SUBCLASSES OF α -SPIRALLIKE FUNCTIONS ASSOCIATED WITH RUSCHEWEYH DERIVATIVES

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Making use of the Ruscheweyh derivatives, we introduce the subclasses $T(n, \alpha, \lambda)$ ($n \in \{0, 1, 2, 3, \dots\}$, $-\pi/2 < \alpha < \pi/2$, and $0 \leq \lambda \leq \cos^2 \alpha$) of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic in $|z| < 1$. Subordination and inclusion relations are obtained. The radius of α -spirallikeness of order ρ is calculated. A convolution property and a special member of $T(n, \alpha, \lambda)$ are also given.

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1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1.1)$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. Let $S \subset A$ consist of univalent functions in U . For $0 \leq \rho < 1$, a function $f \in S$ is said to be starlike of order ρ if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \rho \quad (z \in U). \quad (1.2)$$

The class of such functions we denote by $S^*(\rho)$ ($0 \leq \rho < 1$). A function $f \in S$ is said to be convex in U if

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in U). \quad (1.3)$$

We denote by K the class of all convex functions in U . For $-\pi/2 < \alpha < \pi/2$ and $0 \leq \rho < 1$, a function $f \in S$ is said to be α -spirallike of order ρ in U if

$$\operatorname{Re} \left\{ e^{i\alpha} \frac{zf'(z)}{f(z)} \right\} > \rho \cos \alpha \quad (z \in U). \quad (1.4)$$

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Further let $UCV \subset K$ be the class of functions introduced by Goodman [2] called uniformly convex in U . It was shown in [4, 7] that $f \in A$ is in UCV if and only if

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in U). \quad (1.5)$$

In [7], Ronning investigated the class S_p defined by

$$S_p = \{f \in S^*(0) : f(z) = zg'(z), g \in UCV\}. \quad (1.6)$$

The uniformly convex and related functions have been studied by several authors (see, e.g., [1–4, 7, 6, 8, 12]).

If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in A$, then the Hadamard product or convolution of f and g is defined by $(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$. Let

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z), \quad (1.7)$$

for $f \in A$ and $n \in N_0 = \{0, 1, 2, 3, \dots\}$. Then

$$D^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}. \quad (1.8)$$

This symbol $D^n f$ is called the Ruscheweyh derivative of order n of f . It was introduced by Ruscheweyh [9].

In this paper we introduce and investigate the subclasses $T(n, \alpha, \lambda)$ of A as follows.

Definition 1.1. A function $f \in A$ is said to be in $T(n, \alpha, \lambda)$ if

$$\left(\operatorname{Re} \left\{ e^{i\alpha} \frac{z(D^n f(z))'}{D^n f(z)} \right\} \right)^2 + \lambda > \left| \frac{z(D^n f(z))'}{D^n f(z)} - 1 \right|^2 \quad (z \in U), \quad (1.9)$$

where $n \in N_0$, $-\pi/2 < \alpha < \pi/2$, and $0 \leq \lambda \leq \cos^2 \alpha$.

Note that, for $\lambda = 0$,

$$T(n, \alpha, 0) = \left\{ f \in A : \operatorname{Re} \left\{ e^{i\alpha} \frac{z(D^n f(z))'}{D^n f(z)} \right\} > \left| \frac{z(D^n f(z))'}{D^n f(z)} - 1 \right| \quad (z \in U) \right\}. \quad (1.10)$$

In particular, $T(0, 0, 0) = S_p$ and $T(1, 0, 0) = UCV$.

2. Properties of $T(n, \alpha, \lambda)$

Let f and g be analytic in U . Then we say that f is subordinate to g in U , written $f \prec g$, if there exists an analytic function w in U such that $|w(z)| \leq |z|$ and $f(z) = g(w(z))$ for $z \in U$. If g is univalent in U , then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

THEOREM 2.1. Let $n \in N_0$, $\alpha \in (-\pi/2, \pi/2)$, and $\lambda \in [0, \cos^2 \alpha]$. A function $f \in A$ belongs to $T(n, \alpha, \lambda)$ if and only if

$$e^{i\alpha} \frac{z(D^n f(z))'}{D^n f(z)} \prec h(z) \cos \alpha + i \sin \alpha \quad (z \in U), \quad (2.1)$$

where

$$h(z) = 1 - \frac{\lambda}{2 \cos^2 \alpha} + \frac{2}{\pi^2} \left(\log \frac{1 + \sqrt{(z + \beta)/(1 + \beta z)}}{1 - \sqrt{(z + \beta)/(1 + \beta z)}} \right)^2, \quad (2.2)$$

with

$$\beta = \left(\frac{e^\mu - 1}{e^\mu + 1} \right)^2, \quad \mu = \frac{\sqrt{\lambda} \pi}{2 \cos \alpha}. \quad (2.3)$$

Proof. Let us define $w = u + iv$ by

$$e^{i\alpha} \frac{z(D^n f(z))'}{D^n f(z)} = w(z) \cos \alpha + i \sin \alpha \quad (z \in U). \quad (2.4)$$

Then $w(0) = 1$ and the inequality (1.9) can be rewritten as

$$u > \frac{1}{2} \left(v^2 + 1 - \frac{\lambda}{\cos^2 \alpha} \right). \quad (2.5)$$

Thus

$$w(U) \subset G = \{w = u + iv : u \text{ and } v \text{ satisfy (2.5)}\}. \quad (2.6)$$

It follows from (2.2) that

$$h(0) = 1 - \frac{\lambda}{2 \cos^2 \alpha} + \frac{2}{\pi^2} \left(\log \frac{1 + \sqrt{\beta}}{1 - \sqrt{\beta}} \right)^2 = 1. \quad (2.7)$$

In order to prove the theorem, it suffices to show that the function $w = h(z)$ defined by (2.2) maps U conformally onto the parabolic region G .

Note that

$$0 \leq \frac{1}{2} \left(1 - \frac{\lambda}{\cos^2 \alpha} \right) < 1 - \frac{\lambda}{2 \cos^2 \alpha} \leq 1, \quad (2.8)$$

for $0 \leq \lambda \leq \cos^2 \alpha$. Consider the transformations

$$w_1 = \sqrt{w - \left(1 - \frac{\lambda}{2 \cos^2 \alpha} \right)}, \quad w_2 = e^{\sqrt{2} \pi w_1}, \quad t = \frac{1}{2} \left(w_2 + \frac{1}{w_2} \right). \quad (2.9)$$

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It is easy to verify that the composite function

$$t = \varphi(w) = \operatorname{ch} \left(\pi \sqrt{2w - \left(2 - \frac{\lambda}{\cos^2 \alpha} \right)} \right) \quad (2.10)$$

maps $G^+ = G \cap \{w = u + iv : v > 0\}$ conformally onto the upper half plane $\operatorname{Im}(t) > 0$ so that $w = (1/2)(1 - \lambda/\cos^2 \alpha)$ corresponds to $t = -1$ and $w = 1 - \lambda/2\cos^2 \alpha$ to $t = 1$. Applying the symmetry principle, the function $t = \varphi(w)$ maps G conformally onto $\Omega = \{t : |\arg(t+1)| < \pi\}$. Since $t = 2((1+\zeta)/(1-\zeta))^2 - 1$ maps the unit disk $|\zeta| < 1$ onto Ω , we see that

$$\begin{aligned} w = \varphi^{-1}(t) &= 1 - \frac{\lambda}{2\cos^2 \alpha} + \frac{1}{2\pi^2} \left(\log(t + \sqrt{t^2 - 1}) \right)^2 \\ &= 1 - \frac{\lambda}{2\cos^2 \alpha} + \frac{2}{\pi^2} \left(\log \frac{1 + \sqrt{\zeta}}{1 - \sqrt{\zeta}} \right)^2 = g(\zeta) \end{aligned} \quad (2.11)$$

maps $|\zeta| < 1$ conformally onto G so that $\zeta = \beta$ ($0 \leq \beta < 1$) corresponds to $w = 1$. Therefore the function

$$w = h(z) = g\left(\frac{z + \beta}{1 + \beta z}\right) \quad (z \in U) \quad (2.12)$$

maps U conformally onto G and the proof of the theorem is complete. $\square$

COROLLARY 2.2. *Let $f \in T(n, \alpha, \lambda)$, $n \in \mathbb{N}_0$, $\alpha \in (-\pi/2, \pi/2)$, $\lambda \in [0, \cos^2 \alpha]$, and h be given by (2.2). Then*

$$\frac{D^n f(z)}{z} \prec \exp \left(e^{-i\alpha} \cos \alpha \int_0^z \frac{h(t) - 1}{t} dt \right), \quad (2.13)$$

$$\exp \left(\int_0^1 \frac{h(-\rho|z|) - 1}{\rho} d\rho \right) \leq \left| \left(\frac{D^n f(z)}{z} \right)^{e^{i\alpha} \sec \alpha} \right| \leq \exp \left(\int_0^1 \frac{h(\rho|z|) - 1}{\rho} d\rho \right), \quad (2.14)$$

for $z \in U$. The bounds in (2.14) are sharp with the extremal function $f_0 \in A$ defined by

$$D^n f_0(z) = z \exp \left(e^{-i\alpha} \cos \alpha \int_0^z \frac{h(t) - 1}{t} dt \right). \quad (2.15)$$

Proof. From Theorem 2.1 we have

$$\frac{e^{i\alpha}}{\cos \alpha} \left(\frac{z(D^n f(z))'}{D^n f(z)} - 1 \right) \prec h(z) - 1, \quad (2.16)$$

for $f \in T(n, \alpha, \lambda)$. Since the function $h - 1$ is univalent and starlike (with respect to the origin) in U , using (2.16) and the result of Suffridge [11, Theorem 3], we obtain

$$\frac{e^{i\alpha}}{\cos \alpha} \log \frac{D^n f(z)}{z} = \frac{e^{i\alpha}}{\cos \alpha} \int_0^z \left(\frac{(D^n f(t))'}{D^n f(t)} - \frac{1}{t} \right) dt < \int_0^z \frac{h(t) - 1}{t} dt. \quad (2.17)$$

This implies (2.13).

Noting that the univalent function h maps the disk $|z| < \rho$ ($0 < \rho \leq 1$) onto a region which is convex and symmetric with respect to the real axis, we get

$$h(-\rho|z|) \leq \operatorname{Re} h(\rho z) \leq (\rho|z|) \quad (z \in U). \quad (2.18)$$

Now, (2.17) and (2.18) lead to

$$\int_0^1 \frac{h(-\rho|z|) - 1}{\rho} d\rho \leq \log \left| \left(\frac{D^n f(z)}{z} \right)^{e^{i\alpha} \sec \alpha} \right| \leq \int_0^1 \frac{h(\rho|z|) - 1}{\rho} d\rho, \quad (2.19)$$

for $z \in U$, which yields (2.14).

The bounds in (2.14) are best possible since the equalities are attained for the function f_0 in $T(n, \alpha, \lambda)$ defined by (2.15). $\square$

THEOREM 2.3. *Let $f \in T(n, \alpha, \lambda)$, $n \in \mathbb{N}_0$, $\alpha \in (-\pi/2, \pi/2)$, $\lambda \in [0, \cos^2 \alpha]$. Then $D^n f$ is α -spirallike of order ρ in $|z| < r$, where*

$$\begin{aligned} r = r(\rho, \alpha, \lambda) &= \frac{\beta + \left(\tan \left((\pi/4) \sqrt{2(1-\rho) - \lambda/\cos^2 \alpha} \right) \right)^2}{1 + \beta \left(\tan \left((\pi/4) \sqrt{2(1-\rho) - \lambda/\cos^2 \alpha} \right) \right)^2} \\ &\times \left(\frac{1}{2} \left(1 - \frac{\lambda}{\cos^2 \alpha} \right) \leq \rho < 1 - \frac{\lambda}{2\cos^2 \alpha} \right) \end{aligned} \quad (2.20)$$

and β is given by (2.2). The result is sharp.

Proof. It follows from (2.20) and (2.2) that

$$0 < 2(1-\rho) - \frac{\lambda}{\cos^2 \alpha} \leq 1, \quad 0 \leq \beta < r \leq 1. \quad (2.21)$$

Let h be given by (2.2). Then

$$\begin{aligned} h(-r) &= 1 - \frac{\lambda}{2\cos^2 \alpha} + \frac{2}{\pi^2} \left(\log \frac{1 + i\sqrt{(r-\beta)/(1-\beta r)}}{1 - i\sqrt{(r-\beta)/(1-\beta r)}} \right)^2 \\ &= 1 - \frac{\lambda}{2\cos^2 \alpha} - \frac{8}{\pi^2} \left(\arctan \sqrt{\frac{r-\beta}{1-\beta r}} \right)^2 \end{aligned} \quad (2.22)$$

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and hence

$$\inf_{|z|<r} \operatorname{Re} h(z) = h(-r) = \rho. \quad (2.23)$$

If $f \in T(n, \alpha, \lambda)$, then from Theorem 2.1 and (2.23) we have

$$\operatorname{Re} \left\{ e^{i\alpha} \frac{z(D^n f(z))'}{D^n f(z)} \right\} > \rho \cos \alpha \quad (|z| < r), \quad (2.24)$$

that is, $D^n f$ is α -spirallike of order ρ in $|z| < r$. Further, the result is sharp with the extremal function f_0 defined by (2.15).

Taking $\rho = (1/2)(1 - \lambda/\cos^2 \alpha)$, Theorem 2.3 yields. $\square$

COROLLARY 2.4. *Let $f \in T(n, \alpha, \lambda)$, $n \in N_0$, $\alpha \in (-\pi/2, \pi/2)$, $\lambda \in [0, \cos^2 \alpha]$. Then $D^n f$ is α -spirallike of order $(1/2)(1 - \lambda/\cos^2 \alpha)$ in U and the result is sharp.*

THEOREM 2.5. *Let $f \in T(n, \alpha, \lambda)$, $n \in N_0$, $\alpha \in (-\pi/2, \pi/2)$, $\lambda \in [0, \cos^2 \alpha]$. Then $D^n f \in S^*((1 - \lambda)/2)$ and the order $(1 - \lambda)/2$ is sharp.*

Proof. Let h be given by (2.2). Then it follows from the proof of Theorem 2.1 that

$$\partial h(U) = \left\{ w = u + iv : u = \frac{1}{2} \left(v^2 + 1 - \frac{\lambda}{\cos^2 \alpha} \right) \right\}. \quad (2.25)$$

Hence

$$\min_{|z|=1 (z \neq 1)} \operatorname{Re} \{ e^{-i\alpha} (h(z) \cos \alpha + i \sin \alpha) \} = \min_{u \geq (1/2)(1 - \lambda/\cos^2 \alpha)} g(u) \cos \alpha + \sin^2 \alpha, \quad (2.26)$$

where

$$g(u) = u \cos \alpha - |\sin \alpha| \sqrt{2u - 1 + \frac{\lambda}{\cos^2 \alpha}} \quad \left(u \geq \frac{1}{2} \left(1 - \frac{\lambda}{\cos^2 \alpha} \right) \right). \quad (2.27)$$

Since

$$g'(u) = \cos \alpha - \frac{|\sin \alpha|}{\sqrt{2u - 1 + \lambda/\cos^2 \alpha}} \quad \left(u > \frac{1}{2} \left(1 - \frac{\lambda}{\cos^2 \alpha} \right) \right), \quad (2.28)$$

the function g attains its minimum value at $u = (1 - \lambda)/2\cos^2 \alpha$. Thus

$$\min_{|z|=1 (z \neq 1)} \operatorname{Re} \{ e^{-i\alpha} (h(z) \cos \alpha + i \sin \alpha) \} = g\left(\frac{1 - \lambda}{2\cos^2 \alpha}\right) \cos \alpha + \sin^2 \alpha = \frac{1 - \lambda}{2}. \quad (2.29)$$

Let $f \in T(n, \alpha, \lambda)$. Then, by Theorem 2.1 and (2.29), we conclude that $D^n f$ is starlike of order $(1 - \lambda)/2$ in U , and the function f_0 defined by (2.15) shows that the order $(1 - \lambda)/2$ is sharp. $\square$

THEOREM 2.6. $T(n + 1, \alpha, \lambda) \subset T(n, \alpha, \lambda)$, where $n \in N_0$, $\alpha \in (-\pi/2, \pi/2)$, $\lambda \in [0, \cos^2 \alpha]$.

Proof. It follows from (1.7) that

$$z(D^n f(z))' = (n+1)D^{n+1}f(z) - nD^n f(z) \quad (z \in U), \quad (2.30)$$

for $f \in A$. Set

$$p(z) = e^{i\alpha} \frac{z(D^n f(z))'}{D^n f(z)} \quad (z \in U). \quad (2.31)$$

Then (2.30) and (2.31) lead to

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{e^{-i\alpha}p(z) + n}{n+1} \quad (z \in U). \quad (2.32)$$

Differentiating both sides of (2.32) logarithmically and using (2.31), we get

$$e^{i\alpha} \frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} = p(z) + \frac{zp'(z)}{e^{-i\alpha}p(z) + n} \quad (z \in U). \quad (2.33)$$

If $f \in T(n+1, \alpha, \lambda)$, then by Theorem 2.1 and (2.33) we have

$$p(z) + \frac{zp'(z)}{e^{-i\alpha}p(z) + n} \prec h(z) \cos \alpha + i \sin \alpha \quad (z \in U), \quad (2.34)$$

where h is given by (2.2). The function $Q(z) = e^{-i\alpha}(h(z) \cos \alpha + i \sin \alpha) + n$ is univalent and convex in U and

$$\operatorname{Re} Q(z) > \frac{1-\lambda}{2} + n \geq 0 \quad (z \in U) \quad (2.35)$$

because of (2.29). Hence an application of the result of Miller and Mocanu [5, Corollary 1.1] yields

$$p(z) = e^{i\alpha} \frac{z(D^n f(z))'}{D^n f(z)} \prec h(z) \cos \alpha + i \sin \alpha \quad (z \in U). \quad (2.36)$$

Now, by Theorem 2.1, we know that $f \in T(n, \alpha, \lambda)$ and the theorem is proved. □

Remark 2.7. Combining Theorem 2.6 with Corollary 2.4, we see that each function in $T(n, \alpha, \lambda)$ is α -spirallike of order $(1/2)(1 - \lambda/\cos^2 \alpha)$ in U . In view of Theorems 2.5 and 2.6 we have $T(n, \alpha, \lambda) \subset S^*((1 - \lambda)/2)$.

THEOREM 2.8. *A function $f \in A$ is in $T(n, \alpha, \lambda)$ if and only if*

$$F(z) = \frac{n+1}{z^n} \int_0^z t^{n-1} f(t) dt \quad (2.37)$$

is in $T(n+1, \alpha, \lambda)$, where $n \in \mathbb{N}_0$, $\alpha \in (-\pi/2, \pi/2)$, $\lambda \in [0, \cos^2 \alpha]$.

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Proof. It follows from (2.37) that $F \in A$ and

$$(n+1)f(z) = nF(z) + zF'(z) \quad (z \in U), \quad (2.38)$$

for $f \in A$. By using (2.30) and (2.38), we obtain

$$D^n f(z) = \frac{nD^n F(z) + z(D^n F(z))'}{n+1} = D^{n+1} F(z) \quad (z \in U), \quad (2.39)$$

which proves the assertions of the theorem. $\square$

Let $R(\rho)$ be the class of prestarlike functions of order ρ in U consisting of functions $f \in A$ satisfying

$$\frac{z}{(1-z)^{2-2\rho}} * f(z) \in S^*(\rho), \quad (2.40)$$

for some ρ ($0 \leq \rho < 1$). The following lemma is due to Ruscheweyh [10].

LEMMA 2.9. *If $f \in S^*(\rho)$ and $g \in R(\rho)$ ($0 \leq \rho < 1$), then for any analytic function F in U ,*

$$\frac{g * (Ff)}{g * f}(U) \subset \overline{\text{co}}(F(U)), \quad (2.41)$$

where $\overline{\text{co}}(F(U))$ stands for the convex hull of $F(U)$.

Applying the lemma, we derive the following.

THEOREM 2.10. *Let $f \in T(n, \alpha, \lambda)$ and $g \in R((1-\lambda)/2)$. Then*

$$f * g \in T(n, \alpha, \lambda), \quad (2.42)$$

where $n \in \mathbb{N}_0$, $\alpha \in (-\pi/2, \pi/2)$, $\lambda \in [0, \cos^2 \alpha]$.

Proof. Let $f \in T(n, \alpha, \lambda)$. Making use of Theorems 2.1 and 2.5, we have

$$F(z) = \frac{z(D^n f(z))'}{D^n f(z)} \prec e^{-i\alpha}(h(z)\cos\alpha + i\sin\alpha), \quad D^n f \in S^*\left(\frac{1-\lambda}{2}\right). \quad (2.43)$$

If we put $\varphi = f * g$, then for $z \in U$,

$$\begin{aligned} \frac{z(D^n \varphi(z))'}{D^n \varphi(z)} &= \frac{z(g(z) * D^n f(z))'}{g(z) * D^n f(z)} = \frac{g(z) * (z(D^n f(z)))'}{g(z) * D^n f(z)} \\ &= \frac{g(z) * (F(z)D^n f(z))}{g(z) * D^n f(z)}. \end{aligned} \quad (2.44)$$

Since the univalent function $e^{-i\alpha}(h(z)\cos\alpha + i\sin\alpha)$ is convex in U and $g \in R((1-\lambda)/2)$, from (2.43), (2.44), and the lemma we deduce that

$$\frac{z(D^n\varphi(z))'}{D^n\varphi(z)} \prec e^{-i\alpha}(h(z)\cos\alpha + i\sin\alpha). \quad (2.45)$$

Therefore, by using Theorem 2.1, $\varphi \in T(n, \alpha, \lambda)$ and the proof is complete. $\square$

Note that $R(1/2) = S^*(1/2)$. Since $R(\rho_1) \subset R(\rho_2)$ for $0 \leq \rho_1 < \rho_2 < 1$ (see [10]), we have $K = R(0) \subset R((1-\lambda)/2)$. Thus Theorem 2.10 yields the following.

COROLLARY 2.11. (i) If $f \in T(n, \alpha, 0)$, $n \in N_0$, $\alpha \in (-\pi/2, \pi/2)$, and $g \in S^*(1/2)$, then $f * g \in T(n, \alpha, 0)$.

(ii) If $f \in T(n, \alpha, \lambda)$, $n \in N_0$, $\alpha \in (-\pi/2, \pi/2)$, $\lambda \in [0, \cos^2\alpha]$, and $g \in K$, then $f * g \in T(n, \alpha, \lambda)$.

THEOREM 2.12. Let $n \in N_0$, $\alpha \in (-\pi/2, \pi/2)$, $\lambda \in [0, \cos^2\alpha]$. The function $f \in A$ defined by

$$D^n f(z) = \frac{z}{(1-bz)^{2e^{-i\alpha}\cos\alpha}} \quad (z \in U) \quad (2.46)$$

is in $T(n, \alpha, \lambda)$, where b is complex and

$$|b| = \begin{cases} \frac{\cos^2\alpha + \lambda}{3\cos^2\alpha - \lambda} & (0 \leq \lambda \leq (3 - 2\sqrt{2})\cos^2\alpha), \\ \sqrt{\frac{\sqrt{\lambda}}{2\cos\alpha + \sqrt{\lambda}}} & ((3 - 2\sqrt{2})\cos^2\alpha \leq \lambda \leq \cos^2\alpha). \end{cases} \quad (2.47)$$

The result is sharp, that is, $|b|$ cannot be increased.

Proof. Let $f \in A$ be given by (2.46). Then

$$e^{i\alpha} \frac{z(D^n f(z))'}{D^n f(z)} = \frac{1+bz}{1-bz} \cos\alpha + i\sin\alpha. \quad (2.48)$$

Hence, by Theorem 2.1, $f \in T(n, \alpha, \lambda)$ if and only if

$$\frac{1+bz}{1-bz} \prec h(z), \quad (2.49)$$

where h is given by (2.2), or, equivalently, when

$$\left\{ w : \left| w - \frac{1+|b|^2}{1-|b|^2} \right| < \frac{2|b|}{1-|b|^2} \right\} \subset h(U), \quad (2.50)$$

for $0 < |b| < 1$.

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Let δ denote the minimum distance from the point $(1 + |b|^2)/(1 - |b|^2)$ to the points on the parabola $\partial h(U)$ given by (2.25). Then

$$\delta = \min_{u \geq (1/2)(1-\lambda/\cos^2 \alpha)} \sqrt{g(u)}, \quad g(u) = \left(u - \frac{1 + |b|^2}{1 - |b|^2}\right)^2 + 2u - 1 + \frac{\lambda}{\cos^2 \alpha}. \quad (2.51)$$

Note that

$$\frac{1 + |b|^2}{1 - |b|^2} > \frac{1}{2} \left(1 - \frac{\lambda}{\cos^2 \alpha}\right), \quad g'(u) = 2 \left(u - \frac{2|b|^2}{1 - |b|^2}\right). \quad (2.52)$$

(i) If

$$0 \leq \lambda \leq (3 - 2\sqrt{2})\cos^2 \alpha, \quad |b| = \frac{\cos^2 \alpha + \lambda}{3\cos^2 \alpha - \lambda}, \quad (2.53)$$

then $\lambda^2 - 6\lambda\cos^2 \alpha + \cos^4 \alpha \geq 0$. Thus

$$|b|^2 = \left(\frac{\cos^2 \alpha + \lambda}{3\cos^2 \alpha - \lambda}\right)^2 \leq \frac{\cos^2 \alpha - \lambda}{5\cos^2 \alpha - \lambda}, \quad \frac{2|b|^2}{1 - |b|^2} \leq \frac{1}{2} \left(1 - \frac{\lambda}{\cos^2 \alpha}\right). \quad (2.54)$$

From (2.51), (2.52) and (2.54), we have $g'(u) \geq 0$ and hence

$$\delta = \sqrt{g\left(\frac{1}{2} \left(1 - \frac{\lambda}{\cos^2 \alpha}\right)\right)} = \frac{1 + |b|^2}{1 - |b|^2} - \frac{1}{2} \left(1 - \frac{\lambda}{\cos^2 \alpha}\right) = \frac{2|b|}{1 - |b|^2}. \quad (2.55)$$

(ii) If $0 \leq \lambda < (3 - 2\sqrt{2})\cos^2 \alpha$ and

$$\frac{\cos^2 \alpha + \lambda}{3\cos^2 \alpha - \lambda} < |b| < \sqrt{\frac{\cos^2 \alpha - \lambda}{5\cos^2 \alpha - \lambda}}, \quad (2.56)$$

then $g'(u) > 0$ and

$$\delta = \frac{1 + |b|^2}{1 - |b|^2} - \frac{1}{2} \left(1 - \frac{\lambda}{\cos^2 \alpha}\right) < \frac{2|b|}{1 - |b|^2}. \quad (2.57)$$

(iii) If

$$(3 - 2\sqrt{2})\cos^2 \alpha \leq \lambda \leq \cos^2 \alpha, \quad |b| = \sqrt{\frac{\sqrt{\lambda}}{2\cos \alpha + \sqrt{\lambda}}}, \quad (2.58)$$

then $\lambda^2 - 6\lambda \cos^2 \alpha + \cos^4 \alpha \leq 0$ and so

$$|b|^2 = \frac{\sqrt{\lambda}}{2 \cos \alpha + \sqrt{\lambda}} \geq \frac{\cos^2 \alpha - \lambda}{5 \cos^2 \alpha - \lambda}, \quad \frac{2|b|^2}{1 - |b|^2} \geq \frac{1}{2} \left(1 - \frac{\lambda}{\cos^2 \alpha} \right). \quad (2.59)$$

Thus we have

$$\delta = \sqrt{g\left(\frac{2|b|^2}{1 - |b|^2}\right)} = \sqrt{\frac{4|b|^2}{1 - |b|^2} + \frac{\lambda}{\cos^2 \alpha}} = \frac{2|b|}{1 - |b|^2}. \quad (2.60)$$

(iv) If $(3 - 2\sqrt{2}) \cos^2 \alpha \leq \lambda \leq \cos^2 \alpha$ and $\sqrt{\lambda/(2 \cos \alpha + \sqrt{\lambda})} < |b| < 1$, then

$$\delta = \sqrt{\frac{4|b|^2}{1 - |b|^2} + \frac{\lambda}{\cos^2 \alpha}} < \frac{2|b|}{1 - |b|^2}. \quad (2.61)$$

By virtue of (2.49), (2.50), (2.55), (2.57), (2.60), and (2.61), the proof of the theorem is now complete. $\square$

Letting $n = \alpha = 0$ in Theorem 2.12, we have the following.

COROLLARY 2.13. *The function $f(z) = z/(1 - bz)^2$ is in $T(0, 0, \lambda)$, where $\lambda \in [0, 1]$, b is complex and*

$$|b| = \begin{cases} \frac{1 + \lambda}{3 - \lambda} & (0 \leq \lambda \leq 3 - 2\sqrt{2}), \\ \sqrt{\frac{\sqrt{\lambda}}{2 + \sqrt{\lambda}}} & (3 - 2\sqrt{2} \leq \lambda \leq 1). \end{cases} \quad (2.62)$$

The result is sharp, that is, $|b|$ cannot be increased.

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