We give here a geometric proof of the existence of certain local coordinates on a pseudo-Riemannian manifold admitting a closed conformal vector field.

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1. Introduction

A vector field $V$ on a pseudo-Riemannian manifold $(M, g)$ is called conformal if

$$\mathcal{L}_V g = 2\lambda g$$

(1.1)

for a scalar field $\lambda$, where $\mathcal{L}$ denotes the Lie derivative on $M$. It is easy to see that if $V$ is locally a gradient field, then (1.1) is equivalent to

$$\nabla_X V = \lambda X$$

(1.2)

for every vector field $X$. Here $\nabla$ denotes the Levi-Civita connection of $g$. We call vector fields satisfying (1.2) closed conformal vector fields. They appear in the work of Fialkow [3] about conformal geodesics, in the works of Yano [7–11] about concircular geometry in Riemannian manifolds, and in the works of Tashiro [6], Kerbrat [4], Kühnel and Rademacher [5], and many other authors.

If $V$ is lightlike on $(M, g)$, then from (1.2), we get

$$X g(V, V) = 2 g(\nabla_X V, V) = 2 \lambda g(X, V) = 0$$

(1.3)

for every vector field $X$. Thus $\lambda \equiv 0$ and $V$ is parallel. About lightlike parallel vector fields, we have the following theorem.
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Theorem 1.1 (Brinkmann [2]). If \((M,g)\) admits a lightlike parallel vector field \(V\), then there are local coordinates \(u^1,u^2,...,u^n\) \((n := \dim M > 2)\) such that \(V = \partial/\partial u^1\) and

\[
(g_{ij}) = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & & & \\
\vdots & \vdots & & & \alpha_{\beta} \\
0 & 0 & & & 
\end{pmatrix},
\]

where \(\alpha,\beta \in \{3,...,n\}\) and \(\partial g_{\alpha\beta}/\partial u^1 = 0\).

Brinkmann’s proof is purely analytical. We will give, in the next section, geometric tools which will allow us to generalize Brinkmann’s theorem.

2. Geometric constructions

Let \((M,g)\) be a connected pseudo-Riemannian manifold of dimension \(n\) and signature \((k,n-k)\) with \(0 < k < n\). Given a vector field \(W\) on \(M\), we denote by \(W^\flat\) the one-form defined by \(W^\flat(X) = g(W,X)\). Then \(W\) is locally a gradient field if and only if \(dW^\flat = 0\). In the following, a vector field \(W\) satisfying \(\nabla_W W = 0\) will be called geodesic.

Lemma 2.1. If \(W\) is a geodesic vector field, then \(dW^\flat\) is invariant under the flow of \(W\).

Proof. Let \((\nabla \nabla^e)(X,Y) = (\nabla_X \nabla^e)(Y) = g(\nabla_X W, Y)\). Then, from the fact that \(W\) is geodesic, it follows that

\[
(\mathcal{L}_W \nabla W^\flat)(X,Y) = Wg(\nabla_X W, Y) - g(\nabla_{[W,X]} W, Y) - g(\nabla_X W, [W,Y])
\]

\[
= g(R(W,X)W,Y) + g(\nabla_X W, \nabla_Y W),
\]

where \(R\) denotes the Riemannian curvature tensor,

\[
R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\]

Since \(g(R(W,X)W,Y)\) is symmetric with respect to \(X, Y\), from

\[
dW^\flat(X,Y) = (\nabla W^\flat)(X,Y) - (\nabla W^\flat)(Y,X),
\]

we get \((\mathcal{L}_W dW^\flat)(X,Y) = (\mathcal{L}_W \nabla W^\flat)(X,Y) - (\mathcal{L}_W \nabla W^\flat)(Y,X) = 0\).

Lemma 2.2. If \(W\) is a lightlike geodesic vector field, then \(dW^\flat(X,W) = 0\).
Proof. We have the following.

\[ W \text{ lightlike } \Rightarrow (\nabla W^\perp)(X, W) = g(\nabla_X W, W) = 0 \] \( \Rightarrow \) \( dW^\perp(X, W) = 0. \)

\[ W \text{ geodesic } \Rightarrow (\nabla W^\perp)(W, X) = g(\nabla_W W, X) = 0 \] \( \Rightarrow \) \( dW^\perp(X, W) = 0. \)

A nontangent vector field \( \tilde{W} \) on a pseudo-Riemannian hypersurface \( \tilde{M} \) can be extended to a geodesic vector field \( W \) in a neighbourhood of \( \tilde{M} \) in the following way. Let \( c(s, p) \) be the geodesic starting at \( p = c(0, p) \in \tilde{M} \) with \( c(0, p) = \tilde{W}(p) \) and \( W(c(s, p)) = c(s, p) \). Then, taking into account the fact that \( \tilde{W} \) is transversal (i.e. non-tangent) to \( \tilde{M} \), we conclude that \( W \) is a geodesic vector field on a neighbourhood of \( \tilde{M} \) extending \( \tilde{W} \). Moreover, if \( \tilde{W} \) is lightlike, then so is \( W \). Denoting with \( \tilde{W}^+ \), \( \tilde{W}^\perp \) the tangent and normal component of \( \tilde{W} \), for vector fields \( X, Y \) on \( \tilde{M} \) tangent to \( \tilde{M} \), we have the following lemma.

**Lemma 2.3.** \( dW^\perp(X, Y) = d(\tilde{W}^+)^h(X, Y). \)

**Proof.** The statement follows from \( g(\nabla_X \tilde{W}^+, Y) − g(\nabla_Y \tilde{W}^+, X) = −g(\tilde{W}^+, [X, Y]) = 0. \)

The following remark will be used in the proof of the next proposition.

**Remark 2.4.** Let \( V \) be a vector field and let \( \varphi \) be a function on \( M \). At a point \( p_0 \in M \), the gradient of the solutions of \( Vf = \varphi \) span an affine hyperplane \( H \) of \( T_{p_0}M \). Let \( v := V(p_0) \), then \( H = \{ x \in T_{p_0}M \mid g(x, v) = \varphi(p_0) \} \) and

(a) if \( \varphi(p_0) \neq 0 \), then \( H \) contains lightlike, spacelike, and timelike vectors,

(b) if \( \varphi(p_0) = 0 \), then \( H \) contains only lightlike vectors and the zero vector if and only if \( n = 2 \) and \( v \) is lightlike.

**Proposition 2.5.** If \( V \) is a closed conformal vector field on \( (M, g) \), then in a neighbourhood of a point \( p_0 \) where \( V(p_0) \neq 0 \), there is a lightlike geodesic gradient field \( W \) such that \( g(V, W) = 1. \)

**Proof.** We divide the proof into two cases.

**Case 1.** \( n > 2 \) or \( n = 2 \) and \( V(p_0) \) is non-lightlike.

Let \( u \) be a solution of \( Vu = 0 \) with \( g(p_0)(\nabla u, \nabla u) \neq 0 \) (here \( \nabla u \) denotes the gradient of \( u \)). According to **Remark 2.4(b)**, such a solution exists. Let \( \mathcal{U} \) be an open neighbourhood of \( p_0 \) on which \( g(\nabla u, \nabla u) \neq 0 \), and let \( \tilde{M} \) be the pseudo-Riemannian hypersurface \( u^{-1}(u(p_0)) \cap \mathcal{U} \). Then \( \nabla u \) is a normal vector field on \( \tilde{M} \) and, from \( Vu = 0 \), we have that \( \tilde{V} := V|_{\tilde{M}} \) is a tangent vector field on \( \tilde{M} \). Let \( \tilde{f} : \tilde{M} \to \mathbb{R} \) be a solution of \( \tilde{V} \tilde{f} = 1 \) such that \( g(p_0)(\nabla f, \nabla f) \) and \( g(p_0)(\nabla u, \nabla u) \) have opposite sign (see **Remark 2.4(a)**). Without loss of generality, we assume that \( g(\nabla f, \nabla f) < 0 \) on \( \tilde{M} \). Setting \( \tilde{W} := \nabla f + h\nabla u \), where 

\[ h^2 := −g(\nabla f, \nabla f)/g(\nabla u, \nabla u) > 0, \quad g(\tilde{W}, \tilde{W}) = \tilde{W} \tilde{f} = 1. \]

Let now \( \tilde{W} \) be the geodesic vector field extending \( \tilde{W} \) in a neighbourhood of \( \tilde{M} \). Then \( \tilde{W} \) is lightlike. From \( Wg(V, W) = g(\nabla_W V, W) + g(V, \nabla_W W) = 0 \) and \( g(\tilde{W}, \tilde{W}) = 1 \), we conclude that \( g(V, W) = 1. \) It remains to show that \( W \) is locally a gradient.
For vector fields $X, Y$ on $\tilde{M}$ (not necessarily tangent to $\tilde{M}$), we can write

$$X = X^\top + \alpha \tilde{W}, \quad Y = Y^\top + \beta \tilde{W},$$

(2.5)

where $\alpha$ and $\beta$ are certain functions on $\tilde{M}$ and $X^\top, Y^\top$ are tangent to $\tilde{M}$. Using Lemma 2.2, we get

$$0 = dW^b(X, W) = dW^b(X^\top + \alpha W, W) = dW^b(X^\top, W).$$

(2.6)

In the same way, we get $dW^b(W, Y^\top) = 0$, and therefore $dW^b(X, Y) = dW^b(X^\top, Y^\top)$. Now Lemma 2.3 and $\tilde{W}^\top = \nabla \tilde{f}$ imply that $dW^b(X, Y) = 0$ on $\tilde{M}$. Using Lemma 2.1, we conclude that $dW^b = 0$.

Case 2. $n = 2$ and $V(p_0)$ is lightlike.

According to Remark 2.4(b), we cannot proceed as in Case 1 since the gradient at $p_0$ of a solution of $Vu = 0$ is a lightlike vector. Remarking that along an integral curve $\alpha$ of $V$ through $p_0$ $V$ is lightlike, we set $\tilde{M} := \text{Im} \alpha$. Let now $\tilde{W}$ be a lightlike vector field along $\alpha$ such that $V$ and $\tilde{W}$ are linearly independent. Then, since $g$ is nondegenerate, $g(V, V)g(\tilde{W}, \tilde{W}) - g(V, \tilde{W})^2 = -g(V, \tilde{W})^2 \neq 0$. Therefore we can assume that $g(V, \tilde{W}) = 1$. Since $\tilde{W}$ is not tangent to $\alpha$, we can extend it to a geodesic vector field $W$ on a neighbourhood $\mathcal{U}$ of $p_0$. Then $Wg(W, W) = 0$ which, together with $\tilde{W}$ lightlike, implies $W$ lightlike, and $Wg(V, W) = g(\nabla_V V, W) = 0$ which, together with $g(V, \tilde{W}) = 1$, implies $g(V, W) = 1$. Since every vector field on $\mathcal{U}$ can be written as a linear combination of $V$ and $W$, we have $g(\nabla_V W, Y) - g(\nabla_Y W, X) = 0$ for every vector field $X, Y$ on $\mathcal{U}$ if and only if $g(\nabla_Y W, V) - g(\nabla_Y W, V) = 0$.

Thus $W$ being lightlike and geodesic implies that $W$ is a gradient vector field.

It remains to show that $V$ is lightlike along an integral curve $\alpha$ through $p_0 := \alpha(0)$. This follows from $(d/dt)g(V, V) = 2g(\nabla_V V, V) = 2\lambda g(V, V)$, since its general solution is $g(\alpha(t))(V, V) = g(p_0(V, V)e^{2\int_0^t \lambda(u) du}$.

For example, let $M = \mathbb{R}^n_k$ be the pseudo-Euclidean space of dimension $n$ and signature $(k, n - k)$ with $0 < k < n$, that is, $(x, x) = -(x_1^2 + \cdots + x_k^2) + (x_{k+1}^2 + \cdots + x_n^2)$. The position vector field $V(x) = \sum_{i=1}^n x_i(\partial/\partial x_i)$ satisfies $\nabla_x V = X$, and therefore it is a closed conformal vector field. We will construct, following the proof of Proposition 2.5, a lightlike geodesic gradient field $W$ with $\langle V, W \rangle = 1$ in a neighbourhood of a point $x_0 \neq 0$ ($V(x) = 0$ if and only if $x = 0$). We take for simplicity $x_0 = (1, 0, \ldots, 0)$, then $u(x_1, \ldots, x_n) := x_n/x_1$ is a solution of $Vu = 0$ with $\langle \nabla u, \nabla u \rangle |_{x_0} = 1$. The hypersurface $\tilde{M} := u^{-1}(u(x_0)) = u^{-1}(0)$ is the hyperplane $x_n = 0$. Let $\tilde{V} := V|_{\tilde{M}}$, then $\tilde{f}(x_1, \ldots, x_{n-1}) := \ln x_1$ is a solution of $\tilde{V} \tilde{f} = 1$ with $\langle \nabla \tilde{f}, \nabla \tilde{f} \rangle |_{x_0} = -1$. Defining for every $x \in \tilde{M}$ that

$$\tilde{W}(x) := \nabla \tilde{f}(x) + \nabla u(x) = \frac{1}{x_1} \left(- \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_n} \right),$$

(2.7)

it is easy to see that

$$W(x) := \frac{1}{x_1 + x_n} \left(- \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_n} \right)$$

(2.8)
is a geodesic vector field on $M$ extending $\tilde{W}$. Moreover $W$ is lightlike, $\langle V, W \rangle = 1$, and $W = \nabla \ln |x_1 + x_n|$. It is clear that $W$ is not unique and not everywhere defined. More generally, for an arbitrary point $x_0 \neq 0$, we have, for instance, that

$$W = \nabla \ln |\langle a, x \rangle|, \quad \text{where } a \text{ is a lightlike vector in } \mathbb{R}^n \text{ with } \langle a, x_0 \rangle \neq 0,$$

(2.9)

is a lightlike geodesic gradient field satisfying $\langle V, W \rangle = 1$.

Finally we remark that a nontrivial conformal vector field (a vector field $V$ is nontrivial if there is a point $p \in M$ with $V(p) \neq 0$) has isolated zeros (see [4]). This is in general not true if the conformal vector field is not closed (see, e.g., an example in [1]).

3. Local coordinates

Let $V$ and $W$ be vector fields as in Proposition 2.5 and let $E_1 = V - g(V, V) W$, $E_2 = W$.

It is easy to see that

(i) $E_1, E_2$ are linearly independent;

(ii) the distribution $\mathcal{D}$ spanned by $E_1, E_2$ is integrable and the metric $g$ is nondegenerate on $\mathcal{D}$;

(iii) the distribution $\mathcal{D}^\perp$ spanned by the vector fields orthogonal to $E_1, E_2$ is integrable and $g$ is nondegenerate on $\mathcal{D}^\perp$;

(iv) $[E_1, E_2] = 0$.

We can now state the following theorem.

**Theorem 3.1.** If $(M, g)$ admits a closed conformal vector field $V$, then in a neighbourhood of a point $p_0$ where $V(p_0) \neq 0$, there are local coordinates $u^1, u^2, \ldots, u^n$ such that $V = \partial/\partial u^1 + a(\partial/\partial u^2)$, for some function $a = a(u^2)$, and

$$\begin{vmatrix}
-a & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \cdots & 0 \\
0 & 0 & \cdots & (g_{\alpha\beta}) \\
\end{vmatrix}, \quad (3.1)
$$

where $\alpha, \beta \in \{3, \ldots, n\}$, $\det(g_{\alpha\beta}) \neq 0$, and $\partial g_{\alpha\beta}/\partial u^1 + a(\partial g_{\alpha\beta}/\partial u^2) = a' g_{\alpha\beta}$ ($a' := da/du^2$).

**Proof.** From Frobenius theorem, we know that there are local coordinates $u^1, u^2, \ldots, u^n$ such that

$$\frac{\partial}{\partial u^1} = E_1, \quad \frac{\partial}{\partial u^2} = E_2, \quad g_{1\alpha} = g_{2\alpha} = 0, \quad \alpha = 3, \ldots, n.$$

(3.2)
Hence \( g_{11} = g(E_1, E_1) = g(V, V) - 2g(V, V)g(V, W) = -g(V, V) \), \( g_{12} = g(V, W) = 1 \), \( g_{22} = g(W, W) = 0 \) and, setting \( E_i = \partial/\partial u^i \), \( i = 1, \ldots, n \), we have that

\[
\frac{\partial g_{\alpha\beta}}{\partial u^i} + a \frac{\partial g_{\alpha\beta}}{\partial u^2} = g(E_\alpha, \nabla E_\beta) + g(V, \nabla E_\beta - 2g(V, V) \nabla E_\xi E_\eta)
\]

\[
= g(E_\alpha, \nabla E_\beta + g(V, V) \nabla E_\xi E_\eta)
\]

\[
= g(E_\alpha, \nabla E_\beta + g(V, V) \nabla E_\xi E_\eta)
\]

\[
= g(E_\alpha, \nabla E_\beta + g(V, V) \nabla E_\xi E_\eta)
\]

\[
= g(E_\alpha, \nabla E_\xi E_\eta)
\]

\[
= g(E_\alpha, \nabla E_\xi E_\eta) + g(E_\alpha, \nabla E_\xi E_\eta) = 2\lambda g_{\alpha\beta},
\]

where \( a = g(V, V) \). From \( X g(V, V) = 2\lambda g(X, V) \) and \( g(E_1, V) = g(E_3, V) = \cdots = g(E_n, V) = 0 \), we conclude that \( a = a(u^2) \). Furthermore

\[
a' = W g(V, V) = 2\lambda
\]

and \( a = 0 \) if and only if \( V \) is lightlike (cf. with Brinkmann’s theorem).

On the other hand, we have the following proposition.

**Proposition 3.2.** If on a neighbourhood \( \mathcal{U} \) of a point \( p_0 \in M \), there are local coordinates as in Theorem 3.1, then \( V = \partial/\partial u^1 + a(\partial/\partial u^2) \) is a closed conformal vector field on \( \mathcal{U} \).

**Proof.** The statement follows from

\[
g(\nabla E_i, V_j) = g(\nabla E_i E_1, E_j) + a' \delta_{2i} \delta_{1j} + a g(\nabla E_2 E_j, E_i)
\]

\[
= \frac{1}{2} \left( \frac{\partial g_{1j}}{\partial u^1} + \frac{\partial g_{ij}}{\partial u^1} - \frac{\partial g_{1i}}{\partial u^j} + a \frac{\partial g_{ij}}{\partial u^1} \right) + a' \delta_{2i} \delta_{1j}
\]

\[
= \frac{1}{2} \left( \frac{\partial g_{ij}}{\partial u^1} + a \frac{\partial g_{ij}}{\partial u^2} \right) + \frac{1}{2} a' (\delta_{1i} \delta_{2j} + \delta_{2i} \delta_{1j}),
\]

where \( \delta \) is the Kronecker delta. Namely, for every pair \((i, j)\), we get \( g(\nabla E_i, V_j) = (1/2)a' g_{ij} \). Moreover, \( V \) is lightlike if and only if \( a = 0 \).

**Remark 3.3.** If in Proposition 3.2 we assume that \( a \neq 0 \), then according to Fialkow results, see [3, formulas (12.9) and (12.10)], we must be able to prove that \((\mathcal{U}, g)\) is locally isometric to a warped product with a one-dimensional base manifold. This can be seen in
the following way: take local coordinates $\bar{u}^1, \ldots, \bar{u}^n$ in $\mathcal{U}$ such that
\[
\frac{\partial}{\partial a^1} = \frac{1}{\sqrt{|a|}} \left( \frac{\partial}{\partial u^1} + a \frac{\partial}{\partial u^2} \right), \quad \frac{\partial}{\partial a^2} = \frac{\partial}{\partial u^1}, \quad \frac{\partial}{\partial u^a} = \frac{\partial}{\partial u^a}, \quad \alpha = 3, \ldots, n. \tag{3.6}
\]

This is reached by the coordinate transformation
\[
\bar{u}^1 = \int \frac{\sqrt{|a|}}{a} du^2, \quad \bar{u}^2 = u^1 - \int \frac{1}{a} du^2, \quad \bar{u}^a = u^a, \quad \alpha = 3, \ldots, n. \tag{3.7}
\]

Then it is easy to see that $a = a(\bar{u}^1)$ and that
\[
(g_{ij}) := \left( g \left( \frac{\partial}{\partial \bar{u}^i}, \frac{\partial}{\partial \bar{u}^j} \right) \right) = \begin{pmatrix}
\pm 1 & 0 & 0 & \cdots & 0 \\
0 & -a & 0 & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & \cdots & (g_{ab})
\end{pmatrix}. \tag{3.8}
\]

Furthermore, from $\partial g_{ab}/\partial u^1 + a(\partial g_{ab}/\partial u^2) = a' g_{ab}$, we get
\[
\frac{\partial g_{ab}}{\partial \bar{u}^1} = \frac{1}{\sqrt{|a|}} \left( \frac{\partial g_{ab}}{\partial u^1} + a \frac{\partial g_{ab}}{\partial u^2} \right) = \frac{1}{\sqrt{|a|}} \frac{da}{du^2} g_{ab} = \frac{1}{a} \frac{da}{\bar{u}^1} g_{ab}, \tag{3.9}
\]
and therefore $g_{ab} = a \bar{g}_{ab}$, where $\partial \bar{g}_{ab}/\partial \bar{u}^1 = 0$. Thus $(\mathcal{U}, g)$ is locally isometric to a warped product with a one-dimensional base manifold and warped factor $a$. In these local coordinates, the metric of the fiber manifold is given by
\[
\begin{pmatrix}
-1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & (\bar{g}_{ab})
\end{pmatrix} \tag{3.10}
\]
which means, in other words, that $\bar{u}^2, \ldots, \bar{u}^n$ are Fermi coordinates on the fiber manifold.

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