A VORONOVSKAYA-TYPE THEOREM FOR A POSITIVE LINEAR OPERATOR

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We consider a sequence of positive linear operators which approximates continuous functions having exponential growth at infinity. For these operators, we give a Voronovskaya-type theorem.

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1. Introduction

Sequences of positive linear operators are often used in approximation theory. Let $(L_n)_{n \geq 1}$ be such a sequence, where the operators L_n are defined on a suitable linear subspace E of $C(I)$, $I \subset \mathbb{R}$ an interval. An important problem is the investigation of the limit

$$\lim_{n \rightarrow \infty} n(L_n f - f) \quad (1.1)$$

in order to obtain information about the rate of convergence and the saturation properties of the sequence (L_n) .

The above formula is called *Voronovskaya's formula* for the sequence $(L_n)_{n \geq 1}$.

This paper is devoted to establishing a Voronovskaya-type formula for the sequence of positive linear operators introduced in [1], which approximate continuous functions of exponential order. To obtain the operators, we consider $g(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(1) \neq 0$, an analytic function in the disk $|z| < R$, $R > 1$, and we define the polynomials p_k by the relation

$$g(u) \cosh(ux) = \sum_{k=0}^{\infty} p_k(x) u^k, \quad (1.2)$$

where $\cosh x = \sum_{k=0}^{\infty} (x^{2k}/(2k)!)$ is the hyperbolic cosine of x . Therefore, the polynomials are

$$p_k(x) = \sum_{\nu=0}^k a_{\nu} \frac{x^{k-\nu}}{(k-\nu)!} \cdot \frac{1 + (-1)^{k-\nu}}{2}. \quad (1.3)$$

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Let $C[0, \infty)$ be the set of all real-valued functions continuous on $[0, \infty)$ and $w_p(x) = e^{-px}$, $x \geq 0$, $p > 0$, the weight function. We will work in the space of functions $C_p = \{f \in C[0, \infty) : w_p f \text{ is uniformly continuous and bounded on } [0, \infty)\}$, with the norm $\|f\|_p = \sup_{x \in [0, \infty)} w_p(x) |f(x)|$.

We define the operator $P_n : C_p \rightarrow C_r$, $r > p$, by the relation

$$P_n(f; x) = \frac{1}{g(1) \cosh(nx)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right). \quad (1.4)$$

We consider that $a_n/g(1) \geq 0$, $n = 0, 1, \dots$, which implies that the P_n operator is positive.

We proved in [1] the following theorem.

THEOREM 1.1. *If $f \in C_p$, then for each $x \geq 0$, $\lim_{n \rightarrow \infty} P_n(f; x) = f(x)$, the convergence being uniform in each interval $[0, a]$.*

Remark 1.2. (1) If in (1.2) we consider $g(u) = \cosh u$, the operator P_n becomes

$$L_n(f; x) = \frac{1}{\cosh 1 \cosh(nx)} \sum_{k=0}^{\infty} p_{2k}(nx) f\left(\frac{2k}{n}\right), \quad (1.5)$$

where

$$p_{2k}(x) = \frac{(1+x)^{2k} + (1-x)^{2k}}{2(2k)!}, \quad (1.6)$$

which was studied in [2].

(2) If instead of (1.2) we consider the relation

$$\cosh(ux) = \sum_{k=0}^{\infty} p_k(x) u^k, \quad (1.7)$$

we obtain

$$p_{2k}(x) = \frac{x^{2k}}{(2k)!}. \quad (1.8)$$

The operator

$$L_n^*(f; x) = \frac{1}{\cosh(nx)} \sum_{k=0}^{\infty} \frac{(nx)^{2k}}{(2k)!} f\left(\frac{2k}{n}\right) \quad (1.9)$$

was studied by Leśniewicz and Rempulska [3].

2. Auxiliary results

In order to prove a Voronovskaya-type theorem, we need some auxiliary results.

LEMMA 2.1. For $x \in [0, \infty)$ and $n \in \mathbb{N}$,

$$\begin{aligned} P_n(e_0; x) &= 1, \\ P_n(e_1; x) &= x \tanh(nx) + \frac{1}{n} \cdot \frac{g'(1)}{g(1)}, \\ P_n(e_2; x) &= x^2 + \frac{x}{n} \tanh(nx) \frac{2g'(1) + g(1)}{g(1)} + \frac{1}{n^2} \cdot \frac{g''(1) + g'(1)}{g(1)}, \end{aligned} \quad (2.1)$$

where $e_i(x) = x^i$, $i \in \{0, 1, 2\}$, and $\tanh x$ is the hyperbolic tangent of x .

LEMMA 2.2. For $x \in [0, \infty)$ and $n \in \mathbb{N}$, the following hold:

$$\begin{aligned} P_n(t - x; x) &= -x(1 - \tanh(nx)) + \frac{1}{n} \cdot \frac{g'(1)}{g(1)}, \\ P_n((t - x)^2; x) &= (1 - \tanh(nx)) \left[2x^2 - \frac{x}{n} \left(1 + \frac{2g'(1)}{g(1)} \right) \right] + \frac{x}{n} + \frac{1}{n^2} \cdot \frac{g''(1) + g'(1)}{g(1)}, \\ P_n((t - x)^4; x) &= (1 - \tanh(nx)) \left(a_1 x^4 - a_2 \frac{x^3}{n} + a_3 \frac{x^2}{n^2} - a_4 \frac{x}{n^3} \right) + a_5 \frac{x^2}{n^2} + a_6 \frac{x}{n^3} + a_7 \frac{1}{n^4}, \end{aligned} \quad (2.2)$$

where a_i , $i = \overline{1, 7}$, are positive constants:

$$\begin{aligned} a_1 &= 8, \quad a_2 = 12 + \frac{16g'(1)}{g(1)}, \quad a_3 = 4 \left(1 + \frac{6g'(1) + 3g''(1)}{g(1)} \right), \\ a_4 &= 1 + \frac{14g'(1) + 18g''(1) + 4g^{(3)}(1)}{g(1)}, \quad a_5 = 3, \\ a_6 &= 1 + \frac{6g''(1) + 10g'(1)}{g(1)}, \quad a_7 = \frac{g'(1) + 7g''(1) + 6g^{(3)}(1) + g^{(4)}(1)}{g(1)}. \end{aligned} \quad (2.3)$$

Lemmas 2.1 and 2.2 can be proved by means of successive partial differentiation with respect to u in the generating relation (1.2), and putting then $u = 1$.

LEMMA 2.3. For every fixed point $x_0 \in [0, \infty)$,

$$\lim_{n \rightarrow \infty} nP_n(t - x_0; x_0) = \frac{g'(1)}{g(1)}, \quad \lim_{n \rightarrow \infty} nP_n((t - x_0)^2; x_0) = x_0. \quad (2.4)$$

Proof. Because $1 - \tanh(nx) = 2/(e^{2nx} + 1)$, by Lemma 2.2 we have

$$\begin{aligned} nP_n(t - x_0; x_0) &= \frac{-2nx_0}{e^{2nx_0} + 1} + \frac{g'(1)}{g(1)}, \\ nP_n((t - x_0)^2; x_0) &= \frac{2nx_0}{e^{2nx_0} + 1} \left[2x_0^2 - \frac{x_0}{n} \left(1 + 2 \frac{g'(1)}{g(1)} \right) \right] + x_0 + \frac{1}{n} \cdot \frac{g''(1) + g'(1)}{g(1)}. \end{aligned} \quad (2.5)$$

Therefore Lemma 2.3 holds. $\square$

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LEMMA 2.4. *For each fixed point $x_0 \in [0, \infty)$, there is a positive constant $M_1(x_0)$, depending only on x_0 such that*

$$P_n((t - x_0)^4; x_0) \leq M_1(x_0) \frac{1}{n^2} \quad (2.6)$$

for all $n \in \mathbb{N}$.

Proof. For $x \geq 0$ and $r, n \in \mathbb{N}$, we have

$$x^r (1 - \tanh(nx)) \leq \frac{2^{1-r}}{n^r} r!. \quad (2.7)$$

By Lemma 2.2, it results that

$$\begin{aligned} P_n((t - x_0)^4; x_0) &\leq a_1 \frac{2^{-3}}{n^4} 4! - a_2 \frac{2^{-2}}{n^4} 3! + f_3 \frac{2^{-1}}{n^4} 2! \\ &\quad - a_4 \frac{1}{n^4} + a_5 \frac{x_0^2}{n^2} + a_6 \frac{x_0}{n^3} + a_7 \frac{1}{n^4} \leq M_1(x_0) \frac{1}{n^2}. \end{aligned} \quad (2.8)$$

We proved in [1] the following lemma. □

LEMMA 2.5. *Let $p > 0$, let $r > p$, and let n_0 be a natural number such that $n_0 > p/(\ln r - \ln p)$. Then there exists a positive constant $M_{p,r}$ depending only on p and r such that*

$$e^{-rx} (P_n(t - x)^2 e^{pt}; x) \leq M_{p,r} \frac{x+1}{n} \quad (2.9)$$

for all $x \geq 0$ and $n \geq n_0$.

LEMMA 2.6. *Let $x_0 \in [0, \infty)$ be a fixed point and $\varphi(\cdot; x_0) \in C_p$ a function such that*

$$\lim_{t \rightarrow x_0} \varphi(t; x_0) = 0. \quad (2.10)$$

Then

$$\lim_{n \rightarrow \infty} P_n(\varphi(t; x_0); x_0) = 0. \quad (2.11)$$

Proof. Let $r > p > 0$. For every fixed $x_0 \geq 0$ and $n \in \mathbb{N}$, we have

$$e^{-rx_0} P_n(\varphi(t; x_0); x_0) = \frac{e^{-rx_0}}{g(1) \cosh(nx)} \sum_{k=0}^{\infty} p_k(nx_0) \varphi\left(\frac{k}{n}; x_0\right). \quad (2.12)$$

By the properties of function $\varphi(\cdot; x_0)$, it results that for all $\varepsilon > 0$ there exists a positive constant $\delta(\varepsilon)$ such that if $|t - x_0| < \delta$, then $|\varphi(t; x_0)| < \varepsilon/2$, $t \geq 0$. Moreover, there exists a positive constant $M_2 \equiv M_2(p)$ such that

$$e^{-pt} |\varphi(t; x_0)| \leq M_2 \quad \forall t \geq 0. \quad (2.13)$$

Now we can write

$$\begin{aligned}
 e^{-rx_0} P_n(\varphi(t; x_0); x_0) &\leq \frac{e^{-rx_0}}{g(1) \cosh(nx_0)} \sum_{|k/n - x_0| < \delta} p_k((nx_0)) \left| \varphi\left(\frac{k}{n}; x_0\right) \right| \\
 &+ \frac{e^{-rx_0}}{g(1) \cosh nx_0} \sum_{|k/n - x_0| \geq \delta} p_k(nx_0) \left| \varphi\left(\frac{k}{n}; x_0\right) \right| := S_1 + S_2.
 \end{aligned} \tag{2.14}$$

By the above properties of function $\varphi(\cdot; x_0)$, it follows that

$$\begin{aligned}
 S_1 &< \frac{\varepsilon}{2} \cdot \frac{e^{-rx_0}}{g(1) \cosh(nx_0)} \sum_{|k/n - x_0| < \delta} p_k(nx_0) < \frac{\varepsilon}{2} e^{-rx_0} P_n(1, x_0) < \frac{\varepsilon}{2}, \\
 S_2 &= \frac{e^{-rx_0}}{g(1) \cosh(nx_0)} \sum_{|k/n - x_0| \geq \delta} p_k((nx_0)) \left| \varphi\left(\frac{k}{n}; x_0\right) \right| e^{-pk/n} e^{pk/n} \\
 &\leq M_2 \frac{e^{-rx_0}}{g(1) \cosh nx_0} \sum_{|k/n - x_0| \geq \delta} p_k(nx_0) e^{pk/n}.
 \end{aligned} \tag{2.15}$$

But if

$$\left| \frac{k}{n} - x_0 \right| \geq \delta, \tag{2.16}$$

then

$$1 \leq \frac{1}{\delta^2} \left(\frac{k}{n} - x_0 \right)^2, \tag{2.17}$$

and by Lemma 2.5, we can write

$$\begin{aligned}
 S_2 &\leq M_2 \frac{1}{\delta^2} \cdot \frac{e^{-rx_0}}{g(1) \cosh(nx_0)} \sum_{|k/n - x_0| \geq \delta} p_k(nx_0) \left(\frac{k}{n} - x_0 \right)^2 e^{pk/n} \\
 &\leq M_2 \frac{e^{-rx_0}}{\delta^2} P_n((t - x_0)^2 e^{pt}; x_0) \leq M_2 \frac{1}{\delta^2} M_{p,r} \frac{x_0 + 1}{n}
 \end{aligned} \tag{2.18}$$

for $n \geq n_0$, $n_0 > p/(\ln r - \ln p)$. It results that for a fixed x_0 , ε , δ there exists a natural number $n_0 = n_0(x_0, \varepsilon, \delta, M_2, p, r)$ such that for all $n > n_0$, we have $S_2 < \varepsilon/2$.

Therefore, for all $n > n_0$, we have

$$e^{-rx_0} P_n(\varphi(t; x_0); x_0) < \varepsilon \quad (\text{i.e., } \lim_{n \rightarrow \infty} e^{-rx_0} P_n(\varphi(t; x_0); x_0) = 0). \tag{2.19}$$

It results that $\lim_{n \rightarrow \infty} P_n(\varphi(t; x_0); x_0) = 0$. □

3. A Voronovskaya-type theorem

Now we are in the position to state the main result of this paper.

For a fixed $p > 0$, let

$$C_p^2 = \{f \in C_p \text{ such that } f', f'' \in C_p\}. \tag{3.1}$$

THEOREM 3.1. *If $f \in C_p^2$, then*

$$\lim_{n \rightarrow \infty} n\{P_n(f; x) - f(x)\} = \frac{x}{2} f''(x) + f'(x) \frac{g'(1)}{g(1)} \quad (3.2)$$

for every fixed $x \in [0, \infty)$.

Proof. We use the Taylor formula for a fixed point $x_0 \in [0, \infty)$. For all $t \in [0, \infty)$, we have

$$f(t) = f(x_0) + (t - x_0) f'(x_0) + \frac{1}{2} (t - x_0)^2 f''(x_0) + g(t; x_0) (t - x_0)^2, \quad (3.3)$$

where $g(t; x_0)$ is the Peano form of the remainder, $g(\cdot; x_0) \in C_p$, and $\lim_{t \rightarrow x_0} g(t; x_0) = 0$. Because $P_n(e_0; x) = 1$, we can write

$$\begin{aligned} P_n(f; x_0) - f(x_0) &= f'(x_0) P_n(t - x_0; x_0) + \frac{1}{2} f''(x_0) P_n((t - x_0)^2; x_0) \\ &\quad + P_n(g(t; x_0) (t - x_0)^2; x_0). \end{aligned} \quad (3.4)$$

By Cauchy's inequality, we have

$$P_n(g(t; x_0) (t - x_0)^2; x_0) \leq \{P_n(g^2(t; x_0); x_0)\}^{1/2} \{P_n((t - x_0)^4; x_0)\}^{1/2}. \quad (3.5)$$

The function $\varphi(t; x_0) = g^2(t; x_0)$, $t \geq 0$, satisfies the conditions of Lemma 2.6; therefore

$$\lim_{n \rightarrow \infty} P_n(g^2(t; x_0); x_0) = 0. \quad (3.6)$$

Moreover, by Lemma 2.4, we have

$$nP_n(g(t; x_0) (t - x_0)^2; x_0) \leq \{P_n(g^2(t; x_0); x_0)\}^{1/2} \left(n^2 M_1(x_0) \frac{1}{n^2} \right)^{1/2}. \quad (3.7)$$

It results that $\lim_{n \rightarrow \infty} nP_n(g(t; x_0) (t - x_0)^2; x_0) = 0$. By the above results and by Lemma 2.3, we obtain

$$\lim_{n \rightarrow \infty} n(P_n(f; x_0) - f(x_0)) = f'(x_0) \frac{g'(1)}{g(1)} + \frac{x_0}{2} f''(x_0). \quad (3.8)$$

□

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