ON THE CONVERGENCE OF A NEWTON-LIKE METHOD IN $\mathbb{R}^n$ AND THE USE OF BERINDE'S EXIT CRITERION

RABINDRANATH SEN, SULEKHA MUKHERJEE, AND RAJESH PATRA

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Berinde has shown that Newton's method for a scalar equation $f(x) = 0$ converges under some conditions involving only f and f' and not f'' when a generalized stopping inequality is valid. Later Sen et al. have extended Berinde's theorem to the case where the condition that $f'(x) \neq 0$ need not necessarily be true. In this paper we have extended Berinde's theorem to the class of n -dimensional equations, $F(x) = 0$, where $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\mathbb{R}^n$ denotes the n -dimensional Euclidean space. We have also assumed that $F'(x)$ has an inverse not necessarily at every point in the domain of definition of F .

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1. Introduction

Let F be a nonlinear continuous operator mapping $D_0 \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$. D_0 is an open convex subset of $\mathbb{R}^n$, the n -dimensional Euclidean space. We introduce componentwise partial ordering in $\mathbb{R}^n$.

Componentwise partial ordering in $\mathbb{R}^n$ is defined as follows. For $x, y \in \mathbb{R}^n$, $x \leq y$ if and only if $x_i \leq y_i$, $i = 1, 2, \dots, n$. Let $\langle a, b \rangle$ denote the order interval $\{x \in \mathbb{R}^n \mid a \leq x \leq b\}$.

We seek the solution of

$$F(x) = 0 \quad \text{in } \langle a, b \rangle \subset D_0. \quad (1.1)$$

In case the finite derivative $F'(x)$ has an inverse at the iteration points, Newton's method is given by the iteration

$$x_{m+1} = x_m - [F'(x_m)]^{-1}F(x_m), \quad x_0 \in \langle a, b \rangle, \quad m \geq 0. \quad (1.2)$$

The importance of Newton's method lies in the fact that it offers a quadratic convergence. Nevertheless this quadratic convergence is achieved after making certain assumptions about F , F' , F'' (Ortega and Rheinboldt [6]). In case $F(x) = 0$ is a scalar equation, Berinde (Berinde [1–4]) without making any assumption about the existence of $F''(x)$

2 On convergence of a Newton-like method

has achieved linear convergence under the condition that $F'(x) \neq 0$ in the interval under consideration. In the case of a scalar equation we have not assumed the conditions that $F'(x) \neq 0$ (Sen et al. [9]). It has been shown that the sequence $\{x_m\}$ given by

$$x_{m+1} = x_m - \frac{2F(x_m)}{F'(x_m) + M'_m}, \quad x_0 \text{ prechosen}, m = 0, 1, 2, \dots, \quad (1.3)$$

is convergent. In the above expression, $M = \sup_{x \in [a,b]} |F'(x)|$, $M'_m = M \cdot \text{Sign} F'(x_m)$.

In case $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ we proceed as follows. Let F be differentiable in $D_0 \subseteq \mathbb{R}^n$ and let $F'(x) = (a_{ij}(x))$.

Let

$$A = (C_{ij}), \quad \text{where } 0 \geq C_{ij} = -\sup_x |a_{ij}(x)|, \quad i \neq j, \quad C_{ii} \geq \sup_x \sum_{j=1}^n |a_{ij}(x)|. \quad (1.4)$$

We can show that under certain conditions $A + F'(x)$ has an inverse even if $F'(x)$ may not have an inverse. The Newton-like iterative sequence $\{x_m\}$ is given by

$$x_{m+1} = x_m - 2[A + F'(x_m)]^{-1} F(x_m), \quad x_0 \in \langle a, b \rangle, \quad m = 0, 1, 2, \dots \quad (1.5)$$

We show that under certain assumptions $\{x_m\}$ given by (1.2) converges to a solution of $F(x) = 0$. The exit criterion has been established. Section 2 presents the mathematical preliminaries. Section 3 contains the convergence theorem, Section 4 an extension of Newton's method, Section 5 contains a numerical example, and Section 6 the discussion.

2. Preliminaries

By componentwise partial ordering in $\mathbb{R}^n$, we mean for $x, y \in \mathbb{R}^n$,

$$\begin{aligned} x < y &\iff x_i < y_i, \quad \forall i \\ x \leq y &\iff x_i \leq y_i, \quad \forall i, \text{ but } x \neq y, \\ x \leq y &\iff x_i \leq y_i. \end{aligned} \quad (2.1)$$

Let $L(\mathbb{R}^n, \mathbb{R}^n)$ be the class of matrices which maps $\mathbb{R}^n \rightarrow \mathbb{R}^n$.

Let $P \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $P = (\tilde{a}_{ij})$.

$$\begin{aligned} P > 0 &\iff \tilde{a}_{ij} > 0, \quad \forall i, j, \\ P \geq 0 &\iff \tilde{a}_{ij} \geq 0, \quad \forall i, j, \\ P \geq 0 &\iff \tilde{a}_{ij} \geq 0, \quad \forall i, j, \text{ but } P \neq 0. \end{aligned} \quad (2.2)$$

Definition 2.1 (*M-matrix, a Stieltjes matrix*). A matrix $Q = (q_{ij}) \in L(\mathbb{R}^n, \mathbb{R}^n)$ is said to be an *M-matrix* if $q_{ij} \leq 0$, for all i, j , $i \neq j$. Q^{-1} exists and $Q^{-1} \geq 0$ (Ortega and Rheinboldt [6]). A symmetric *M-matrix* is called a *Stieltjes matrix*.

Definition 2.2 (*spectral radius*). A spectral radius of a matrix H is denoted by $\rho(H)$ and $\rho(H) = \sup_i |\lambda_i|$, where $\lambda_1, \dots, \lambda_n$ are eigenvalues of H .

THEOREM 2.3 (Ortega and Rheinboldt [6]). $A = (C_{ij})$ is an M -matrix if and only if (i) the diagonal elements of A are positive, and (ii) the matrix $H = I - D^{-1}A$ where $D = \text{diag}(a_{11}, \dots, a_{nn})$ satisfies $\rho(H) < 1$.

LEMMA 2.4. *The following results hold:*

- (i) $A = (C_{ij})$ is an M -matrix,
- (ii) $[A + F'(x)]$ is an M -matrix, provided the condition

$$\sup_x \left(\sum_{k=1}^n |a_{ik}(x)| \right) + a_{ii}(x) > \sum_{\substack{j=1 \\ j \neq i}}^n \left(\sup_x |a_{ij}(x)| - a_{ij}(x) \right) \quad (2.3)$$

holds.

Proof. (i) Let D denote the diagonal matrix (C_{ii}) . Then $D > 0$. Moreover $I - D^{-1}A$ is a matrix with zero diagonal elements and (i, j) th element $= -C_{ij}/C_{ii}$, $i \neq j$.

Therefore $\|I - D^{-1}A\|_{l_1} = \sup_i (\sum_{j, i \neq j} |C_{ij}|/C_{ii}) < 1$, where $\|\cdot\|_{l_1}$ denotes l_1 -norm.

Thus $\rho(I - D^{-1}A)$, the spectral radius (Ortega and Rheinboldt [6]) of $(I - D^{-1}A)$, is less than 1. Hence A is an M -matrix.

(ii) We write $F'(x) = D_1(x) - B_1(x)$ where the diagonal matrix $D_1(x) = (a_{ii}(x))$ and the matrix $B_1(x) = (b_{ij}(x))$, where $b_{ii}(x) = 0$ and $b_{ij}(x) = -a_{ij}(x)$, for $i \neq j$.

Let $\tilde{A} = A + F'(x) = D_2(x) - B_2(x)$, where the diagonal matrix is given by $D_2(x) = (C_{ii} + a_{ii}(x))$, $C_{ii} + a_{ii}(x) > 0$, and the matrix $B_2(x) = (b'_{ij}(x))$, where $b'_{ij}(x) = (-C_{ij} - a_{ij}(x))$, $b'_{ii} = 0$. Moreover, $b'_{ij}(x) = \sup_x |a_{ij}(x)| - a_{ij}(x) \geq 0$.

Let $I - D_2(x)^{-1}\tilde{A} = (e_{ij}(x))$, where $e_{ii}(x) = 0$ and

$$e_{ij}(x) = \frac{-C_{ij} - a_{ij}(x)}{C_{ii} + a_{ii}(x)} \leq \frac{\sup_x |a_{ij}(x)| - a_{ij}(x)}{\sup_x \sum_{k=1}^n |a_{ik}(x)| + a_{ii}(x)}, \quad i \neq j. \quad (2.4)$$

Therefore,

$$\|I - D_2(x)^{-1}\tilde{A}\|_{l_1} = \sup_i \sum_{\substack{j=1 \\ j \neq i}}^n |e_{ij}(x)| = \sup_i \frac{\sum_{j \neq i} |C_{ij} + a_{ij}(x)|}{|C_{ii} + a_{ii}(x)|} \quad (2.5)$$

$$\leq \sup_i \frac{\sum_{j \neq i} (\sup_x |a_{ij}(x)| - a_{ij}(x))}{\sup_x \sum_{k=1}^n |a_{ik}(x)| + a_{ii}(x)} < 1$$

provided that

$$\sup_x \sum_{k=1}^n |a_{ik}(x)| + a_{ii}(x) > \sum_{\substack{j=1 \\ j \neq i}}^n (\sup_x |a_{ij}(x)| - a_{ij}(x)), \quad \forall i. \quad (2.6)$$

Thus $\rho(I - D_2(x)^{-1}\tilde{A}) < 1$ provided the relation (2.6) holds for all i, j . □

3. Convergence

THEOREM 3.1. *Let the following conditions be fulfilled.*

- (i) $F : D_0 \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$.
- (ii) D_0 is an open convex subset $\subseteq \mathbb{R}^n$.
- (iii) F is continuously differentiable in D_0 .
- (iv) F has a solution in the order-interval $\langle a, b \rangle \subseteq D_0$.
- (v) $x_0 \in \langle a, b \rangle$ is an initial approximation to the solution.
- (vi) $A = (C_{ij})$ is a matrix defined by (1.4).
- (vii) $[A + F'(x)]$ is a Stieltjes matrix.
- (viii) $F'(x)$ is symmetric for each x and $F'(x) \geq 0$.
- (ix) The eigenspaces of $[A + F'(x)]^{-1}$ and of $F'(x)$, respectively, have nonempty intersection.
- (x) $\rho(F'(x)) < \rho([A + F'(x)]^{-1})^{-1}$, where ρ stands for the spectral radius.
- (xi) $\rho(C(x)) < 1$ and $\rho(C(x))(1 - \rho(C(x))) < 1/2$, where

$$C(x) = [A + F'(x)]^{-1} F'(x + t(x^* - x)), \quad 0 < t < 1. \quad (3.1)$$

Then $\{x_m\}$ given by

$$x_{m+1} = x_m - 2[A + F'(x_m)]^{-1} F(x_m) \quad (3.2)$$

will converge to a solution x^* of $F(x) = 0$, provided that $\{x_m\} \subseteq \langle a, b \rangle$.

Proof. It follows from (3.2) and the application of mean-value theorem in $\mathbb{R}^n$ that

$$\begin{aligned} x_{m+1} - x^* &= x_m - x^* - 2[A + F'(x_m)]^{-1} (F(x_m) - F(x^*)) \\ &= \int_0^1 [I - 2[A + F'(x_m)]^{-1} F'(x_m + t(x^* - x_m))] \cdot (x_m - x^*) dt \end{aligned} \quad (3.3)$$

or

$$\|x_{m+1} - x^*\| \leq \sup_{0 < t < 1} \left\| I - 2[A + F'(x_m)]^{-1} F'(x_m + t(x^* - x_m)) \right\| \|x_m - x^*\|. \quad (3.4)$$

Since $F'(x) \geq 0$, $F'(x_m + t(x^* - x_m)) \geq 0$.

Also, $[A + F'(x)]$ being a Stieltjes matrix is both symmetric and an M -matrix, $[A + F'(x)]^{-1} \geq 0$. Therefore,

$$C(x) = [A + F'(x)]^{-1} F'(x + t(x^* - x)) \geq 0. \quad (3.5)$$

Let $B(x) = [I - 2[A + F'(x)]^{-1} F'(x + t(x^* - x))]$.

Since $C(x) \geq 0$, by the Perron-Frobenius theorem (Varga [12]), for a given x , $C(x)$ has an eigenvalue $\lambda(x) = \rho(C(x))$, with ρ being the spectral radius. Therefore, for a given x ,

$B(x)$ has an eigenvalue $\mu(x) = 1 - 2\lambda'(x)$, where $\lambda'(x)$ is an eigenvalue of $C(x)$. If we use Euclidean norm of a vector and then matrix norm induced by the vector norm,

$$\|B\| = [\rho(B^T B)]^{1/2}, \quad \rho(B^T B) = \sup |\mu^* \mu| = \sup_{\lambda'} |(1 - 2\lambda')^* (1 - 2\lambda')| < 1 \quad (3.6)$$

if $|1 - 4\operatorname{Re}\lambda' + 4|\lambda'|^2| < 1$ for all λ' , or, if $|\lambda'|^2 < \operatorname{Re}\lambda' < 1/2 + |\lambda'|^2$.

Thus $\|B\| < 1$ if

$$\rho(C(x))^2 < \rho(C(x)) < \frac{1}{2} + \rho(C(x))^2. \quad (3.7)$$

The left-hand inequality in (3.7) yields $\rho(C(x)) < 1$, and the right-hand inequality in (3.7) yields $\rho(C(x))(1 - \rho(C(x))) < 1/2$ always true.

Thus by condition (xi) $\|B\| < 1$.

If

$$\|B\| = \alpha < 1, \quad \|x_{m+1} - x^*\| \leq \alpha \|x_m - x^*\|. \quad (3.8)$$

□

Remark 3.2. The sequence $\{x_m\}$ given by (3.2) is convergent if conditions (ii)–(iv) of Theorem 3.1 are satisfied, if $[A + F'(x)]$ is an M -matrix and $\|I - 2[A + F'(x_m)]^{-1}F'(x)\| < 1$.

The following theorem determines the stopping inequality.

THEOREM 3.3. *Let the conditions of Theorem 3.1 be true. Then the numerical computation of the sequence $\{x_m\}$ given by (3.2) is stopped when the following inequality is valid:*

$$\|x_m - x^*\| \leq \|[I + A^{-1}F'(x_m)]\| \|A\| \cdot \left(\sup_{0 < t < 1} \|F'(x_m + t(x^* - x_m))\| \right)^{-1} \|x_{m+1} - x_m\|. \quad (3.9)$$

Proof. Equation (3.2) can be written as

$$\begin{aligned} \|x_{m+1} - x_m\| &= 2 \left\| [A + F'(x_m)]^{-1} [F(x_m) - F(x^*)] \right\| \\ &= \sup_{0 < t < 1} \left\| \int_0^1 [A + F'(x_m)]^{-1} F'(x_m + t(x^* - x_m)) \cdot (x^* - x_m) dt \right\| \\ &\leq \left\| [A + F'(x_m)]^{-1} \right\| \sup_{0 < t < 1} \|F'(x_m + t(x^* - x_m))\| \cdot \|x^* - x_m\|. \end{aligned} \quad (3.10)$$

Since A and $A + F'(x_m)$ are both M -matrices, $[I + A^{-1}F'(x_m)]^{-1}$ exists, and therefore $\|([I + A^{-1}F'(x_m)]^{-1})^{-1}\| \leq \|I + A^{-1}F'(x_m)\|$.

6 On convergence of a Newton-like method

Thus,

$$\begin{aligned} \|x^* - x_m\| &\leq \left\| \left([A + F'(x_m)]^{-1} \right)^{-1} \left\| \left(\sup_{0 < t < 1} \|F'(x_m + t(x^* - x_m))\| \right) \right\|^{-1} \|x^* - x_m\| \right. \\ &\leq \|I + A^{-1}F'(x_m)\| \|A\| \left\| \left(\sup_{0 < t < 1} \|F'(x_m + t(x^* - x_m))\| \right) \right\|^{-1} \cdot \|x_{m+1} - x_m\|. \end{aligned} \quad (3.11)$$

□

Note 3.4. The inequality (3.11) may be termed as “the exit criterion” because if $\|x_{m+1} - x_m\| < \varepsilon$, ε a small positive quantity, then $\|x^* - x_m\| \leq C_m \varepsilon$, where

$$C_m = \|[I + A^{-1}F'(x_m)]\| \|A\| \left\| \left(\sup_{0 < t < 1} \|F'(x_m + t(x^* - x_m))\| \right) \right\|^{-1}. \quad (3.12)$$

4. The extension of Newton’s method

However $\langle a, b \rangle$ is generally not an invariant set with respect to iterations (3.2); that is, it is possible to obtain a certain p such that $x_p \notin \langle a, b \rangle$. In case $x_p < a$ or $x_p > b$, the mapping $F(x)$ is extended throughout $\mathbb{R}^n$ in the light of Berinde’s extension, and the sequence $\{x_m\}$ given by (3.2) is extended throughout $\mathbb{R}^n$.

THEOREM 4.1. *Let the following conditions be fulfilled:*

- (i) $F(a) \leq 0$,
- (ii) $F(x)$ is differentiable at a and A is an M -matrix (Ortega and Rheinboldt [6]),
- (iii) $F(b) \geq 0$,
- (iv) $F(x)$ is differentiable at b .

Then if x_p goes out of $\langle a, b \rangle$, x_{p+1} will lie in $\langle a, b \rangle$.

Extend $F(x)$ throughout $\mathbb{R}^n$ as follows:

$$\tilde{F}(x) = \begin{cases} A(x - a) + F(a) & x \leq a, \\ F(x) & x \in \langle a, b \rangle, \\ A(x - b) + F(b) & x \geq b. \end{cases} \quad (4.1)$$

Proof. If some iteration x_p does not lie in $\langle a, b \rangle$ we have either $x_p < a$ or $x_p > b$. In the first case applying (3.2) after extension to $\tilde{F}(x)$ we get

$$\begin{aligned} x_{p+1} &= x_p - 2[A + \tilde{F}'(x_p)]^{-1} \tilde{F}(x_p) = x_p - 2[2A]^{-1}[A(x_p - a) + F(a)] \\ &= x_p - (x_p - a) - A^{-1}F(a) = a - A^{-1}F(a) > a, \end{aligned} \quad (4.2)$$

since A is an M -matrix and $F(a) \leq 0$. Therefore, $x_{p+1} \in \langle a, b \rangle$.

If $x_p > b$, repeating the same steps as above we get

$$x_{p+1} = x_p - 2[2A]^{-1}[A(x_p - b) + F(b)] = x_p - (x_p - b) - A^{-1}F(b) = b - A^{-1}F(b). \quad (4.3)$$

Since A is an M -matrix and $F(b) \geq 0$, $A^{-1}F(b) \geq 0$. Hence $x_{p+1} \in \langle a, b \rangle$.

Thus beginning from a step $p_0 \geq 0$, we necessarily have $x_m \in [a, b]$. If Theorems 3.1 and 4.1 are valid, $x_m \in \langle a, b \rangle$ for $m \geq p_0$, and the convergence of $\{x_m\}$ to a solution x^* in $\langle a, b \rangle$ is guaranteed. Furthermore, the error estimate (3.8) and the exit criterion or the stopping inequality (3.9) are both valid. $\square$

5. Numerical example

Let $z = [x, y]^T$, $D_0 = \langle -(\pi/2), \pi \rangle \times \langle 0, 1 \rangle$.

$$F(z) = \begin{cases} f_1(x, y) \\ f_2(x, y) \end{cases} = \begin{cases} \left(x - \frac{\pi}{2}\right)^3 + \left(\left(x - \frac{\pi}{2}\right) \sin\left(x - \frac{\pi}{2}\right)\right)y - 0.752 \\ \pi^2 y + \pi^2 y^3 - \left(x - \frac{\pi}{2}\right) \cos\left(x - \frac{\pi}{2}\right) + \sin\left(x - \frac{\pi}{2}\right) - \frac{5\pi^2}{8} - 0.152. \end{cases} \quad (5.1)$$

We are interested in solving $F(z) = 0$ for $z \in D_0$.

Initial approximation $z_0 = (x_0, y_0)^T$.

$$\begin{aligned} \frac{\partial f_1}{\partial x} &= 3\left(x - \frac{\pi}{2}\right)^2 + \left(\sin\left(x - \frac{\pi}{2}\right)\right)y + \left(\left(x - \frac{\pi}{2}\right) \cos\left(x - \frac{\pi}{2}\right)\right)y, \\ \frac{\partial f_1}{\partial y} &= \left(x - \frac{\pi}{2}\right) \sin\left(x - \frac{\pi}{2}\right), \\ \frac{\partial f_2}{\partial x} &= \left(x - \frac{\pi}{2}\right) \sin\left(x - \frac{\pi}{2}\right), \\ \frac{\partial f_2}{\partial y} &= \pi^2 + 3\pi^2 y^2. \end{aligned} \quad (5.2)$$

$F'(z)$ is symmetric.

$F'(z) \geq 0$ for all $x \in \langle -\pi, \pi \rangle$ and $y \in \langle 0, 1 \rangle$ except at $x = -\pi, y = \pi$.

$$\begin{aligned} F(a) = F(x)|_{\substack{x=-\pi/2 \\ y=0}} &= \begin{cases} -\pi^3 - 0.752 \\ -\frac{5\pi^2}{8} - \pi - 0.152 \end{cases} \leq 0, \\ F(b) = F(x)|_{\substack{x=\pi \\ y=1}} &= \begin{cases} \frac{\pi^3}{8} + \frac{\pi}{2} - 0.752 \\ \frac{3\pi^2}{8} + 1 - 0.152 \end{cases} \geq 0. \end{aligned} \quad (5.3)$$

$F'(z)|_{\substack{x=\pi/2 \\ y=0}}$ does not have an inverse.

We choose A as

$$A = \begin{pmatrix} 2 & -1.5708 \\ -1.5708 & 22 \end{pmatrix} \quad (5.4)$$

and $\epsilon = 10^{-11}$, the desired accuracy is achieved in 17 iterations.

Starting from $x_0 = 1.5708$ and $y_0 = 0$, we obtain $x_1 = 2.67715530056448$ and $y_1 = 0.45117812654687$; and $x_{17} = 2.35205300236830$ and $y_{17} = 0.50014720328245$.

6. Discussion

(i) Convergence of Newton's method as proposed by Kantorovich (see [11]) is based on majorization principle which ensures that all the members of the sequence $\{x_m\}$ will lie in a small neighborhood of the initial approximation x_0 . Hence majorization principle has not been used. But in order to ensure that $\{x_m\}$ does not go beyond $\langle a, b \rangle$, barring a finite number of members, an extended formula of the mapping F is taken.

(ii) Here the condition that $F'(x) \neq 0$ has been relaxed and the extended method is called Newton-like method.

(iii) The convergence is linear.

(iv) The numerical equation under consideration being nonlinear has more than one solution, the x -component of one solution being $x^* = 1.087961617$. In our case, the initial point taken is close to a point where the Jacobian becomes singular and the purpose is to show that the sequence of iterations (1.5) with the initial point mentioned above still converges to a solution of the given equation.

(v) For other modifications of Newton's method please see (Ortega and Rheinboldt [6], Keller [5], Sen [7, 8], Sen and Guhathakurta [10]).

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Rabindranath Sen: Department of Applied Mathematics, University College of Science,
92 A.P.C. Road, Calcutta 700009, India
E-mail address: rabindra33@rediffmail.com

Sulekha Mukherjee: Department of Mathematics, University of Kalyani, Kalyani 741 235,
West Bengal, India
E-mail address: sulekhams33@rediffmail.com

Rajesh Patra: Department of Mathematics, University of Kalyani, Kalyani 741 235,
West Bengal, India
E-mail address: rajeshpatra33@rediffmail.com