LAGUERRE-TYPE BELL POLYNOMIALS

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We develop an extension of the classical Bell polynomials introducing the Laguerre-type version of this well-known mathematical tool. The Laguerre-type Bell polynomials are useful in order to compute the \(n\)th Laguerre-type derivatives of a composite function. Incidentally, we generalize a result considered by L. Carlitz in order to obtain explicit relationships between Bessel functions and generalized hypergeometric functions.

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1. Introduction

The Bell polynomials [1] appear in different frameworks. They are often used in combinatorial analysis [20], and even in statistics [14], although without explicit references. Moreover these polynomials have been applied even in many other contexts, such as the Blissard problem (see [20, page 46]), the representation of Lucas polynomials of the first and second kinds [4, 9], the representation formulas of Newton sum rules for polynomials’ zeros [12, 13], the recurrence relations for a class of Freud-type polynomials [3], the representation of symmetric functions of a countable set of numbers, generalizing the classical algebraic Newton-Girard formulas [15]. Consequently they were also used [6] in order to find reduction formulas for the orthogonal invariants of a strictly positive compact operator, deriving in a simple way the so-called Robert formulas [21].

Some generalized forms of Bell polynomials already appeared in literature (see, e.g., [11, 17, 19]). A generalization of the Bell polynomials suitable for the differentiation of multivariable composite functions can also be found in [18]. Lastly, in [2], the so-called multidimensional Bell polynomials of higher order were introduced, which are suitable for representing the derivative of a composite function of several (say \(m\)) variables \(f(q^{(1)}(t), q^{(2)}(t), \ldots, q^{(m)}(t))\), where \(q^{(i)}(t) = \phi^{(i,1)}(\phi^{(i,2)}(\ldots \phi^{(i,n)}(t)))\), \(i = 1, 2, \ldots, m\).

In this article we find explicit representation formulas for the \(n\)th Laguerre-type derivatives of a composite function. The case of the first Laguerre derivative \(DxD\), \(D := d/dx\) is essentially related to an article by Carlitz [5], originated by a preceding paper by Lardner [16] in which the powers \((DxD)^n\) of this derivative appear.
2 Laguerre-type Bell polynomials

2. Recalling the Bell polynomials

We recall that the Bell polynomials are a classical mathematical tool for representing the \( n \)th derivative of a composite function. In fact by considering the composite function \( \Phi(t) := f(g(t)) \) of functions \( x = g(t) \) and \( y = f(x) \) defined in suitable intervals of the real axis and \( n \) times differentiable with respect to the relevant independent variables and by using the following notations:

\[
\Phi_h := D^h_t \Phi(t), \quad f_h := D^h_x f(x)|_{x=g(t)}, \quad g_h := D^h_t g(t),
\]

\[
([f,g]_n) := (f_1,g_1; f_2,g_2; \cdots; f_n,g_n),
\]

they are defined as follows:

\[
Y_n([f,g]_n) := \Phi_n.
\]  

(2.2)

For example one has

\[
Y_1([f,g]_1) = f_1g_1,
\]

\[
Y_2([f,g]_2) = f_1g_2 + f_2g_1^2,
\]

\[
Y_3([f,g]_3) = f_1g_3 + f_2(3g_2g_1) + f_3g_1^3.
\]  

(2.3)

Further examples can be found in [20, page 49].

Inductively, we can write

\[
Y_n([f,g]_n) = \sum_{k=1}^{n} A_{n,k}(g_1,g_2,\ldots,g_n) f_k,
\]

(2.4)

where the coefficient \( A_{n,k} \), for any \( k = 1, \ldots, n \), is a polynomial in \( g_1,g_2,\ldots,g_n \), homogeneous of degree \( k \) and isobaric of weight \( n \) (i.e., it is a linear combination of monomials \( g_1^{k_1}g_2^{k_2}\cdots g_n^{k_n} \) whose weight is constantly given by \( k_1 + 2k_2 + \cdots + nk_n = n \)).

For them the following result holds true.

Proposition 2.1. The Bell polynomials satisfy the recurrence relation

\[
Y_0([f,g]_0) := f_1,
\]

\[
Y_{n+1}([f,g]_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}([f_1,g]_{n-k}) g_{k+1},
\]

(2.5)
where

\[
([f_1, g]_{n-k}) := (f_2, g_1; f_3, g_2; \ldots; f_{n-k+1}, g_{n-k}).
\]

(2.6)

An explicit expression for the Bell polynomials is also given by the Faà di Bruno formula [10]:

\[
\Phi_n = Y_n([f, g]_n) = \sum_{\pi(n)} \frac{n!}{j_1! j_2! \cdots j_n!} [\frac{g_1}{1!}]^{j_1} [\frac{g_2}{2!}]^{j_2} \cdots [\frac{g_n}{n!}]^{j_n},
\]

(2.7)

where the sum runs over all partitions \(\pi(n)\) of the integer \(n\), that is, \(n = j_1 + 2j_2 + \cdots + nj_n\). In (2.7) \(j_h\) denotes the number of parts of size \(h\), and \(j = j_1 + j_2 + \cdots + j_n\) denotes the number of parts of the considered partition. A proof of the Faà di Bruno formula can be found in [20]. In [22] the proof is based on the umbral calculus (see [23] and the references therein).

3. Laguerre-type derivatives

The Laguerre-type derivatives were introduced in [7, 8] in connection with a differential isomorphism denoted by the symbol \(\mathcal{T} := \mathcal{T}_x\), acting onto the space \(\mathcal{A} := \mathcal{A}_x\) of analytic functions of the \(x\) variable by means of the correspondence

\[
D := \frac{d}{dx} \longrightarrow \hat{D}_L := DxD; \quad x \cdot \longrightarrow \hat{D}_x^{-1},
\]

(3.1)

where

\[
\hat{D}_x^{-1} f(x) := \int_0^x f(\xi)d\xi,
\]

\[
\hat{D}_x^{-n} f(x) := \frac{1}{(n-1)!} \int_0^x (x - \xi)^{n-1} f(\xi)d\xi,
\]

(3.2)

so that

\[
\mathcal{T}_x(x^n) = \hat{D}_x^{-n}(1) := \frac{1}{(n-1)!} \int_0^x (x - \xi)^{n-1} d\xi = \frac{x^n}{n!}.
\]

(3.3)

According to this isomorphism, the exponential operator \(e^x\) is transformed into the first Laguerre-type exponential \(e_1(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2}\) which is an eigenfunction of the Laguerre derivative operator \(D_L := DxD\). We have, in fact,

\[
\mathcal{T}_x(e^x) = \sum_{k=0}^{\infty} \frac{\mathcal{T}_x(x^k)}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2} = e_1(x),
\]

(3.4)

\[
\hat{D}_Le_1(ax) = ae_1(ax), \quad \forall a \in \mathbb{C}.
\]
This result can be generalized by considering the $r$th Laguerre-type exponential $e_r(x) := \sum_{k=0}^{\infty} x^k / (k!)^r$, the $r$th Laguerre-type derivative operator $D_{L} := DxDx \cdots Dx$ (containing $r + 1$ ordinary derivatives), and the iterated isomorphism $\mathcal{T}^r$, since

$$\mathcal{T}_x^r(e^x) = \sum_{k=0}^{\infty} \mathcal{T}_x(x^k) / (k!)^r = \sum_{k=0}^{\infty} x^k / (k!)^{r+1} = e_r(x),$$

(3.5)

$$\hat{D}_r e_r(ax) = a e_r(ax), \quad \forall a \in \mathbb{C}.\quad \text{(3.6)}$$

**Remark 3.1.** The above results show that, for every positive integer $r$, we can define a Laguerre-type exponential function $e_r(x)$, satisfying an eigenfunction property, which is an analog of the elementary property of the exponential. This function reduces to the exponential function when $r = 0$, so that we can put by definition

$$e_0(x) := e^x, \quad \hat{D}_0 := D.\quad \text{(3.6)}$$

Obviously, $\hat{D}_{1L} := \hat{D}_L$.

For this reason we will refer to such functions as $L$-exponential functions, or shortly $L$-exponentials.

### 4. Laguerre-type Bell polynomials

The problem of constructing Bell polynomials can be extended in the natural way to the case of the Laguerre-type derivatives.

To this aim, by using notations in (2.1), we introduce the following definition.

**Definition 4.1.** The $n$th Laguerre-type Bell polynomial, denoted by $r_L Y_n(x; [f, g])_n$, represents the $n$th $r_L$-Laguerre-type derivative of the composite function $f(g(t))$.

We will show that $r_L Y_n$ can be expressed as a polynomial in the independent variable $x$, depending on $f_1, g_1; f_2, g_2; \ldots; f_n, g_n$ in terms of the classical Bell polynomials.

We start noting that, according to a general result due to Viskov [24], the Laguerre derivative satisfy

$$(D_L)^n = (DxD)^n = D^n x^n D^n,$$ (4.1)

and furthermore, for any order $r$, it turns out that

$$(D_{rL})^n = (DxDx \cdots Dx)^n = D^n x^n D^n x^n \cdots D^n x^n D^n.\quad \text{(4.2)}$$

According to the above equations, the proof of Carlitz [5] can be reduced to a simple application of the Leibnitz rule, since

$$(DxD)^n = D^n (x^n D^n) = \sum_{k=0}^{n} \binom{n}{k} D^{n-k} x^n D^{n+k}$$

$$= \sum_{k=0}^{n} \left( \begin{pmatrix} n \\ k \end{pmatrix} \right)^2 (n-k)! x^k D^{n+k} = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n}{k} x^k D^{n+k}.\quad \text{(4.3)}$$
Therefore, the following representation formula for the Laguerre-type Bell polynomials, denoted by $L Y_n$, holds true.

**Theorem 4.2.** The $L Y_n$ polynomials are expressed in terms of the ordinary Bell polynomials according to the equation

$$L Y_n(x; [f, g]_n) = \sum_{k=0}^{n} \frac{n!}{k!} \left( \begin{array}{c} n \\ k \end{array} \right) x^{k} Y_{n+k}([f, g]_{n+k}).$$

(4.4)

The above results can be easily generalized, since

$$(D_{2L})^n = (DxDx)^n = D^n x^n (D^n x^n D^n)$$

$$= \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \frac{n!}{k_1! (k_1 + k_2)!} \left( \begin{array}{c} n \\ k_1 \end{array} \right) \left( \begin{array}{c} n \\ k_2 \end{array} \right) x^{k_1+k_2} D^{n+k_1+k_2}.$$  

(4.5)

5. The general case

The following result follows by induction.

**Theorem 5.1.** The powers of the $r$th Laguerre-type derivative operator $D_{rL} := DxDx \cdots DxD$ (containing $r + 1$ ordinary derivatives) can be expanded in the form

$$(D_{rL})^n = (DxDx \cdots DxD)^n = D^n x^n D^n x^n \cdots D^n x^n D^n$$

$$= \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \cdots \sum_{k_r=0}^{n} \frac{n!}{k_1! (k_1 + k_2)! \cdots (k_1 + k_2 + \cdots + k_r)!} \left( \begin{array}{c} n \\ k_1 \end{array} \right) \left( \begin{array}{c} n \\ k_2 \end{array} \right) \cdots \left( \begin{array}{c} n \\ k_r \end{array} \right) x^{k_1+k_2+\cdots+k_r} D^{n+k_1+k_2+\cdots+k_r}.$$  

(5.1)

Therefore, for the $r$th Laguerre-type Bell polynomials denoted by $rL Y_n$, the following result holds true.

**Theorem 5.2.** The $rL Y_n$ polynomials are expressed in terms of the ordinary Bell polynomials according to the equation

$$rL Y_n(x; [f, g]_n) = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \cdots \sum_{k_r=0}^{n} \frac{n!}{k_1! (k_1 + k_2)! \cdots (k_1 + k_2 + \cdots + k_r)!} \left( \begin{array}{c} n \\ k_1 \end{array} \right) \left( \begin{array}{c} n \\ k_2 \end{array} \right) \cdots \left( \begin{array}{c} n \\ k_r \end{array} \right) x^{k_1+k_2+\cdots+k_r} Y_{n+k_1+k_2+\cdots+k_r}([f, g]_{n+k_1+k_2+\cdots+k_r}).$$  

(5.2)
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References


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