq \mu_i(x)$ for all $x \in X$. For $x, y \in X$, we have

$$\left( \bigvee_{i \in I} \mu_i \right)(x) = \sup \{\mu_i(x) \mid i \in I\} \leq \sup \{S(\mu_i(x \ast y), \mu_i(y)) \mid i \in I\}$$

$$= S\left(\sup \{\mu_i(x \ast y) \mid i \in I\}, \sup \{\mu_i(y) \mid i \in I\}\right)$$

(3.3)

Hence $\bigvee_{i \in I}$ is a fuzzy ideal of $X$ with respect to $S$. \hfill $\Box$

**Proposition 3.8.** Every sensible fuzzy ideal of $X$ with respect to $S$ is order preserving.

**Proposition 3.9.** Let $\mu$ be a sensible fuzzy ideal of $X$ with respect to $S$. If the inequality $x \ast y \leq z$ holds in $X$, then $\mu(x) \leq S(\mu(y), \mu(z))$ for all $x, y, z \in X$.

**Definition 3.10** [1]. A fuzzy set $\mu$ is called a fuzzy subalgebra of $X$ with respect to a $t$-conorm $S$ if $\mu(x \ast y) \leq S(\mu(x), \mu(y))$ for all $x, y \in X$.

**Theorem 3.11.** Let $S$ be a $t$-conorm. Then every sensible fuzzy ideal of $X$ with respect to $S$ is a sensible fuzzy subalgebra of $X$ with respect to $S$.

**Proof.** Straightforward. \hfill $\Box$

The converse of Theorem 3.11 is not true in general as seen in the following example.
Example 3.12. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th>$\ast$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Define a fuzzy set $\mu : X \rightarrow [0,1]$ by $\mu(0) = \mu(b) = \mu(c) = 0$ and $\mu(a) = 1$ and let $S_m : [0,1] \times [0,1] \rightarrow [0,1]$ be a function defined by $S_m(x,y) = \min\{x+y,1\}$ which is a $t$-conorm for all $x, y \in [0,1]$. By routine computation, we can easily check that $\mu$ is a sensible fuzzy subalgebra of $X$ with respect to $S_m$. But $\mu$ is not a sensible fuzzy ideal of $X$ with respect to $S_m$ because $\mu(a) = 1 \geq 0 = S_m(\mu(a \ast b),\mu(b))$.

Remark 3.13. In Example 3.12, we observe that a sensible fuzzy subalgebra with respect to $S$ is not a sensible fuzzy ideal with respect to $S$. So, a question arises: under what condition(s) a sensible fuzzy subalgebra with respect to $S$ is a sensible fuzzy ideal with respect to $S$? We answer this question in the following theorems without proofs.

Theorem 3.14. Let $S$ be a $t$-conorm. A sensible fuzzy subalgebra $\mu$ of $X$ with respect to $S$ is a sensible fuzzy ideal of $X$ with respect to $S$ if and only if for all $x,y,z \in X$, the inequality $x \ast y \leq z$ implies that $\mu(x) \leq S(\mu(y),\mu(z))$.

Theorem 3.15. Let $S$ be a $t$-conorm and let $X$ be a BCK-algebra in which the equality $x = (x \ast y) \ast y$ holds for all distinct elements $x$ and $y$ of $X$. Then every sensible fuzzy subalgebra of $X$ with respect to $S$ is a sensible fuzzy ideal of $X$ with respect to $S$.

Definition 3.16. Let $f : X \rightarrow Y$ be a mapping, where $X$ and $Y$ are nonempty sets, and $\mu$ is fuzzy set of $Y$. The preimage of $\mu$ under $f$ written $\mu^f$ is a fuzzy set of $X$ defined by $\mu^f(x) = \mu(f(x))$ for all $x \in X$.

Theorem 3.17. Let $f : X \rightarrow Y$ be a homomorphism of BCK-algebras. If $\mu$ is a fuzzy ideal of $Y$ with respect to $S$, then $\mu^f$ is a fuzzy ideal of $X$ with respect to $S$.

Proof. For any $x \in X$, we have $\mu^f(x) = \mu(f(x)) \geq \mu(0) = \mu^f(0) = \mu^f(0)$. Let $x, y \in X$. Then we have

\[
S(\mu^f(x \ast y),\mu^f(y)) = S(\mu(f(x \ast y)),\mu(f(y)))
\]

\[
= S(\mu(f(x) \ast f(y)),\mu(f(y)))
\]

\[
\leq \mu(f(x)) = \mu^f(x).
\]

Hence $\mu^f$ is a fuzzy ideal of $X$ with respect to $S$. \qed

Theorem 3.18. Let $f : X \rightarrow Y$ be an epimorphism of BCK-algebras. If $\mu^f$ is a fuzzy ideal of $X$ with respect to $S$, then $\mu$ is a fuzzy ideal of $Y$ with respect to $S$.

Proof. Let $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Then $\mu(y) = \mu(f(x)) = \mu^f(x) \geq \mu^f(0) = \mu(f(0)) = \mu(0)$, where $0 = f(0)$. Let $x, y \in Y$. Then there exist $a, b \in X$ such that
f(a) = x and f(b) = y. It follows that

\[ \mu(x) = \mu(f(a)) = \mu(f(a)) = \mu_f(a) \]

\[ \leq S(\mu_f(a \ast b), \mu_f(b)) = S(\mu(f(a \ast b)), \mu(f(b))) \]  \hfill (3.5)

\[ = S(\mu(f(a) \ast f(b)), \mu(f(b))) = S(\mu(x \ast y), \mu(y)). \]

Hence \( \mu \) is a fuzzy ideal of \( Y \) with respect to \( S \).

**Definition 3.19.** Let \( f \) be a mapping defined on \( X \). If \( \nu \) is a fuzzy set in \( f(X) \), then the fuzzy set \( \mu = \nu \circ f \) in \( X \) (i.e., the fuzzy set defined by \( \mu(x) = \nu(f(x)) \) for all \( x \in X \)) is called the **preimage** of \( \nu \) under \( f \).

**Theorem 3.20.** Let \( S \) be a \( t \)-conorm and let \( f : X \to Y \) be an epimorphism of BCK-algebras, \( \nu \) sensible fuzzy ideal of \( Y \) with respect to \( S \) and \( \mu \), the preimage of \( \nu \) under \( f \). Then \( \mu \) is a sensible fuzzy ideal of \( X \) with respect to \( S \).

**Proof.** The proof is obtained dually by using the notion of \( t \)-conorm \( S \) instead of \( t \)-norm \( T \) in [4].

**Theorem 3.21.** Let \( \mu \) be a fuzzy set in \( X \) and \( \text{Im}(\mu) = \{\alpha_0, \alpha_1, \ldots, \alpha_k\} \), where \( \alpha_i < \alpha_j \) whenever \( i > j \). Let \( \{A_n \mid n = 0,1,\ldots,k\} \) be a family of ideals of \( X \) with respect to a \( t \)-conorm \( S \) such that

(i) \( A_0 \subset A_1 \subset \cdots \subset A_k = X \),

(ii) \( \mu(A^*) = \alpha_n \), where \( A_n^* = A_n \setminus A_{n-1}, A_{-1} = \emptyset \) for \( n = 0,1,\ldots,k \).

Then \( \mu \) is a fuzzy ideal of \( X \) with respect to \( S \).

**Proof.** Since \( 0 \in A_0 \), we have \( \mu(0) = \alpha_0 \leq \mu(x) \) for all \( x \in X \). Let \( x, y \in X \). Then we discuss the following cases: if \( x \ast y \in A_n^* \) and \( y \in A_n^* \), then \( x, y \in A_n \) because \( A_n \) is an ideal of \( X \). Thus

\[ \mu(x) \leq \alpha_n = S(\mu(x \ast y), \mu(y)). \]  \hfill (3.6)

If \( x \ast y \not\in A_n^* \) and \( y \not\in A_n^* \), then the following four cases arise:

1. \( x \ast y \in X \setminus A_n \) and \( y \in X \setminus A_n \),
2. \( x \ast y \in A_{n-1} \) and \( y \in A_{n-1} \),
3. \( x \ast y \in X \setminus A_n \) and \( y \in A_{n-1} \),
4. \( x \ast y \in A_{n-1} \) and \( y \in X \setminus A_n \).

But, in either case, we know that

\[ \mu(x) \leq S(\mu(x \ast y), \mu(y)). \]  \hfill (3.7)

If \( x \ast y \in A_n^* \) and \( y \not\in A_n^* \), then either \( y \in A_{n-1} \) or \( y \in X \setminus A_n \). It follows that either \( x \in A_n \) or \( x \in X \setminus A_n \). Thus

\[ \mu(x) \leq S(\mu(x \ast y), \mu(y)). \]  \hfill (3.8)
If \( x \neq y \in A_n^* \) and \( y \in A_n^* \), then by similar process, we have
\[
\mu(x) \leq S(\mu(x \ast y), \mu(y)).
\] (3.9)

This completes the proof. \( \square \)

**Definition 3.22** [9]. A BCK-algebra \( X \) is said to satisfy the ascending (resp., descending) chain condition (ACC (resp., DCC)) if for every ascending (resp., descending) sequence \( A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \) (resp., \( A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \)) of ideals of \( X \) there exists a natural number \( n \) such that \( A_n = A_k \) for all \( n \geq k \). If \( X \) satisfies DCC, \( X \) is an Artin BCK-algebras.

**Theorem 3.23.** Let \( S \) be a \( t \)-conorm. If \( \mu \) is a fuzzy ideal of \( X \), with respect to \( S \), having finite image, then \( X \) is an Artin BCK-algebra.

**Proof.** Suppose that there exists a strictly descending chain \( A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \) of fuzzy ideals of \( X \) which does not terminate at finite step. Define a fuzzy set \( \mu \) in \( X \) by
\[
\mu(x) := \begin{cases} 
\frac{1}{n+1} & \text{if } x \in A_n \setminus A_{n+1}, \ x = 0, 1, 2, \ldots, \\
0 & \text{if } x \in \bigcap_{n=0}^{\infty} A_n,
\end{cases}
\] (3.10)

where \( A_0 = X \). We prove that \( \mu \) is a fuzzy ideal of \( X \) with respect to \( S \). Clearly, \( \mu(0) \leq \mu(x) \) for all \( x \in X \). Let \( x, y \in X \). Assume that \( x \neq y \in A_n \setminus A_{n+1} \) and \( y \in A_k \setminus A_{k+1} \) for \( n = 0, 1, 2, \ldots; k = 0, 1, 2, \ldots \). Without loss of generality, we may assume that \( n \leq k \). Then obviously \( y \in A_n \), and so \( x \in A_n \) because \( A_n \) is a fuzzy ideal of \( X \). Hence
\[
\mu(x) \leq \frac{1}{n+1} = S(\mu(x \ast y), \mu(y)).
\] (3.11)

If \( x \neq y \), \( y \in \bigcap_{n=0}^{\infty} A_n \), then \( x \in \bigcap_{n=0}^{\infty} A_n \). Thus
\[
\mu(x) = 0 = S(\mu(x \ast y), \mu(y)).
\] (3.12)

If \( x \neq y \notin \bigcap_{n=0}^{\infty} A_n \) and \( y \notin \bigcap_{n=0}^{\infty} A_n \), then there exists \( k \in \mathbb{N} \) such that \( x \neq y \in A_k \setminus A_{k+1} \). It follows that \( x \in A_k \) so that
\[
\mu(x) \leq \frac{1}{k+1} = S(\mu(x \ast y), \mu(y)).
\] (3.13)

Finally, suppose that \( x \neq y \subseteq \bigcap_{n=0}^{\infty} A_n \) and \( y \notin \bigcap_{n=0}^{\infty} A_n \). Then \( y \in A_r \setminus A_{r+1} \) for some \( r \in \mathbb{N} \). Hence \( x \in A_r \), and so
\[
\mu(x) \leq \frac{1}{r+1} = S(\mu(x \ast y), \mu(y)).
\] (3.14)

Consequently, we conclude that \( \mu \) is a fuzzy ideal of \( X \) with respect to \( S \) and \( \mu \) has infinite number of different values. This is a contradiction, and the proof is complete. \( \square \)

**Theorem 3.24.** Let \( S \) be a \( t \)-conorm. The following statements are equivalent:

(i) every ascending chain of ideals of \( X \) with respect to \( S \) terminates at finite step,

(ii) the set of values of any fuzzy ideal with respect to \( S \) is a well-ordered subset of \([0, 1]\).
Sensible fuzzy ideals of BCK-algebras with $t$-conorms

Proof. Let $\mu$ be a fuzzy ideal of $X$ with respect to $S$. Suppose that the set of values of $\mu$ is not a well-ordered subset of $[0,1]$. Then there exists a strictly increasing sequence $\{\alpha_n\}$ such that $\mu(x) = \alpha_n$. Let $G_n := \{x \in X \mid \mu(x) \leq \alpha\}$. Then

$$G_1 \subset G_2 \subset G_3 \subset \cdots$$

(3.15)

is a strictly ascending chain of ideals of $X$ which is not terminating. This is a contradiction.

Conversely, suppose that there exists a strictly ascending chain

$$G_1 \subset G_2 \subset G_3 \subset \cdots$$

(*)

of ideals of $X$ with respect to $S$ which does not terminate at finite step. Define a fuzzy set $\mu$ in $X$ by

$$\mu(x) := \begin{cases} 1/k, & \text{where } k = \max\{n \in \mathbb{N} \mid x \in G_n\}, \\ 1, & \text{if } x \in G_n, \end{cases}$$

(3.16)

where $G = \bigcup_{n \in \mathbb{N}} G_n$. Since $0 \in G_n$ for all $n = 0, 1, \ldots$, therefore, $\mu(0) \leq \mu(x)$ for all $x \in X$. Let $x, y \in X$. If $x \ast y, y \in G_n \setminus G_{n-1}$ for $n = 2, 3, \ldots$, then $x \in G_n$. Thus, we obtain

$$\mu(x) \leq \frac{1}{n} = S(\mu(x \ast y), \mu(y)).$$

(3.17)

Assume that $x \ast y \in G_n$ and $y \in G_n \setminus G_m$ for all $m < n$. Since $\mu$ is an ideal of $X$, therefore, $x \in G_n$. Thus

$$\mu(x) \leq \frac{1}{n} \leq \frac{1}{m+1} \leq \mu(y),$$

(3.18)

and hence

$$\mu(x) \leq S(\mu(x \ast y), \mu(y)).$$

(3.19)

Similarly, for the case $x \ast y \in G_n \setminus G_m$ and $y \in G_n$, we have

$$\mu(x) \leq S(\mu(x \ast y), \mu(y)).$$

(3.20)

Hence $\mu$ is an ideal of $X$ with respect to $t$-conorm $S$. Since the chain (*) is not terminating, $\mu$ has strictly descending sequence of values. This contradicts that the value of any set of fuzzy ideal with respect to $S$ is well ordered. This ends the proof. 

\[\Box\]

Lemma 3.25. Let $T$ be a $t$-norm. Then $t$-conorm $S$ can be defined as

$$S(x, y) = 1 - T(1 - x, 1 - y).$$

(3.21)

Proof. Straightforward. 

\[\Box\]

Theorem 3.26. A fuzzy set $\mu$ of a BCK-algebra $X$ is a $T$-fuzzy ideal of $X$ if and only if its complement $\mu^c$ is an $S$-fuzzy ideal of $X$. 

Proof. Let \( \mu \) be a \( T \)-fuzzy ideal of \( X \). For \( x, y \in X \), we have
\[
\begin{align*}
\mu^c(0) &= 1 - \mu(0) \leq 1 - \mu(x) = \mu^c(x), \\
\mu^c(x) &= 1 - \mu(x) \leq 1 - T\mu((x \ast y), \mu(y)) \\
&= 1 - T1 - \mu^c((x \ast y), 1 - \mu^c(y)) \\
&= S(\mu^c(x \ast y), \mu^c(y)).
\end{align*}
\]
Hence \( \mu^c \) is an \( S \)-fuzzy ideal of \( X \).
The converse is proved similarly. \(\square\)

4. \( S \)-product and direct product with respect to a \( t \)-conorm

In this section, we discuss properties of \( S \)-product and direct product of fuzzy ideals of a BCK-algebra with respect to a \( t \)-conorm.

Definition 4.1. Let \( S \) be a \( t \)-conorm and let \( \lambda \) and \( \mu \) be two fuzzy sets in \( X \). Then the \( S \)-product of \( \lambda \) and \( \mu \) is denoted by \( [\lambda \cdot \mu]_S \) and defined by \( [\lambda \cdot \mu]_S(x) = S(\lambda(x), \mu(x)) \), for all \( x \in X \).

Theorem 4.2. Let \( \lambda \) and \( \mu \) be two fuzzy ideals of \( X \) with respect to \( S \). If a \( t \)-conorm \( S^* \) dominates \( S \), that is, if \( S^*(\lambda(x), \mu(x)) \leq S(S^*(\alpha, \beta), S^*(\gamma, \delta)) \) for all \( \alpha, \beta, \gamma, \delta \in [0,1] \), then \( S^* \)-product \( [\lambda \cdot \mu]_{S^*} \) is a fuzzy ideal of \( X \) with respect to \( S \).

Proof. For any \( x \in X \), we have
\[
[\lambda \cdot \mu]_{S^*}(0) = S^*(\lambda(0), \mu(0)) \leq S^*(\lambda(x), \mu(x)) = [\lambda \cdot \mu]_{S^*}(x).
\]
Let \( x, y \in X \). Then
\[
[\lambda \cdot \mu]_{S^*}(x) = S^*(\lambda(x), \mu(x)) \\
\leq S^*(S(\lambda(x \ast y), \lambda(y)), S(\mu(x \ast y), \mu(y))) \\
\leq S(S^*(\lambda(x \ast y), \mu(x \ast y)), S^*(\lambda(y), \mu(y))) \\
= S([\lambda \cdot \mu]_{S^*}(x \ast y), [\lambda \cdot \mu]_{S^*}(y)).
\]
Hence \( [\lambda \cdot \mu]_{S^*} \) is a fuzzy ideal of \( X \) with respect to \( S \). \(\square\)

Theorem 4.3. Let \( S \) and \( S^* \) be \( t \)-conorms in which \( S^* \) dominates \( S \). Let \( f : X \to Y \) be an epimorphism of BCK-algebras. If \( \lambda \) and \( \mu \) are fuzzy ideals of \( Y \) with respect to \( S \), then \( f^{-1}([\lambda \cdot \mu]_{S^*}) = [f^{-1}(\lambda), f^{-1}(\mu)]_{S^*} \).

Proof. For any \( x \in X \), we have
\[
f^{-1}([\lambda \cdot \mu]_{S^*})(x) = [\lambda \cdot \mu]_{S^*}(f(x)) = S^*(\lambda(f(x)), \mu(f(x))) \\
= S^*([f^{-1}(\lambda)](x), [f^{-1}(\mu)](x)) = [f^{-1}(\lambda), f^{-1}(\mu)]_{S^*}(x).
\]
\(\square\)

Theorem 4.4. Let \( S \) be a \( t \)-conorm. Let \( X_1 \) and \( X_2 \) be BCK-algebras and let \( X = X_1 \times X_2 \) be the direct product BCK-algebra of \( X_1 \) and \( X_2 \). Let \( \lambda \) be a fuzzy ideal of a BCK-algebra \( X_1 \) with
10 Sensible fuzzy ideals of BCK-algebras with \( t \)-conorms

respect to \( S \) and let \( \mu \) be a fuzzy ideal of a BCK-algebra \( X_2 \) with respect to \( S \). Then \( \nu = \lambda \times \mu \) is a fuzzy ideal of \( X = X_1 \times X_2 \) with respect to \( S \) defined by

\[
\nu(x_1, x_2) = (\lambda \times \mu)(x_1, x_2) = S(\lambda(x_1), \mu(x_2)).
\]  
(4.4)

**Proof.** For any \((x, y) \in X_1 \times X_2 = X\), we have

\[
\nu(0, 0) = (\lambda \times \mu)(0, 0) = S(\lambda(0), \mu(0))
\leq S(\lambda(x), \mu(y)) = (\lambda \times \mu)(x, y) = \nu(x, y).
\]  
(4.5)

Let \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \in X_1 \times X_2 = X \). Then we have

\[
\nu(x) = (\lambda \times \mu)(x) = (\lambda \times \mu)(x_1, x_2) = S(\lambda(x_1), \mu(x_2))
\leq S(S(\lambda(x_1 \ast y_1), \lambda(y_1)), S(\mu(x_2 \ast y_2), \mu(y_2)))
= S(S(\lambda(x_1 \ast y_1), \mu(x_2 \ast y_2)), S(\lambda(y_1), \mu(y_2)))
= S(\lambda \times \mu)(x_1 \ast y_1, x_2 \ast y_2), (\lambda \times \mu)(y_1, y_2)
= S(\lambda \times \mu)((x_1, x_2) \ast (y_1, y_2)), (\lambda \times \mu)(y_1, y_2)
= S(\lambda \times \mu)(x \ast y), (\lambda \times \mu)(y)) = S(\nu(x \ast y), \nu(y)).
\]  
(4.6)

Hence \( \nu \) is a fuzzy ideal of \( X \) with respect to \( S \). \( \square \)

The relationship between fuzzy ideals \( \mu_1 \times \mu_2 \) and \([\mu_1 \cdot \mu_2]_S \) with respect to \( S \) can be viewed via the following diagram:

\[\begin{array}{ccc}
X & \rightarrow & X \times X \\
\downarrow S & & \downarrow \times \nu \\
I & \leftarrow & I \times I
\end{array}\]  
(4.7)

where \( I = [0, 1] \) and \( d: X \rightarrow X \times X \) is defined by \( d(x) = (x, x) \). It is easy to see that \([\mu_1 \cdot \mu_2]_S \) is the preimage of \( \mu_1 \times \mu_2 \) under \( d \).

Converse of Theorem 4.4 may not be true as seen in the following example.

**Example 4.5.** Let \( X \) be a BCK-algebra and let \( s, t \in [0, 1] \). Define fuzzy sets \( \mu_1 \) and \( \mu_2 \) in \( X \) by \( \mu_1(x) = 1 \) and

\[
\mu_2(x) = \begin{cases} 
1 & \text{if } x = 0, \\
\ \ \ \ \ t & \text{otherwise}
\end{cases}
\]  
(4.8)

for all \( x \in X \), respectively.

If \( x = 0 \), then \( \mu_2(x) = 1 \), and thus

\[
(\mu_1 \times \mu_2)(x, x) = S(\mu_1(x), \mu_2(x)) = S(1, 1) = 1.
\]  
(4.9)
If $x \neq 0$, then $\mu_2(x) = t$, and thus

$$(\mu_1 \times \mu_2)(x, x) = S(\mu_1(x), \mu_2(x)) = S(1, t) = 1.$$  \hspace{1cm} (4.10)

That is, $\mu_1 \times \mu_2$ is a constant function and so $\mu_1 \times \mu_2$ is a fuzzy ideal of $X_1 \times X_2$. Now $\mu_1$ is a fuzzy ideal of $X$, but $\mu_2$ is not a fuzzy ideal of $X$ since for $x \neq 0$, we have $\mu_2(0) = 1 > t = \mu_2(x)$.

Now we generalize the product of two fuzzy ideals with respect to $S$ to the product of $n$ fuzzy ideals with respect to $S$. We first need to generalize the domain of $t$-conorm $S$ to $\prod_{i=1}^{n}[0,1]$ as follows.

**Definition 4.6.** The function $S_n : \prod_{i=1}^{n}[0,1] \to [0,1]$ is defined by

$$S_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = S(\alpha_i, S_{n-1}(\alpha_1, \alpha_2, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_n))$$  \hspace{1cm} (4.11)

for all $1 \leq i \leq n$, $n \geq 2$, $S_2 = S$, and $S_1 = \text{identity}.$

**Lemma 4.7.** For a $t$-conorm $S$ and every $\alpha_i, \beta_i \in [0,1]$, where $1 \leq i \leq n$, $n \geq 2$,

$$S_n(S(\alpha_1, \beta_1), S(\alpha_2, \beta_2), \ldots, S(\alpha_n, \beta_n)) = S(S_n(\alpha_1, \alpha_2, \ldots, \alpha_n), S_n(\beta_1, \beta_2, \ldots, \beta_n)).$$  \hspace{1cm} (4.12)

**Theorem 4.8.** Let $S$ be a $t$-conorm and let $X = \prod_{i=0}^{n}X_i$ be the direct product of $BCK$-algebras. If $\mu_i$ is a fuzzy ideal of $X_i$ with respect to $S$, where $1 \leq i \leq n$, then $\mu = \prod_{i=1}^{n}\mu_i$ defined by

$$\mu(x) = \left(\prod_{i=1}^{n}\mu_i\right)(x_1, x_2, \ldots, x_n) = S_n(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n))$$  \hspace{1cm} (4.13)

for all $x = (x_1, x_2, \ldots, x_n) \in X$ is a fuzzy ideal of $X$ with respect to $S$.

**Proof.** Clearly, $\mu(0) \leq \mu(x)$ for all $x = (x_1, x_2, \ldots, x_n) \in X = \prod_{i=1}^{n}X_i$.

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be the elements of $X = \prod_{i=1}^{n}X_i$. Then

$$\mu(x) = \left(\prod_{i=1}^{n}\mu_i\right)(x_1, x_2, \ldots, x_n) = S_n(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n))$$

$$\leq S_n(S(\mu_1(x_1 \ast y_1), \mu(y_1)), S(\mu_2(x_2 \ast y_2), \mu(y_2)), \ldots, S(\mu_n(x_n \ast y_n), \mu(y_n)))$$

$$= S(S_n(\mu_1(x_1 \ast y_1), \mu_2(x_2 \ast y_2), \ldots, \mu_n(x_n \ast y_n)), S_n(\mu(y_1), \mu(y_2), \ldots, \mu(y_n)))$$

$$= S(S\left(\prod_{i=1}^{n}\mu_i\right)(x_1 \ast y_1, x_2 \ast y_2, \ldots, x_n \ast y_n), \left(\prod_{i=1}^{n}\mu_i\right)(y_1, y_2, \ldots, y_n))$$

$$= S(\mu(x \ast y), \mu(y)).$$  \hspace{1cm} (4.14)

Hence $\mu = \prod_{i=1}^{n}\mu_i$ is a fuzzy ideals of $X$ with respect to $S$. \hfill \blacksquare
Sensible fuzzy ideals of BCK-algebras with \( t \)-conorms

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References


M. Akram: Punjab University College of Information Technology, University of The Punjab, Allama Iqbal Campus (Old Campus), P. O. Box 54000, Lahore, Pakistan

E-mail address: m.akram@pucit.edu.pk

Jianming Zhan: Department of Mathematics, Hubei Institute for Nationalities, Enshi, Hubei Province 445000, China

E-mail address: zhanjianming@hotmail.com