The paper deals with the existence of solutions of some parabolic bilateral problems approximated by the renormalized solutions of some parabolic equations.

Copyright © 2006 Hindawi Publishing Corporation. All rights reserved.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ and $T > 0$. We denote by $Q$ the cylinder $\Omega \times (0, T)$ and $\Gamma = \partial Q$.

Let

$$A(u) = -\text{div}(a(x, t, u, \nabla u))$$  \hspace{1cm} (1.1)

be a Leray-Lions operator acting on $L^p(0, T; W^{1,p}_0(\Omega))$, $1 < p < \infty$, into its dual $L^{p'}(0, T; W^{-1,p'}(\Omega))(1/p + 1/p' = 1)$. Consider the following parabolic problem:

$$u \in \mathcal{H} = \{ v \in L^p(0, T; W^{1,p}_0(\Omega)) : v(t) \in K \text{ a.e.} \},$$

$$\int_0^T \left\langle \frac{\partial v}{\partial t}, u - v \right\rangle dt + \int_\Omega a(x, t, u, \nabla u)(\nabla u - \nabla v) dx dt \leq \int_0^T \langle f, u - v \rangle dt,$$

$$\forall v \in \mathcal{H} \cap \left\{ v \in L^p(0, T; W^{1,p}_0(\Omega)) : \frac{\partial v}{\partial t} \in L^{p'}(0, T; W^{-1,p'}(\Omega)); v(0) = 0 \right\},$$

where $K$ is a given convex in $W^{1,p}_0(\Omega)$ and $f \in L^{p'}(0, T; W^{-1,p'}(\Omega))$.

It is well known that (P) admits at least one solution via a classical penalty method (see Lions [5] for $p \geq 2$ and Landes-Mustonen [4] for $1 < p < 2$). Recently in [6], the authors
2 Parabolic inequalities in $L^1$

approximated $(P)$ by the following sequence of parabolic equations:

$$\frac{\partial u_n}{\partial t} + A(u_n) + \|h(x,u_n)\|_{\mathcal{L}^1}^{n-1}h(x,u_n)\|G(x,t,u_n,\nabla u_n)\| = f \quad \text{in } Q,$$

$$u_n(x,t) = 0 \quad \text{on } \partial Q,$$

$$u_n(x,0) = 0 \quad \text{in } \Omega,$$

where $h$ and $G$ are two Carathéodory functions satisfying some natural growth conditions. The obtained convex $K$ depends on two obstacles constructed from $h$.

In the $L^1$ case, that is, $f \in L^1(\Omega \times [0,T])$, the formulations $(P)$ and $(P_n)$ are not appropriate. So, we introduce the renormalized problem $(R_n)$ associated to $(P_n)$ (see the definition below). The study of the asymptotic behavior of $(R_n)$ as $n \to \infty$ leads to some bilateral parabolic problem. Our approach allows us also to prove the existence of solutions for general parabolic inequalities of type

$$T_k(u) \in \mathcal{K},$$

$$\int_0^T \left\langle \frac{\partial v}{\partial t}, T_k(u-v) \right\rangle dt + \int_Q a(x,t,u,\nabla u)\nabla T_k(u-v) dx dt$$

$$+ \int_Q H(x,t,u,\nabla u)T_k(u-v) dx dt \leq \int_Q fT_k(u-v) dx dt, \quad \forall v \in \mathcal{K} \cap D \cap L^\infty(Q),$$

where $D = \{ v \in L^p(0,T;W_0^{1,p}(\Omega)) : \partial v/\partial t \in L^{p'}(0,T,W_0^{-1,p'}(\Omega)) + L^1(Q), \ v(0) = 0 \}$ and where $H$ is a given Carathéodory function satisfying some natural growth assumption.

For some recent and classical results for some parabolic inequalities problems, the reader can refer to [2, 7, 9, 10].

2. Main result

Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$ and $1 < p < +\infty$.

We denote by $Q$ the cylinder $\Omega \times (0,T)$ and $\Gamma = \partial Q$.

Let $A(u) = -\text{div}(a(x,t,\nabla u))$ be a Leray-Lions operator defined on $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$, where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $x \in \Omega$, for all $t \in \mathbb{R}$ and for all $\zeta, \zeta' \in \mathbb{R}^N$, ($\zeta \neq \zeta'$) the following hold:

$$|a(x,t,\zeta)| \leq \beta(k(x,t) + |\zeta|^{p-1}),$$

$$(a(x,t,\zeta) - a(x,t,\zeta'))(\zeta - \zeta') > 0, \quad \alpha > 0, \quad \beta > 0, \quad k \in L^{p'}(Q).$$

Furthermore, let $h : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$h(x,0) = 0, \quad h(x,s) \text{ is nondecreasing with respect to } s.$$
$G$ is a Carathéodory function satisfying the following assumptions:

$$
|G(x,t,s,ξ)| \leq b(|s|)(c(x,t) + |ξ|^p), \quad G(x,t,s,0) = 0, \quad (2.3)
$$

$$
\{v \in L^p(0,T;W^{1,p}_0(\Omega)) : G(x,t,v,\nabla v) = 0 \text{ a.e. in } Q\}
\subset \{v \in L^p(0,T;W^{1,p}_0(\Omega)) : |h(x,v)| \leq 1 \text{ a.e. in } Q\}. \quad (2.4)
$$

Let us suppose

for almost $x \in \Omega \setminus \Omega_+^\infty$ there exists $\epsilon = \epsilon(x) > 0$ such that

$$
h(x,s) > 1, \quad \forall s \in [q_+(x),q_+(x) + \epsilon[), \quad (2.5)
$$

for almost $x \in \Omega \setminus \Omega_-^\infty$ there exists $\epsilon = \epsilon(x) > 0$ such that

$$
h(x,s) < -1, \quad \forall s \in [q_-(x) - \epsilon,q_-(x)[, \quad (2.5)
$$

where $b$ is a continuous nondecreasing function and $c(x,t) \in L^1(Q)$, $c \geq 0$, and

$$
q_+(x) = \inf \{s > 0, h(x,s) \geq 1\},
$$

$$
q_-(x) = \sup \{s > 0, h(x,s) \leq -1\},
$$

$$
\Omega_+^\infty = \{x \in \Omega : q_+(x) = +\infty\},
$$

$$
\Omega_-^\infty = \{x \in \Omega : q_-(x) = -\infty\}.
$$

We define for all $s$ and $k$ in $\mathbb{R}$, $k \geq 0$, $T_k(s) = \max(-k,\min(k,s))$.

We will say that $u_n$ is a renormalized solution of $(P_n)$ if

$$
T_k(u_n) \in L^p(0,T;W^{1,p}_0(\Omega)), \quad \forall k > 0,
$$

$$
\lim_{h \to \infty} \int_{h \leq |u_n| \leq h+1} a(x,t,\nabla u_n) \nabla u_n \, dx \, dt = 0,
$$

$u_n$ satisfies in the distributional sense

$$
(A(u_n))_t - \text{div} \left( a(x,t,\nabla u_n) A'(u_n) \right) + a(x,t,\nabla u_n) \nabla u_n A''(u_n)
$$

$$
+ |h(x,u_n)|^{p-1} \left( h(x,u_n) \right) |G(x,t,u_n,\nabla u_n) A'(u_n) = f A'(u_n), \quad (R_n)
$$

$$
\forall A \in C^1(\mathbb{R}), A',A'' \in L^\infty(\Omega), \quad A' \text{ has a compact support and } u_n \text{ satisfies}
$$

the initial condition in the sense that $A(u_n) \in C([0,T],L^1(\Omega))$.

Thanks to [8, Theorem 3.2, page 164], there exists at least one solution $u_n$ of $(R_n)$.

**Theorem 2.1.** Under the hypotheses (2.1)–(2.5), $f \in L^1(Q)$, the problem $(P_n)$ has at least one renormalized solution $(u_n)$ such that

$$
T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in } L^p(0,T;W^{1,p}_0(\Omega)), \quad (2.7)
$$
where $u$ is a solution of the following obstacle problem:

$$q_-(x) \leq u(x,t) \leq q_+(x) \quad \text{a.e.} \ (x,t) \in Q,$$

$$T_k(u) \in L^p(0,T; W^{1,p}_0(\Omega)),$$

\[
\int_0^T \left\langle \frac{\partial v}{\partial t}, T_k(u - v) \right\rangle dt + \int_Q a(x,t,\nabla u) \nabla T_k(u - v) dx \, dt \\
\leq \int_Q f T_k(u - v) \, dx \, dt, \quad \forall v \in \mathcal{H} \cap D \cap L^\infty(Q),
\]

where

\[
D = \left\{ v \in L^p(0,T; W^{1,p}_0(\Omega)), \frac{\partial v}{\partial t} \in L^{p'}(0,T; W^{-1,p'}(\Omega)) + L^1(Q), v(0) = 0 \right\},
\]

\[
\mathcal{H} = \{ v \in L^p(0,T; W^{1,p}_0(\Omega)) \mid v(t) \in K \}, \quad K = \{ v \in W^{1,p}_0(\Omega), q_- \leq v \leq q_+ \}.
\]

Moreover, if $q_-, q_+ \in L^\infty(\Omega)$, then $u \in L^p(0,T; W^{1,p}_0(\Omega)) \cap L^\infty(Q)$.

Remark 2.2. The same result can be obtained when dealing with general operator of Leray-Lions type depending also on $u$, that is, $A(u) = -\text{div}(a(x,t,u,\nabla u))$.

Proof of Theorem 2.1.

Step 1. Let $A(t) = H_m(t)$, $H_m(t) = \int_0^t h_m(s) \, ds$, where

\[
h_m(s) = \begin{cases} 
1 & \text{if } |s| \leq m, \\
\text{affine} & \text{if } m \leq |s| \leq m+1, \\
0 & \text{if } m+1 \leq |s|. 
\end{cases}
\]

Taking now $T_k(H_m(u_n))$ as test function in $(R_n)$, we obtain

\[
\int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, T_k(H_m(u_n)) \right\rangle dt + \int_{|H_m(u_n)| < k} a(x,t,\nabla u_n) \nabla u_n h_m^2(u_n) \, dx \, dt \\
+ \int_Q |h(x,u_n)|^{n-1} h(x,u_n) |G(x,t,u_n,\nabla u_n)| h_m(u_n) T_k(H_m(u_n)) \, dx \, dt \\
+ \int_Q a(\cdot,t,\nabla u_n) \nabla u_n h_m(u_n) T_k(H_m(u_n)) \, dx \, dt = \int_Q f h_m(u_n) T_k(H_m(u_n)) \, dx \, dt.
\]

Since

\[
\int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, T_k(H_m(u_n)) \right\rangle dt = \int_\Omega \left( \int_0^{H_m(u_n(x,T))} T_k(s) \, ds \right) dx - \int_\Omega \left( \int_0^{H_m(u_n(x,0))} T_k(s) \, ds \right) dx
\]
and by using the fact that \( \int_{\Omega} (\int_0^{H_m(u_n(x,T))} T_k(s) ds) \geq 0 \), we obtain

\[
\int_{|H_m(u_n)| < k} a(x, t, \nabla u_n) \nabla u_n h_m^2(u_n) \, dx \, dt \leq C k + \int_{\{m \leq |u_n| \leq m+1\}} a(x, t, \nabla u_n) \nabla u_n \, dx \, dt,
\]

\[
\int_{Q} |h(x, u_n)|^{n-1} h(x, u_n) |G(x, t, u_n, \nabla u_n)| h_m(u_n) T_k(H_m(u_n)) \, dx \, dt \leq C k + \int_{\{m \leq |u_n| \leq m+1\}} a(x, t, \nabla u_n) \nabla u_n \, dx \, dt.
\]  

(2.12)

We have \( H_m(s) \) (resp., \( h_m(s) \)) tends to \( s \) (resp., to 1) as \( m \) goes to \( +\infty \).

Using Fatou's lemma and the definition of the renormalized solution leads to

\[
\int_{Q} |\nabla T_k(u_n)|^p \, dx \, dt \leq C k, \quad (2.13)
\]

\[
\int_{Q} |h(x, u_n)|^{n-1} h(x, u_n) |G(x, t, u_n, \nabla u_n)| T_k(u_n) \, dx \, dt \leq C k, \quad (2.14)
\]

which gives

\[
\int_{Q} |h(x, u_n)|^n |G(x, t, u_n, \nabla u_n)| \frac{|T_k(u_n)|}{k} \, dx \, dt \leq C, \quad (2.15)
\]

and as \( k \to 0 \) we obtain

\[
\int_{Q} |h(x, u_n)|^n |G(x, t, u_n, \nabla u_n)| \, dx \, dt \leq C. \quad (2.16)
\]

Choosing now a \( C^2 \) function \( \rho_k \), such that \( \rho_k(s) = s \) for \( |s| \leq k \) and \( 2k \text{sign}(s) \) for \( |s| > 2k \), we get

\[
(\rho_k(u_n))_t - \text{div} (a(x, t, \nabla u_n) \rho_k'(u_n)) + a(x, t, \nabla u_n) \nabla u_n \rho_k''(u_n) + |h(x, u_n)|^{n-1} h(x, u_n) |G(x, t, u_n, \nabla u_n)| \rho_k'(u_n) = \rho_k(u_n). \quad (2.17)
\]

We deduce that \( (\rho_k(u_n))_t \) is bounded in \( L^1(Q) + L^p(0, T; W^{-1,p'}(\Omega)) \).

Now thanks to the following result.

**Lemma 2.3 [11].** Let \( p > 1 \). If \( (u_n) \) is a bounded sequence of \( L^p(0, T; W^{1,p}_0(\Omega)) \) such that \( \partial u_n/\partial t \) is bounded in \( L^1 + L^p(0, T; W^{-1,p'}(\Omega)) \), then \( u_n \) is relatively compact in \( L^p(Q) \).
6 Parabolic inequalities in $L^1$

We deduce that $\rho_k(u_n)$ is relatively compact in $L^p(Q)$ and so there exists a measurable function $u$ such that $u_n \rightharpoonup u$ a.e. in $Q$.

Finally, we deduce from (2.13) that $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $L^p(0, T; W^{1,p}_0(\Omega))$, and strongly in $L^p(Q)$.

Step 2. We are dealing now with the almost convergence of the gradient.

We have to prove that, for $0 < \theta < 1$,

$$\lim_{n \to \infty} \int_Q \left( [a(x, t, \nabla T_k(u_n)) - a(x, t, \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] \right)^\theta dx \, dt = 0. \quad (2.18)$$

Let $\omega \in L^p(0, T; W^{1,p}_0(\Omega))$, we define for any $\mu > 0$, $\omega_\mu$ the time regularization of $\omega$,

$$\omega_\mu(x, t) = \mu \int_{-\infty}^t \overline{\omega}(x, s) \exp(\mu(s - t)) \, ds, \quad (2.19)$$

where $\overline{\omega}$ is the zero extension of $\omega$ for $s > T$. Furthermore, $\omega_\mu$ satisfies the following properties (see [3]):

$$\omega_\mu \rightharpoonup \omega \quad \text{strongly in } L^p(0, T; W^{1,p}_0(\Omega)), \quad (2.20)$$

where $\overline{\omega}$ is the zero extension of $\omega$ for $s > T$. Furthermore, $\omega_\mu$ satisfies the following properties (see [3]):

$$\omega_\mu \rightharpoonup \omega \quad \text{strongly in } L^p(0, T; W^{1,p}_0(\Omega)), \quad (2.20)$$

Letting $\eta > 0$, we obtain

$$\int_Q \left( [a(x, t, \nabla T_k(u_n)) - a(x, t, \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] \right)^\theta dx \, dt \leq C \text{meas} \{ | T_k(u_n) - T_k(u)_\mu | \geq \eta \}^{1-\theta}$$

$$+ C \left( \int_{\{|T_k(u_n) - T_k(u)_\mu| < \eta\}} [a(x, t, \nabla T_k(u_n))] - a(x, t, \nabla T_k(u)) \right)^\theta. \quad (2.21)$$
On the other hand, we have

\[
\int_{\{|T_k(u_n) - T_k(u)| < \eta\}} \left[ a(x, t, \nabla T_k(u_n)) - a(x, t, \nabla T_k(u)) \right] \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] dx \, dt \\
\leq \int_{\{|T_k(u_n) - T_k(u)| < \eta\}} \left[ a(x, t, \nabla T_k(u_n)) - a(x, t, \nabla T_k(u)) \right] \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] dx \, dt \\
+ \int_{\{|T_k(u_n) - T_k(u)| < \eta\}} a(x, t, \nabla T_k(u_n)) (\nabla T_k(u) - \nabla T_k(u)) dx \, dt \\
+ \int_{\{|T_k(u_n) - T_k(u)| < \eta\}} \left[ a(x, t, \nabla T_k(u)) - a(x, t, \nabla T_k(u)) \right] \nabla T_k(u_n) dx \, dt \\
- \int_{\{|T_k(u_n) - T_k(u)| < \eta\}} a(x, t, \nabla T_k(u)) \nabla T_k(u) dx \, dt \\
+ \int_{\{|T_k(u_n) - T_k(u)| < \eta\}} a(x, t, \nabla T_k(u)) \nabla T_k(u) dx \, dt \\
\leq I_1 + I_2 + I_3 + I_4 + I_5.
\]

Take \( T_\eta(H_m(u_n) - T_k(u)) \) as test function in \((R_n)\) with \( A(t) = H_m(t) \). We obtain

\[
\int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, T_\eta(H_m(u_n) - T_k(u)) \right\rangle \, dt \\
+ \int_{\{|H_m(u_n) - T_k(u)| < \eta\}} a(x, t, \nabla u_n) \nabla u_n h_m^2(u_n) - \nabla T_k(u) h_m(u_n) dx \, dt \\
+ \int_Q |h(x, u_n)|^{n-1} h(x, u_n) \left| G(x, t, u_n, \nabla u_n) \right| h_m(u_n) T_\eta(H_m(u_n) - T_k(u)) dx \, dt \\
+ \int_Q a(x, t, \nabla u_n) \nabla u_n h_m(u_n) T_\eta(H_m(u_n) - T_k(u)) dx \, dt \\
= \int_Q f h_m(u_n) T_\eta(H_m(u_n) - T_k(u)) dx \, dt.
\]

We have

\[
\int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, T_\eta(H_m(u_n) - T_k(u)) \right\rangle \, dt \\
= \int_0^T \left\langle \frac{\partial H_m(u_n) - T_k(u)}{\partial t}, T_k(H_m(u_n) - T_k(u)) \right\rangle \, dt \\
+ \int_0^T \left\langle \frac{\partial T_k(u)}{\partial t}, T_k(H_m(u_n) - T_k(u)) \right\rangle \, dt.
\]
Using the fact that
\[ \int_{T_0}^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, T_\eta(H_m(u_n) - T_k(u)_\mu) \right\rangle dt \geq 0, \]
\[ \int_{T_0}^T \left\langle \frac{\partial T_k(u)_\mu}{\partial t}, T_k(H_m(u_n) - T_k(u)_\mu) \right\rangle dt \]
\[ = \mu \int_Q (T_k(u) - T_k(u)_\mu) T_\eta(T_k(H_m(u_n)) - T_k(u)_\mu) dxdt, \]
consequently,
\[ \limsup_{n \to \infty} \limsup_{m \to \infty} \int_{T_0}^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, T_\eta(H_m(u_n) - T_k(u)_\mu) \right\rangle dt \]
\[ \geq \mu \int_Q (T_k(u) - T_k(u)_\mu) T_\eta(u - T_k(u)_\mu) dxdt = \epsilon(m, n) \geq 0. \]

This implies that
\[ \int_{\{H_m(u_n) - T_k(u)_\mu \leq \eta\}} a(x, t, \nabla u_n) \nabla u_n h_m^2(u_n) - \nabla T_k(u)_\mu h_m(u_n) dxdt \]
\[ + \int_Q |h(x, u_n)|^{n-1} h(x, u_n) |G(x, t, u_n, \nabla u_n) | h_m(u_n) T_\eta(H_m(u_n) - T_k(u)_\mu) dxdt \]
\[ + \int_Q a(x, t, \nabla u_n) \nabla u_n h_m'(u_n) T_\eta(H_m(u_n) - T_k(u)_\mu) dxdt \]
\[ \leq \int_Q f_{h_m(u_n)} T_\eta(H_m(u_n) - T_k(u)_\mu) dxdt + \epsilon(m, n), \]

which gives by using the fact that
\[ \int_Q |h(x, u_n)|^{n-1} h(x, u_n) |G(x, t, u_n, \nabla u_n) | h_m(u_n) T_\eta(H_m(u_n) - T_k(u)_\mu) dxdt \leq C_\eta, \]
\[ \int_{\{H_m(u_n) - T_k(u)_\mu \leq \eta\}} a(x, t, \nabla u_n) \nabla u_n h_m^2(u_n) - \nabla T_k(u)_\mu h_m(u_n) dxdt \]
\[ \leq C_\eta + \epsilon(m, n) + \eta \int_{|u_n| \leq m+1} a(x, t, \nabla u_n) \nabla u_n dxdt, \]

which gives as \( m \to \infty, \)
\[ \int_{\{u_n - T_k(u)_\mu \leq \eta\}} a(x, t, \nabla u_n) \nabla u_n - \nabla T_k(u)_\mu dxdt \leq C_\eta + \epsilon(n). \]
Finally from (2.22),
\[ |I_1| \leq C\eta + \epsilon(n) - \int_{|Tk(u_n) - Tk(u)| < \eta} a(x,t, \nabla T_k(u)) \left( \nabla T_k(u_n) - \nabla T_k(u) \right). \tag{2.30} \]

Since \( a(x,t, \nabla T_k(u)) \chi_{|[Tk(u_n) - Tk(u)| < \eta]} \rightarrow a(x,t, \nabla T_k(u)) \chi_{|[Tk(u) - Tk(u)| < \eta]} \) in \( L^p(Q) \) and \( T_k(u_n) \rightarrow T_k(u) \) weakly in \( L^p(0,T; W^{1,\theta}_0(\Omega)) \), then
\[ - \int_{|Tk(u_n) - Tk(u)| < \eta} a(x,t, \nabla T_k(u)) \left( \nabla T_k(u_n) - \nabla T_k(u) \right) dx \, dt = - \int_{|Tk(u) - Tk(u)| < \eta} a(x,t, \nabla T_k(u)) \left( \nabla T_k(u) - \nabla T_k(u) \right) dx \, dt + \epsilon(n). \tag{2.31} \]
So
\[ |I_1| \leq C\eta + \epsilon(n). \tag{2.32} \]

For what concerns the term \( I_2 \), one has
\[ I_2 = \epsilon(n,\mu), \tag{2.33} \]

since
\[ a(x,t, \nabla T_k(u_n)) \chi_{|[Tk(u_n) - Tk(u)| < \eta]} \rightarrow a(x,t, \nabla T_k(u)) \chi_{|[Tk(u) - Tk(u)| < \eta]} \text{ in } (L^p(Q))^N, \]
\[ \left( \nabla T_k(u_n) - \nabla T_k(u) \right) \chi_{|[Tk(u_n) - Tk(u)| < \eta]} \rightarrow \left( \nabla T_k(u) - \nabla T_k(u) \right) \chi_{|[Tk(u) - Tk(u)| < \eta]} \] \tag{2.34}

In the same way, we show that
\[ I_3 = \epsilon(n,\mu), \quad I_4 = \epsilon(n,\mu), \quad I_5 = \epsilon(n,\mu). \tag{2.35} \]

Combining the above estimates, we get
\[ \lim_{n \to \infty} \int_Q \left( [a(x,t, \nabla T_k(u_n))] - a(x,t, \nabla T_k(u)) \right) \left[ \nabla T_k(u_n) - \nabla T_k(u) \right]^\theta dx \, dt = 0. \tag{2.36} \]

Then there exists a subsequence also denoted by \( (u_n) \) such that
\[ \nabla u_n \rightharpoonup \nabla u \quad \text{a.e. in } Q. \tag{2.37} \]

**Step 3.** From (2.16), we deduce that
\[ \int_Q |h(x,u_n)|^n |G(x,t,u_n,\nabla u_n)| \, dx \, dt \leq C, \tag{2.38} \]
which gives for every \( \beta > 0 \),
\[ \int_{|h(x,T_\beta(u_n))| > k} |G(x,t,T_\beta(u_n),\nabla T_\beta(u_n))| \, dx \, dt \leq \frac{C}{k^n}, \tag{2.39} \]
10 Parabolic inequalities in $L^1$

where $k > 1$. Letting $n \to +\infty$ for $k$ fixed, we deduce by using Fatou's lemma

$$\int_{|h(x, T_\beta(u))| > k} |G(x, t, T_\beta(u), \nabla T_\beta(u))| \, dx \, dt = 0, \quad (2.40)$$

and so, by (2.4)

$$|h(x, T_\beta(u))| \leq 1 \quad \text{a.e. in } Q. \quad (2.41)$$

So

$$q_-(x) \leq T_\beta(u(x)) \leq q_+(x) \quad \text{a.e. in } Q. \quad (2.42)$$

Letting now $\beta \to +\infty$, we deduce also that

$$q_-(x) \leq u(x) \leq q_+(x) \quad \text{a.e. in } Q. \quad (2.43)$$

Step 4. Strong convergence of the truncations.

We will prove that

$$\lim_{n \to \infty} \int_Q \left[ a(x, t, \nabla T_k(u_n)) - a(x, t, \nabla T_k(u)) \right] \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] \, dx \, dt = 0. \quad (2.44)$$

Fix $k > 0$ and let $\varphi(s) = \exp(\delta s^2)$, $\delta > 0$. Let $l > k$ and define the function $R_l(s) = \int_0^s \rho_t(t) \, dt$. Let us consider $\omega^m u = T_k(H_m(u))$, where $\varphi$ is the mollification with respect to time $t$. Letting $v^{m,n}_\nu = \rho_t(H_m(u_n)) \varphi(T_k(H_m(u_n)) - \omega^m)$ as test function in the problem $(R_n)$, we get

$$\int_0^T \left( \frac{\partial H_m(u_n)}{\partial t} - \rho_t(H_m(u_n)) \varphi(T_k(H_m(u_n)) - \omega^m_u) \right) \, dt$$

$$+ \int_Q a(x, t, \nabla u_n) \nabla u_n h^2(\varphi(t)) \rho_t(H_m(u_n)) \varphi(T_k(H_m(u_n)) - \omega^m_u) \, dx \, dt$$

$$+ \int_Q a(x, t, \nabla u_n) \nabla T_k(H_m(u_n)) - \nabla \omega^m_u$$

$$\times h_m(u_n) \rho_t(H_m(u_n)) \varphi'(T_k(H_m(u_n)) - \omega^m_u) \, dx \, dt$$

$$+ \int_Q a(x, t, \nabla u_n) \nabla u_n h'_m(\varphi(t)) \rho_t(H_m(u_n)) \varphi(T_k(H_m(u_n)) - \omega^m_u) \, dx \, dt$$

$$+ \int_Q |h(x, u_n)|^{n-1} h(x, u_n) |G(x, t, u_n, \nabla u_n)|$$

$$\times h_m(u_n) \rho_t(H_m(u_n)) \varphi(T_k(H_m(u_n)) - \omega^m_u) \, dx \, dt$$

$$= \int_Q f v^{m,n}_\nu h_m(u_n) \, dx \, dt. \quad (2.45)$$

We deal now with the estimate of each term of the last equalities.
Since $H_m(u_n) \in L^p(0,T; W^{-1,p}(\Omega))$ and $\partial H_m(u_n)/\partial t \in L^{p'}(0,T; W^{-1,1}^p(\Omega)) + L^1(Q)$, there exists a smooth function $H_m(u_n)_\sigma$ such that as $\sigma \to 0$,

$$H_m(u_n)_\sigma \to H_m(u_n) \quad \text{strongly in } L^p(0,T; W^{-1,p}_0(\Omega)), \quad \frac{\partial H_m(u_n)_\sigma}{\partial t} \to \frac{\partial H_m(u_n)}{\partial t} \quad \text{strongly in } L^{p'}(0,T; W^{-1,1}^p(\Omega)) + L^1(Q).$$  \hfill (2.46)

This implies that

$$I = \int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, \rho_i(H_m(u_n)) \varphi(T_k(H_m(u_n)) - \omega_\mu^m) \right\rangle dt$$

$$= \lim_{\sigma \to 0} \int_Q \left[ R_i(H_m(u_n)_\sigma) - T_k(H_m(u_n)_\sigma) \right] \varphi(T_k(H_m(u_n)_\sigma) - \omega_\mu^m) dx dt$$

$$= \lim_{\sigma \to 0} \int_Q \left[ R_i(H_m(u_n)_\sigma) - T_k(H_m(u_n)_\sigma) \right] \varphi(T_k(H_m(u_n)_\sigma) - \omega_\mu^m) dx dt$$

$$+ \int_Q \left[ T_k(H_m(u_n)_\sigma) \right] \varphi(T_k(H_m(u_n)_\sigma) - \omega_\mu^m) dx dt$$

$$= \lim_{\sigma \to 0} \left\{ \int_\Omega \left[ R_i(H_m(u_n)_\sigma) - T_k(H_m(u_n)_\sigma) \varphi(T_k(H_m(u_n)_\sigma) - \omega_\mu^m) \right] dx dt$$

$$- \int_Q \left[ R_i(H_m(u_n)_\sigma) - T_k(H_m(u_n)_\sigma) \right] \varphi'(T_k(H_m(u_n)_\sigma) - \omega_\mu^m)(T_k(H_m(u_n)_\sigma)$$

$$- \omega_\mu^m)' dx dt + \int_Q \left[ T_k(H_m(u_n)_\sigma) \right] \varphi(T_k(H_m(u_n)_\sigma) - \omega_\mu^m) dx dt \right\}$$

$$= \lim_{\sigma \to 0} \{ I_1(\sigma) + I_2(\sigma) + I_3(\sigma) \}. \hfill (2.47)$$

Observe that for $|s| \leq k$ we have $R_i(s) = T_k(s) = s$ and for $|s| > k$ we have $|R_i(s)| \geq |T_k(s)|$ and, since both $R_i(s)$ and $T_k(s)$ have the same sign of $s$, we deduce that sign$(s)(R_i(s) - T_k(s)) \geq 0$. Consequently,

$$I_1(\sigma) = \int_{\{|H_m(u_n)_\sigma| > k\}} \left[ R_i(H_m(u_n)_\sigma) - T_k(H_m(u_n)_\sigma) \varphi(T_k(H_m(u_n)_\sigma) - \omega_\mu^m) \right] dx dt \geq 0. \hfill (2.48)$$
We have, since $(R_l(s) - T_k(s))(T_k(s))' = 0$, for all $s$,

$$I_2(\sigma) = \int_{\{ |H_m(u_n)_{\sigma}| > k \}} \left[ R_l(H_m(u_n)_{\sigma}) - T_k(H_m(u_n)_{\sigma}) \varphi'(T_k(H_m(u_n)_{\sigma})) \right.$$

$$\left. - \omega^m_{\mu} \right] (\omega^m_{\mu})' \, dx \, dt = \mu \int_{\{ |H_m(u_n)_{\sigma}| > k \}} \left[ R_l(H_m(u_n)_{\sigma}) - T_k(H_m(u_n)_{\sigma}) \varphi'(T_k(H_m(u_n)_{\sigma})) \right.$$

$$\left. - \omega^m_{\mu} \right] (T_k(H_m(u_n)_{\sigma}) - \omega^m_{\mu}) \, dx \, dt,$$

by using the fact that $\varphi' \geq 0$ and that

$$(R_l(H_m(u_n)_{\sigma}) - T_k(H_m(u_n)_{\sigma}))(T_k(H_m(u_n)_{\sigma}) - \omega^m_{\mu}) \chi_{\{ |H_m(u_n)_{\sigma}| > k \}} \geq 0,$$

the last integral is of the form $\epsilon(m,n)$. We deduce that

$$\lim_{\sigma \to 0^+} \sup I_2(\sigma) \geq \epsilon(m,n).$$

For $I_3(\sigma)$, one has

$$I_3(\sigma) = \int_Q \left[ T_k(H_m(u_n)_{\sigma}) \right]' \varphi(T_k(H_m(u_n)_{\sigma}) - \omega^m_{\mu}) \, dx \, dt$$

$$= \int_Q \left[ T_k(H_m(u_n)_{\sigma}) - \omega^m_{\mu} \right]' \varphi(T_k(H_m(u_n)_{\sigma}) - \omega^m_{\mu}) \, dx \, dt$$

$$+ \int_Q (\omega^m_{\mu})' \varphi(T_k(H_m(u_n)_{\sigma}) - \omega^m_{\mu}) \, dx \, dt.$$

Let $\Phi(s) = \int^s_0 \varphi(t) \, dt$. Remark that $(T_k(H_m(u_n)_{\sigma}) - \omega^m_{\mu}) \varphi(T_k(H_m(u_n)_{\sigma}) - \omega^m_{\mu}) \geq 0$.

Integrating by parts, using the fact that $\Phi \geq 0$, and following the same way as above, we have

$$\lim_{\sigma \to 0^+} \sup I_3(\sigma) \geq \epsilon(m,n).$$

Combining these estimates, we conclude that

$$\int_0^T \left< \frac{\partial H_m(u_n)}{\partial t}, p_l(H_m(u_n)) \varphi(T_k(H_m(u_n)) - \omega^m_{\mu}) \right> \, dt \geq \epsilon(m,n).$$
We set

\[ I_4(m) = \int_Q \left| h(x, u_n) \right|^{n-1} h(x, u_n) \rho_l(H_m(u_n)) \varphi(T_k(H_m(u_n)) \omega_m^m) | G(x, t, u_n, \nabla u_n)|, \]

so we have

\[ \limsup_{m \to \infty} I_4(m) \geq I_4^1 + I_4^2, \quad (2.56) \]

where

\[ I_4^1 = \int_{\{ |u_n| < k, 0 \leq u_n \leq T_k(u)_\mu \}} \left| h(x, u_n) \right|^{n-1} h(x, u_n) \varphi(T_k(u_n) - T_k(u)_\mu) \rho_l(u_n) | G(x, t, u_n, \nabla u_n)| \, dx \, dt, \]

\[ I_4^2 = \int_{\{ |u_n| < k, T_k(u)_\mu \leq u_n \leq 0 \}} \left| h(x, u_n) \right|^{n-1} h(x, u_n) \varphi(T_k(u_n) - T_k(u)_\mu) \rho_l(u_n) | G(x, t, u_n, \nabla u_n)| \, dx \, dt. \]

Since \( q_- \leq T_k(u)_\mu \leq q_+ \) (recall that \( q_- \leq T_k(u) \leq q_+ \)) and \( 0 \leq \rho_l(u_n) \leq 1 \), one easily has

\[ |I_4^1| \leq \int_{\{ |u_n| < k \}} c(x, t) \left| \varphi(T_k(u_n) - T_k(u)_\mu) \right| \]

\[ + \frac{b(k)}{a} \int_{\{ |u_n| < k \}} |\nabla u_n|^p | \varphi(T_k(u_n) - T_k(u)_\mu)| \]

\[ \leq \frac{b(k)}{a} \int_Q \left[ a(x, t, \nabla T_k(u_n)) - a(x, t, \nabla T_k(u)_\mu) \right] \left[ \nabla T_k(u_n) - \nabla T_k(u)_\mu \right] \]

\[ \times | \varphi(T_k(u_n) - T_k(u)_\mu) | \, dx \, dt + \epsilon(n, \mu), \]

and also we have the same estimation of \( I_4^2 \).

Then

\[ |I_4^1| + |I_4^2| \leq 2 \frac{b(k)}{a} \int_Q \left[ a(x, t, \nabla T_k(u_n)) - a(x, t, \nabla T_k(u)_\mu) \right] \left[ \nabla T_k(u_n) - \nabla T_k(u)_\mu \right] \]

\[ \times | \varphi(T_k(u_n) - T_k(u)_\mu) | \, dx \, dt + \epsilon(n, \mu). \]

(2.59)
14 Parabolic inequalities in $L^1$

By denoting by $J_1$ the third term of (2.45), one can write

\[ J_1 = \int_Q a(x,t,\nabla u_n)(\nabla T_k(H_m(u_n)) - \nabla T_k(H_m(u))_\mu) \]
\[ \times h_m(u_n)\rho_l(H_m(u_n))\varphi'(T_k(H_m(u_n)) - T_k(H_m(u))_\mu) dx \, dt \]
\[ = \int_Q a(x,t,\nabla T_k(u_n))(\nabla T_k(H_m(u_n)) - \nabla T_k(H_m(u))_\mu) h_m(u_n) \]
\[ \times \rho_l(H_m(u_n))\varphi'(T_k(H_m(u_n)) - T_k(H_m(u))_\mu) dx \, dt \]
\[ + \int_{|u_n|>k} a(x,t,\nabla u_n)(\nabla T_k(H_m(u_n)) - \nabla T_k(H_m(u))_\mu) h_m(u_n) \]
\[ \times \rho_l(H_m(u_n))\varphi'(T_k(H_m(u_n)) - T_k(H_m(u))_\mu) dx \, dt, \] (2.60)

\[ J_1 = \int_Q a(x,t,\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(u)_\mu)\rho_l(u_n)\varphi'(T_k(u_n) - T_k(u)_\mu) dx \, dt \]
\[ - \int_{|u_n|>k} \rho_l(u_n)a(x,t,\nabla u_n)(\nabla T_k(u)_\mu)\varphi'(T_k(u_n) - T_k(u)_\mu) dx \, dt + \epsilon(m). \]

Since $a(x,t,\nabla u_n)\rho_l(u_n)$ is bounded in $L^p(Q)$, we deduce that

\[ a(x,t,\nabla u_n)\rho_l(u_n) \rightharpoonup a(x,t,\nabla u)\rho_l(u) \quad \text{weakly in} \ L^p(Q), \] (2.61)

and so

\[ J_1 = \int_Q (a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u)_\mu))(\nabla T_k(u_n) - \nabla T_k(u)_\mu) \]
\[ \times \varphi'(T_k(u_n) - T_k(u)_\mu) dx \, dt + \epsilon(m,n,\mu). \] (2.62)

Concerning the second term of (2.45), one easily has

\[ \int_Q a(x,t,\nabla u_n)\nabla u_n h_m^2(u_n)\rho_l(H_m(u_n))\varphi(T_k(H_m(u_n)) - \omega^m) dx \, dt \]
\[ \leq \varphi(2k) \int_{|l|<|u_n|<l+1} a(x,t,\nabla u_n)\nabla u_n dx \, dt, \] (2.63)

and since

\[ \int_{|l|<|u_n|<l+1} a(x,t,\nabla u_n)\nabla u_n dx \, dt \leq \int_{|u_n|>l} |f| dx \, dt, \] (2.64)
we deduce that

\[
\left| \int_Q a(x,t,\nabla u_n) \nabla u_n h_m^2(u_n) \rho'(H_m(u_n)) \varphi(T_k(H_m(u_n)) - \omega_k^{\alpha}) \, dx \right| \leq \varphi(2k) \int_{|u_n| > l} |f| \, dx \, dt = \epsilon(n,l).
\]  

(2.65)

The same result can be obtained for the fourth term of (2.45).

Combining (2.45)–(2.65), using the fact that \( \phi' - 2(b(k)/\alpha)|\phi| \geq 1/2 \) for \( \delta \geq (b(k)/\alpha)^2 \), we deduce that

\[
\lim_{n \to \infty} \int_Q [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)] \, dx \, dt = 0.
\]  

(2.66)

On the other hand, we have

\[
\int_Q [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)] \, dx \, dt
\]

\[
- \int_Q [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] \, dx \, dt
\]

\[
= \int_Q [a(x,t,\nabla T_k(u_n)) (\nabla T_k(u) - \nabla T_k(u))] \, dx \, dt
\]

\[
- \int_Q [a(x,t,\nabla T_k(u)) (\nabla T_k(u_n) - \nabla T_k(u))] \, dx \, dt
\]

\[
+ \int_Q [a(x,t,\nabla T_k(u))] (\nabla T_k(u_n) - \nabla T_k(u)) \, dx \, dt = \epsilon(n,k).
\]  

(2.67)

Consequently by [1, Lemma 5], we obtain

\[
T_k(u_n) \to T_k(u) \quad \text{strongly in } L^p(0,T; W^{1,\rho}_0(\Omega)) \quad \text{for every } k > 0.
\]  

(2.68)

Step 5 (passage to the limit). Letting \( v \in D \cap \mathcal{H} \cap L^\infty(Q) \), and using \( T_k(H_m(u_n) - \theta v) \) as test function in the problem \((R_n)\), we obtain

\[
\int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, T_k(H_m(u_n) - \theta v) \right\rangle \, dt + \int_Q a(x,t,\nabla u_n) \nabla T_k(H_m(u_n) - \theta v) h_m(u_n) \, dx \, dt
\]

\[
+ \int_Q a(x,t,\nabla u_n) \nabla u_n T_k(H_m(u_n) - \theta v) h_m'(u_n) \, dx \, dt
\]

\[
+ \int_Q |h(x,u_n)|^{-1} h(x,u_n) |G(x,t,u_n,\nabla u_n) | h_m(u_n) T_k(H_m(u_n) - \theta v) \, dx \, dt
\]

\[
\leq \int_Q f T_k(H_m(u_n) - \theta v) h_m(u_n) \, dx \, dt.
\]  

(2.69)
We have

\[
\int_Q \left| h(x, u_n) \right|^{n-1} h(x, u_n) \left( G(x, t, u_n, \nabla u_n) \right) h_m(u_n) T_k(H_m(u_n) - \theta v) \, dx \, dt \\
\geq \int_{\{0 \leq H_m(u_n) \leq \theta v\}} \left| h(x, u_n) \right|^{n-1} \\
\times h(x, u_n) \left( G(x, t, u_n, \nabla u_n) \right) h_m(u_n) T_k(H_m(u_n) - \theta v) \, dx \, dt \\
+ \int_{\{\theta v \leq H_m(u_n) \leq 0\}} \left| h(x, u_n) \right|^{n-1} \\
\times h(x, u_n) \left( G(x, t, u_n, \nabla u_n) \right) h_m(u_n) T_k(H_m(u_n) - \theta v) \, dx \, dt.
\]

(2.70)

Now we deal with the estimation of the last two terms in the right-hand side of the last inequality which we denote, respectively, by \( J'_1(m, n) \) and \( J'_2(m, n) \). Let us define

\[
\delta_1(x, t) = \sup_{0 \leq s \leq \theta v} h(x, s),
\]

(2.71)

then we get \( 0 \leq \delta_1(x, t) < 1 \) a.e. in \( Q \).

We have

\[
\limsup_{m \to \infty} \left| J'_1(m, n) \right| \leq k \int_{\{0 \leq u_n \leq \theta v\}} (\delta(x, t))'' (c(x, t) + |\nabla u_n|^p) \\
\leq \int_{\{|u_n| \leq \|u\|_{\infty}\}} (\delta(x, t))'' (c(x, t) + |\nabla u_n|^p),
\]

(2.72)

and by using the strong convergence of \( T_{\|v\|_\infty}(u_n) \) in \( L^p(0, T; W^{1,p}_0(\Omega)) \), we deduce that

\[
\limsup_{n \to \infty} \limsup_{m \to \infty} \left| J'_1(m, n) \right| = 0,
\]

(2.73)

with the same technique (taking \( \delta_2(x, t) = \inf_{\theta v \leq s \leq 0} h(x, s) \)), we can see that

\[
\limsup_{m \to \infty} \left| J'_2(n, m) \right| \to 0 \quad \text{as} \quad n \to +\infty.
\]

(2.74)

On the other hand,

\[
\int_Q a(x, t, \nabla u_n) \nabla T_k(H_m(u_n) - \theta v) h_m(u_n) \, dx \, dt \\
= \int_Q a(x, t, \nabla u_n) \nabla (H_m(u_n) - \theta v) \chi_{\{H_m(u_n) - \theta v \leq k\}} h_m(u_n) \, dx \, dt \\
= \int_Q (a(x, t, \nabla u_n) - a(x, t, \theta \nabla v)) \nabla (H_m(u_n) - \theta v) \chi_{\{H_m(u_n) - \theta v \leq k\}} h_m(u_n) \, dx \, dt \\
+ \int_Q a(x, t, \theta \nabla v) \nabla (H_m(u_n) - \theta v) \chi_{\{H_m(u_n) - \theta v \leq k\}} h_m(u_n) \, dx \, dt.
\]

(2.75)
Since \( a(x, t, \theta v) \) belongs to \((L^p(Q))^N\), using Fatou’s lemma in the first term of the last side gives

\[
\liminf_{n,m \to \infty} \int_0^T \langle Au_n, T_k(H_m(u_n) - \theta v) \rangle dt \geq \int_0^T \langle Au, T_k(u) - \theta v \rangle dt.
\] (2.76)

Go back to (2.69) and pass to the limit as \( m, n \to \infty \) to obtain

\[
\int_0^T \langle \theta \frac{\partial v}{\partial t}, T_k(u - \theta \nabla v) \rangle dt + \int_Q a(x, t, \nabla u) \nabla T_k(u - \theta v) dx dt \leq \int_Q f T_k(u - \theta v) dx dt.
\] (2.77)

Letting now \( \theta \) tend to 1, we get

\[
\int_0^T \langle \frac{\partial v}{\partial t}, T_k(u - \theta v) \rangle dt + \int_Q a(x, t, \nabla u) \nabla T_k(u - v) dx dt \leq \int_Q f T_k(u - v) dx dt,
\] (2.78)

which completes the proof. \( \square \)

Remark 2.4. The same technique allows us to prove an existence result for solutions of the following parabolic inequalities:

\[
q_-(x) \leq u(x, t) \leq q_+(x) \quad \text{a.e. in } Q,
\]

\[
T_k(u) \in L^p(0, T; W^{1,p}_0(\Omega)),
\]

\[
\int_0^T \langle \frac{\partial v}{\partial t}, T_k(u - v) \rangle dt + \int_Q a(x, t, \nabla u) \nabla T_k(u - v) dx dt + \int_Q H(x, t, u, \nabla u) T_k(u - v) dx dt \
\leq \int_Q f T_k(u - v) dx dt, \quad \forall v \in H \cap D \cap L^\infty(Q),
\] (2.79)

where \( H \) is a given Carathéodory function satisfying, for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^N\) and a.e. \((x, t) \in Q\), the following conditions:

\[
|H(x, t, s, \xi)| \leq \lambda(|s|) (\delta(x, t) + |\xi|^p),
\]

\[
H(x, t, s, \xi) s \geq 0,
\] (2.80)

with \( \lambda : \mathbb{R}^+ \to \mathbb{R}^+ \) is a continuous increasing function and \( \delta(x, t) \) is a given positive function in \( L^1(Q) \).

Acknowledgments

This work has been supported by LERMA (EMI), LMA (FSR), and Volkswagen Foundation Grant no. 1/79315.
Parabolic inequalities in $L^1$

References


K. Azelmat: LMA, GAN, Département de Mathématiques et d’Informatique, Faculté des Sciences, Université Mohammed V Agdal, Avenue Ibn Battouta, Rabat BP 1014, Morocco

E-mail address: azelmat@fsr.ac.ma

M. Kbiri Alaoui: LERMA, Ecole Mohammadia d’Ingénieurs, Université Mohammed V Agdal, Avenue Ibn Sina, Agdal, Rabat BP 765, Morocco; LMA, GAN, Département de Mathématiques et d’Informatique, Faculté des Sciences, Université Mohammed V Agdal, Avenue Ibn Battouta, Rabat BP 1014, Morocco

E-mail address: mka_la@yahoo.fr

D. Meskine: LERMA, Ecole Mohammadia d’Ingénieurs, Université Mohammed V Agdal, Avenue Ibn Sina, Agdal, Rabat BP 765, Morocco; LMA, GAN, Département de Mathématiques et d’Informatique, Faculté des Sciences, Université Mohammed V Agdal, Avenue Ibn Battouta, Rabat BP 1014, Morocco

E-mail address: driss.meskine@laposte.net

A. Souissi: LMA, GAN, Département de Mathématiques et d’Informatique, Faculté des Sciences, Université Mohammed V Agdal, Avenue Ibn Battouta, Rabat BP 1014, Morocco

E-mail address: souissi@fsr.ac.ma