STARLIKENESS AND CONVEXITY OF A CLASS OF ANALYTIC FUNCTIONS

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Let $\mathcal{A}$ be the class of analytic functions in the unit disk that are normalized with $f(0) = f'(0) - 1 = 0$ and let $-1 \leq B < A \leq 1$. In this paper we study the class $G_{\lambda, \alpha} = \{f \in \mathcal{A} : |(1 - \alpha + \alpha z f''(z)/f'(z))/zf'(z)/f(z) - (1 - \alpha)| < \lambda, z \in \mathcal{U}\}$, $0 \leq \alpha \leq 1$, and give sharp sufficient conditions that embed it into the classes $S^*[A, B] = \{f \in \mathcal{A} : zf'(z)/f(z) \prec (1 + Az)/(1 + Bz)\}$ and $K(\delta) = \{f \in \mathcal{A} : 1 + zf''(z)/f'(z) \prec (1 - \delta)(1 + z)/(1 - z) + \delta\}$, where “ $\prec$ ” denotes the usual subordination. Also, sharp upper bound of $|a_2|$ and of the Fekete-Szegő functional $|a_3 - \mu a_2^2|$ is given for the class $G_{\lambda, \alpha}$.

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1. Introduction and preliminaries

A region Ω from the complex plane $\mathbb{C}$ is called convex if for every two points $\omega_1, \omega_2 \in \Omega$, the closed line segment $[\omega_1, \omega_2] = \{(1 - t)\omega_1 + t\omega_2 : 0 \leq t \leq 1\}$ lies in Ω . Fixing $\omega_1 = 0$ brings the definition of starlike region. If $\mathcal{A}$ denotes the class of functions $f(z)$ that are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$, then a function $f \in \mathcal{A}$ is called *convex* or *starlike* if it maps $\mathcal{U}$ into a convex or starlike region, respectively. Corresponding classes are denoted by K and S^* . It is well known that $K \subset S^*$, and it is well known that both are subclasses of the class of univalent functions and have the following analytical representations:

$$\begin{aligned} f \in K &\iff \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in \mathcal{U}, \\ f \in S^* &\iff \operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathcal{U}. \end{aligned} \tag{1.1}$$

More about these classes may be found in [2].

Further, let $f, g \in \mathcal{A}$. Then we say that $f(z)$ is *subordinate* to $g(z)$, and we write $f(z) \prec g(z)$, if there exists a function $\omega(z)$, analytic in the unit disk $\mathcal{U}$, such that $\omega(0) = 0$,

2 Starlikeness and convexity of analytic functions

$|\omega(z)| < 1$, and $f(z) = g(\omega(z))$ for all $z \in \mathcal{U}$. Specially, if $g(z)$ is univalent in $\mathcal{U}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subseteq g(\mathcal{U})$.

In terms of subordination, we have

$$S^* = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \right\}, \quad K = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z} \right\}. \quad (1.2)$$

If $-1 \leq B < A \leq 1$, then a generalization of class S^* is

$$S^*[A, B] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \right\}. \quad (1.3)$$

Geometrically, this means that the image of $\mathcal{U}$ by $zf'(z)/f(z)$ is inside the open disk centered on the real axis with diameter endpoints $(1-A)/(1-B)$ and $(1+A)/(1+B)$. Special selection of A and B leads us to the following classes: $S^*[1, -1] \equiv S^*$, $S^*[1 - 2\alpha, -1] \equiv S^*(\alpha)$ -class of starlike functions of order α , $0 \leq \alpha < 1$, and $K(\alpha)$ is the class of convex functions of order α , $0 \leq \alpha < 1$, defined by $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$, that is,

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad z \in \mathcal{U}. \quad (1.4)$$

These classes are widely studied during the past decades, mainly in two different directions: for developing criteria for starlikeness or convexity and for obtaining properties of the Maclaurin coefficients of a starlike or convex function. In this paper sufficient conditions (some of them sharp) that embed the class

$$G_{\lambda, \alpha} = \left\{ f \in \mathcal{A} : \left| \frac{1 - \alpha + \alpha z f''(z)/f'(z)}{zf'(z)/f(z)} - (1 - \alpha) \right| < \lambda, z \in \mathcal{U} \right\}, \quad (1.5)$$

$0 < \alpha \leq 1, \lambda > 0$, into the classes $S^*[A, B]$ and $K(\delta)$, $0 \leq \delta < 1$, will be given, together with sharp upper bound of the Fekete-Szegő functional $|a_3 - \mu a_2^2|$, $\mu \in \mathbb{R}$. Sufficient motivation for studying the class $G_{\lambda, \alpha}$ is the fact that it makes close connection between classes,

$$G_{\lambda, 1/2} = \left\{ f \in \mathcal{A} : \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < 2\lambda, z \in \mathcal{U} \right\},$$

$$G_{\lambda, 1} = \left\{ f \in \mathcal{A} : \left| \frac{f(z)f''(z)}{f'^2(z)} \right| < \lambda, z \in \mathcal{U} \right\}, \quad (1.6)$$

$$G_{\lambda, 1/(2-\gamma)} = \left\{ f \in \mathcal{A} : \left| \frac{1 - \gamma + zf''(z)/f'(z)}{zf'(z)/f(z)} - (1 - \gamma) \right| < \lambda(2 - \gamma), z \in \mathcal{U} \right\},$$

studied in [1, 6–9, 11–13] and other references.

2. Conditions for starlikeness and convexity

For obtaining the result for convexity and starlikeness of the class $G_{\lambda, \alpha}$, we will use the method of differential subordinations. Valuable reference on this topic is [5]. The general theory of differential subordinations, as well as the theory of first-order differential

subordinations, was introduced by Miller and Mocanu in [3, 4]. Namely, if $\phi : \mathbb{C}^2 \rightarrow \mathbb{C}$ is analytic in a domain D , if $h(z)$ is univalent in $\mathcal{U}$, and if $p(z)$ is analytic in $\mathcal{U}$ with $(p(z), zp'(z)) \in D$ when $z \in \mathcal{U}$, then $p(z)$ is said to satisfy a first-order differential subordination if

$$\phi(p(z), zp'(z)) \prec h(z). \quad (2.1)$$

The univalent function $q(z)$ is said to be a *dominant* of the differential subordination (2.1) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2.1). If $\tilde{q}(z)$ is a dominant of (2.1) and $\tilde{q}(z) \prec q(z)$ for all dominants of (2.1), then we say that $\tilde{q}(z)$ is the *best dominant* of the differential subordination (2.1).

From the theory of first-order differential subordinations, we will make use of the following lemma.

LEMMA 2.1 (see [4]). *Let $q(z)$ be univalent in the unit disk $\mathcal{U}$, and let $\theta(\omega)$ and $\phi(\omega)$ be analytic in a domain D containing $q(\mathcal{U})$, with $\phi(\omega) \neq 0$ when $\omega \in q(\mathcal{U})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that*

(i) $Q(z) \in S^*$;

(ii) $\operatorname{Re}\{zh'(z)/Q(z)\} = \operatorname{Re}\{\theta'(q(z))/\phi(q(z)) + zQ'(z)/Q(z)\} > 0$, $z \in \mathcal{U}$.

If $p(z)$ is analytic in $\mathcal{U}$, with $p(0) = q(0)$, $p(\mathcal{U}) \subseteq D$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \quad (2.2)$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of (2.2).

In the beginning, using Lemma 2.1 we will prove the following result.

THEOREM 2.2. *Let $f \in \mathcal{A}$, $-1 \leq B < A \leq 1$, and $(1 + |A|)/(3 + |A|) \leq \alpha \leq 1$. If*

$$\frac{1 - \alpha + \alpha zf''(z)/f'(z)}{zf'(z)/f(z)} \prec \alpha + (1 - 2\alpha) \frac{1 + Bz}{1 + Az} + \frac{\alpha z(A - B)}{(1 + Az)^2} \equiv h(z), \quad (2.3)$$

then $f \in S^[A, B]$. This result is sharp.*

Proof. We choose $p(z) = f(z)/zf'(z)$, $q(z) = (1 + Bz)/(1 + Az)$, $\theta(\omega) = (1 - 2\alpha)\omega + \alpha$, and $\phi(\omega) = -\alpha$. Then $q(z)$ is convex, thus univalent, because $1 + zq''(z)/q'(z) = (1 - Az)/(1 + Az)$; $\theta(\omega)$ and $\phi(\omega)$ are analytic in the domain $D = \mathbb{C}$ which contains $q(\mathcal{U})$ and $\phi(\omega)$ when $\omega \in q(\mathcal{U})$. Further,

$$Q(z) = zq'(z)\phi(q(z)) = \frac{\alpha(A - B)z}{(1 + Az)^2} \quad (2.4)$$

is starlike because $zQ'(z)/Q(z) = (1 - Az)/(1 + Az)$. Further,

$$\begin{aligned} h(z) &= \theta(q(z)) + Q(z) = \alpha + (1 - 2\alpha) \frac{1 + Bz}{1 + Az} + \frac{\alpha z(A - B)}{(1 + Az)^2}, \\ \operatorname{Re} \frac{zh'(z)}{Q(z)} &= \operatorname{Re} \left(1 - \frac{1}{\alpha} + \frac{2}{1 + Az} \right) > 1 - \frac{1}{\alpha} + \frac{2}{1 + |A|}, \end{aligned} \quad (2.5)$$

4 Starlikeness and convexity of analytic functions

$z \in \mathcal{U}$, which is greater or equal to zero if and only if $\alpha \geq (1 + |A|)/(3 + |A|)$. Therefore from Lemma 2.1, it follows that $p(z) < q(z)$, that is, $f \in S^*[A, B]$.

The result is sharp as the functions ze^{Az} and $z(1 + Bz)^{A/B}$ show in the cases $B = 0$ and $B \neq 0$, respectively. $\square$

Remark 2.3. According to the definition of subordination, the sharpness of the result of Theorem 2.2 means that $h(\mathcal{U})$ is the greatest region in the complex plane with the property that if

$$\frac{1 - \alpha + \alpha z f''(z)/f'(z)}{z f'(z)/f(z)} \in h(\mathcal{U}) \quad (2.6)$$

for all $z \in \mathcal{U}$, then $f(z) \in S^*[A, B]$.

The following corollary gives sharp sufficient conditions that embed $G_{\lambda, \alpha}$ into $S^*[A, B]$.

COROLLARY 2.4. *Let $-1 \leq B < A \leq 1$ and $(1 + |A|)/(3 + |A|) \leq \alpha \leq 1$. Then*

$$\lambda = (A - B) \cdot \frac{(1 - 2\alpha)|A| - (1 - 3\alpha)}{(1 + |A|)^2} \quad (2.7)$$

is the greatest number such that $G_{\lambda, \alpha} \subseteq S^[A, B]$.*

Proof. In order to prove this corollary, due to Theorem 2.2 it is enough to show that

$$\lambda = \min \{ |h(z) - (1 - \alpha)| : |z| = 1 \} \equiv \hat{\lambda}, \quad (2.8)$$

where $h(z)$ is defined as in the statement of the theorem and

$$h(z) - (1 - \alpha) = -z(A - B) \cdot \frac{A(1 - 2\alpha)z + 1 - 3\alpha}{(1 + Az)^2}. \quad (2.9)$$

Further, let

$$\begin{aligned} \psi(t) &\equiv |h(e^{i\gamma\pi/2}) - (1 - \alpha)|^2 \\ &= (A - B)^2 \cdot \frac{[(1 - 2\alpha)^2 A^2 + 2(1 - 3\alpha)(1 - 2\alpha)At + (1 - 3\alpha)^2]}{(1 + 2At + A^2)^2}, \end{aligned} \quad (2.10)$$

$t = \cos(\gamma\pi/2) \in [-1, 1]$. Thus $\hat{\lambda} = \min\{\sqrt{\psi(t)} : -1 \leq t \leq 1\}$.

If $\alpha \leq 1/2$, then $1 - 2\alpha \geq 0$ and having in mind that $1 - 3\alpha \leq -2|A|/(3 + |A|) \leq 0$, we receive that $\psi(t)$ is a monotone function and

$$\hat{\lambda} = \min \{ \sqrt{\psi(-1)}, \sqrt{\psi(1)} \} = \min \{ |h(-1) - (1 - \alpha)|, |h(1) - (1 - \alpha)| \} = \lambda. \quad (2.11)$$

The last equality holds because $1 - 3\alpha \pm A(1 - 2\alpha) \geq 0$ is equivalent to $\alpha \geq (1 + |A|)/(3 + |A|) \geq (1 - |A|)/(3 - 2|A|)$.

If $\alpha > 1/2$, we have the following analysis. Equation $\psi'_t(t) = 0$ has unique solution

$$t_* = -\frac{A^2(1 - \alpha)(1 - 2\alpha) + (1 - 3\alpha)(1 - 4\alpha)}{2A(1 - 2\alpha)(1 - 3\alpha)}. \quad (2.12)$$

It can be verified that $|t_*| > 1$ is equivalent to

$$\varphi(A, \alpha) \equiv A^2(1 - \alpha)(1 - 2\alpha) - 2|A|(1 - 2\alpha)(1 - 3\alpha) + (1 - 3\alpha)(1 - 4\alpha) > 0. \quad (2.13)$$

Now, $\varphi(A, \alpha)$ is a decreasing function of $|A| \in [0, 1]$ which implies that $\varphi(A, \alpha) \geq \varphi(1, \alpha) = 2\alpha^2 > 0$. Thus, $|t_*| > 1$, which implies that $\psi(t)$ is a monotone function on $[-1, 1]$ leading to $\hat{\lambda} = \min\{\sqrt{\psi(t)} : -1 \leq t \leq 1\} = \min\{\sqrt{\psi(-1)}, \sqrt{\psi(1)}\} = \min\{|h(-1) - (1 - \alpha)|, |h(1) - (1 - \alpha)|\}$. At the end, the function

$$\eta(A, \alpha) \equiv |h(1) - (1 - \alpha)| - |h(-1) - (1 - \alpha)| = 2A \cdot \frac{1 - A^2 - 2\alpha(2 - A^2)}{(1 + A)^2(1 - A)^2} \quad (2.14)$$

has the opposite sign of the sign of coefficient A . Therefore,

$$\hat{\lambda} = \begin{cases} |h(1) - (1 - \alpha)|, & A \geq 0 \\ |h(-1) - (1 - \alpha)|, & A < 0 \end{cases} = \lambda. \quad (2.15)$$

Sharpness of the result follows from the sharpness of Theorem 2.2 (see Remark 2.3) and the fact that the obtained λ is the greatest, which embeds the disk $|\omega - (1 - \alpha)| < \lambda$ in $h(\mathcal{U})$. $\square$

The following example exhibits some concrete conclusions that can be obtained from the results of the previous section by specifying the values α, A, B .

Example 2.5. Let $-1 \leq B < A \leq 1$.

- (i) $G_{\lambda, 1/2} \subseteq S^*[A, B]$ when $\lambda = (A - B)/2(1 + |A|)^2$.
- (ii) $G_{\lambda, 1} \subseteq S^*[A, B]$ when $\lambda = (A - B) \cdot (2 - |A|)/2(1 + |A|)^2$.
- (iii) $G_{\lambda, 1/(2-\gamma)} \subseteq S^*[A, B]$ when $\gamma \geq -(1 - |A|)/(1 + |A|)$ and $\lambda = (A - B) \cdot (1 + \gamma - \gamma|A|)/2(1 + |A|)^2$.
- (iv) $G_{\lambda, \alpha} \subseteq S^*$ when $1/2 \leq \alpha \leq 1$ and $\lambda = \alpha/2$.
- (v) $G_{\lambda, \alpha} \subseteq S^*[0, B] \subset S^*(1/(1 - B))$ when $1/3 \leq \alpha \leq 1$, $-1 \leq B < 0$ and $\lambda = B(1 - 3\alpha)$.

The value of λ in each of the above cases is the greatest that makes the corresponding inclusion true.

Remark 2.6. The result from Example 2.5(i) is the same as in [13, Corollary 2.6]. Also, for $\alpha = 1/2$ in Example 2.5(v), we receive the same result as in [6, Theorem 1]. Finally, for $\alpha = 1$ and $B = -1$ in Example 2.5(v), we receive the same result as in [11, Corollary 2].

Next theorem studies connection between $G_{\lambda, \alpha}$ and the class of convex functions of some order.

THEOREM 2.7. $G_{\lambda, \alpha} \subseteq K(2 - 1/\alpha)$ when $1/2 \leq \alpha < 1$ and $\lambda = (1 - \alpha)(3\alpha - 1)/\sqrt{2(5\alpha^2 - 4\alpha + 1)}$.

Proof. Let $f \in G_{\lambda, \alpha}$ and $B = \lambda/(1 - 3\alpha)$. Then, by Example 2.5(v) we have $f \in S^*[0, B]$, that is, $|f(z)/zf'(z) - 1| < B$, $z \in \mathcal{U}$. Further,

$$1 + \frac{zf''(z)}{f'(z)} - \left(2 - \frac{1}{\alpha}\right) = \frac{zf'(z)}{\alpha f(z)} \cdot \frac{1 - \alpha + \alpha zf''(z)/f'(z)}{zf'(z)/f(z)}, \quad (2.16)$$

6 Starlikeness and convexity of analytic functions

and for all $z \in \mathcal{U}$, we obtain

$$\begin{aligned}
 \left| \arg \left(1 + \frac{zf''(z)}{f'(z)} - 2 + \frac{1}{\alpha} \right) \right| &\leq \left| \arg \frac{zf'(z)}{f(z)} \right| + \left| \arg \frac{1 - \alpha + \alpha zf''(z)/f'(z)}{zf'(z)/f(z)} \right| \\
 &\leq \arcsin |B| + \arcsin \frac{\lambda}{1 - \alpha} \\
 &= \arcsin \left(\frac{\lambda}{1 - \alpha} \sqrt{1 - B^2} + |B| \sqrt{1 - \frac{\lambda^2}{(1 - \alpha)^2}} \right) \\
 &= \arcsin 1 = \frac{\pi}{2},
 \end{aligned} \tag{2.17}$$

that is, $f \in K(2 - 1/\alpha)$. □

Example 2.8. For $\alpha = 1/2$ and $\alpha = 1/(2 - \gamma)$ in the previous theorem, we get

- (i) $G_{\lambda, 1/2} \subseteq K$ when $\lambda = \sqrt{2}/4$;
- (ii) $G_{\lambda, 1/(2-\gamma)} \subseteq K(\gamma)$ when $0 \leq \gamma < 1$ and $\lambda = (1 - \gamma^2)/[(2 - \gamma)\sqrt{2(1 + \gamma^2)}]$.

Remark 2.9. By putting $\alpha = 1/(2 - \gamma)$, $0 \leq \gamma < 1$, we get the result from [10, Theorem 2].

3. Sharp estimate of the Fekete-Szegő functional

In this section we give sharp estimates of $|a_2|$ and of the Fekete-Szegő functional $|a_3 - \mu a_2^2|$ for a function $f \in G_{\lambda, \alpha}$. We will use following lemmas.

LEMMA 3.1 [2, page 41]. *Let $p \in \mathcal{P}$, that is, let p be analytic in $\mathcal{U}$, be given by $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ and $\operatorname{Re} p(z) > 0$ for $z \in \mathcal{U}$. Then $|p_n| \leq 2$ and for all $n \in \mathbb{N}$, $|p_2 - p_1^2/2| \leq 2 - |p_1|^2/2$.*

LEMMA 3.2. *Let $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ be an analytic function in the unit disk $\mathcal{U}$ and $|\omega(z)| < 1$, $z \in \mathcal{U}$. Then $|c_1| \leq 1$ and $|c_2| \leq 1 - |c_1|^2$.*

Proof. Define a function $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$ by $p(z) = (1 - \omega(z))/(1 + \omega(z))$. Then $c_1 = -p_1/2$, $c_2 = (p_1^2/2 - p_2)/2$ and the rest follows from Lemma 3.1. □

THEOREM 3.3. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in G_{\lambda, \alpha}$ for some $\lambda > 0$ and $0 \leq \alpha \leq 1$. Then $|a_2| \leq \lambda/|1 - 3\alpha|$ and for any complex μ , the following bound is sharp:*

$$|a_3 - \mu a_2^2| \leq \max \left\{ \frac{\lambda}{2|4\alpha - 1|}, \frac{\lambda^2 |1 - \mu|}{(1 - 3\alpha)^2} \right\}. \tag{3.1}$$

Proof. If $f \in G_{\lambda, \alpha}$, then

$$(1 - \alpha)f(z)f''(z) + \alpha z f(z)f''(z) = z f'^2(z)[1 - \alpha + \lambda \omega(z)], \tag{3.2}$$

where $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ is such that $|\omega(z)| < 1$, $z \in \mathcal{U}$. After equating the coefficients, we get $a_2 = \lambda c_1/(3\alpha - 1)$ and

$$a_3 = \frac{\lambda c_2}{2(4\alpha - 1)} + \frac{\lambda^2 c_1^2}{(1 - 3\alpha)^2}. \tag{3.3}$$

From Lemma 3.2, we get $|a_2| \leq \lambda/|1 - 3\alpha|$. Further,

$$a_3 - \mu a_2^2 = \frac{\lambda c_2}{2(4\alpha - 1)} + \frac{\lambda^2 c_1^2}{(1 - 3\alpha)^2} (1 - \mu). \quad (3.4)$$

So, for $x = |c_1| \leq 1$,

$$|a_3 - \mu a_2^2| \leq Ax^2 + \frac{\lambda}{2|1 - 4\alpha|} \equiv H(x), \quad (3.5)$$

where $A = \lambda^2|1 - \mu|/(1 - 3\alpha)^2 - \lambda/2|1 - 4\alpha|$ and

$$|a_3 - \mu a_2^2| \leq \begin{cases} H(1) = \frac{\lambda^2|1 - \mu|}{(1 - 3\alpha)^2}, & A \geq 0, \\ H(0) = \frac{\lambda}{2|1 - 4\alpha|}, & A < 0. \end{cases} \quad (3.6)$$

The upper bound is sharp due to the functions $f_1(z) = z(1 - 3\alpha)/(1 - 3\alpha + \lambda z)$ and $f_2(z) = z \cdot \sqrt{(1 - 4\alpha)/(1 - 4\alpha + \lambda z^2)}$. $\square$

Remark 3.4. By putting $\alpha = 1/(2 - \gamma)$, $0 \leq \gamma < 1$, we get the result from [10, Theorem 3].

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