In this note, using the Hausdorff measure of noncompactness, necessary and sufficient conditions are formulated for a linear operator and matrices between the spaces $c$ and $c_0$ to be compact. Among other things, some results of Cohen and Dunford are recovered.

We will write $s$, $c$, and $c_0$, for the set of all complex, convergent, and null sequences, respectively. Let $A = (a_{nm})_{n,m \geq 1}$ be an infinite matrix and consider the sequence $x = (x_n)_{n \geq 1}$. We will define the product

$$Ax = (Ax)_n = (A_n(x))_{n \geq 1} \quad \text{with} \quad A_n(x) = \sum_{m=1}^{\infty} a_{nm}x_m, \quad n = 1, 2, \ldots,$$

whenever the series are convergent for all $n \geq 1$. For any given subsets $X$, $Y$ of $s$, we will say that the operator represented by the infinite matrix $A = (a_{nm})_{n,m \geq 1}$ maps $X$ into $Y$ that is $A \in (X, Y)$, if

(i) the series defined by $A_n(x) = \sum_{m=1}^{\infty} a_{nm}x_m$ are convergent for all $n \geq 1$ and for all $x \in X$;

(ii) $Ax \in Y$ for all $x \in X$.

If $c \subset c_A = \{x : Ax \in c\}$, $A$ is conservative. Well-known necessary and sufficient conditions for $A$ to be conservative are

$$\|A\| = \sup_{n \geq 1} \sum_{m=1}^{\infty} |a_{nm}| < \infty,$$

$$a_{00} = \lim_{n \to \infty} \sum_{m=1}^{\infty} a_{nm} \quad \text{exists},$$

$$a_{0m} = \lim_{n \to \infty} a_{nm} \quad \text{exists for} \quad m = 1, 2, \ldots.$$
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In this case, (2) is the norm of operator $A$. If $A$ is conservative, then $\chi(A) = \lim_n \sum_{m=1}^{\infty} a_{nm} - \sum_{m=1}^{\infty} a_{0m}$ is called the characteristic of $A$, and in the case $\chi(A) = 0$, $A$ is conull. If $\lim_n (Ax)_n = \lim_n x_n$ for all $x \in c$, then $A$ is called regular. A conservative matrix is regular if and only if $a_{00} = 1$ and $a_{0m} = 0$ for all $m$ [5, 6].

Let $B(c)$ be the set of all bounded linear operators on $c$. It is well known (see [6, Theorem 4.51-D]) that each bounded linear operator $A$ on $c$ into $c$ determines and is determined by a matrix of scalars $a_{nm}$, $n = 1, 2, \ldots, m = 0, 1, 2, \ldots$.

Let $X$, $Y$ be Banach spaces, and let $B(X, Y)$ be the set of all linear bounded operators from $X$ to $Y$. If $Q$ is a bounded subset of $X$, then the Hausdorff measure of noncompactness of $Q$ is denoted by $q(Q)$, and

$$q(Q) = \inf \{ \epsilon > 0 : Q \text{ has a finite } \epsilon - \text{net in } X \}. \quad (8)$$

The function $q$ is called the Hausdorff measure of noncompactness (ball measure of noncompactness); it was introduced by Gohberg et al. [4], later studied by Goldenstein and Markus in 1968, Istrătescu in 1972, and others. Let us point out that the notation of the measure of noncompactness has proved useful results in several areas of functional analysis, operator theory, fixed point theory, differential equations, and so forth (see, e.g., [1, 2, 4]). Let us recall that $q(Q) = 0$ if and only if $Q$ is a totally bounded set. For $A \in B(X, Y)$, the Hausdorff measure of noncompactness of $A$, denoted by $\|A\|_q$, is defined by $\|A\|_q = q(AB_1)$, where $B_1 = \{x \in X : \|x\| \leq 1\}$ is the unit ball in $X$. Hence, $A$ is compact if and only if $\|A\|_q = 0$.

Let us recall that if $X$ is a Banach space with a Schauder basis $\{v_1, v_2, \ldots\}$, $Q$ a bounded subset of $X$, $P_n : X \rightarrow X$ the projector onto the linear span of $\{v_1, v_2, \ldots, v_n\}$, and $\mu(Q) = \limsup_{n \rightarrow \infty} \sup_{x \in Q} \|(I - P_n)x\|$, then the following inequality holds:

$$\frac{1}{a} \mu(Q) \leq q(Q) \leq \inf \sup_{x \in Q} \|(I - P_n)x\| \leq \mu(Q), \quad (9)$$

where $a = \limsup_{n \rightarrow \infty} \|I - P_n\|$ [1, 2, 4].

Now, we can state the following main result.
Theorem 1. Let \( A \in B(c) \), let \( \alpha_{00} \) be as in (7), and let \( a_{0n}, n = 1, 2, \ldots \), be as in (4). Then

\[
\frac{1}{2} \lim_{n \to \infty} \sup \left( |a_{n0} - \alpha_{00} + \sum_{m=1}^{\infty} a_{0m} | + \sum_{m=1}^{\infty} |a_{nm} - a_{0m}| \right)
\leq \|A\|_q \leq \lim_{n \to \infty} \sup \left( |a_{n0} - \alpha_{00} + \sum_{m=1}^{\infty} a_{0m} | + \sum_{m=1}^{\infty} |a_{nm} - a_{0m}| \right).
\] (10)

Proof. Suppose that \( x \in c, \lim_{m \to \infty} x_m = x_0 \) and \( y = Ax \). Now \( y_n = a_{n0}x_0 + \sum_{m=1}^{\infty} a_{nm}x_m, n = 1, 2, \ldots \), and \( \lim_{n} y_n = y_0 \). By [6, page 219, (4.51-11)], (5) and (7),

\[
y_0 = x_0a_{00} + \sum_{m=1}^{\infty} (x_m - x_0)a_{0m} = x_0 \left( \alpha_{00} - \sum_{m=1}^{\infty} a_{0m} \right) + \sum_{m=1}^{\infty} x_m a_{0m}.
\] (11)

The elements \( e = (1,1,1,\ldots) \) and \( e_i = \{\delta_{ij}\}, i = 1, 2, \ldots \) form the Schauder basis in \( c \). Let \( P_n : c \to c \) be the projector defined by

\[
P_n(x) = x_0e + \sum_{i=1}^{n} (x_i - x_0) e_i, \quad n = 1, 2, \ldots
\] (12)

It is easy that \( \|I - P_n\| = 2 \), and by (9) we have

\[
\sup_{\|x\| \leq 1} \| (I - P_n) Ax \| = \sup_{\|x\| \leq 1} \| y_k - y_0 \|
\]

\[
= \sup_{\|x\| \leq 1, k \geq n+1} \left| x_0 \left( a_{k0} - \alpha_{00} + \sum_{m=1}^{\infty} a_{0m} \right) + \sum_{m=1}^{\infty} (a_{km} - a_{0m}) x_m \right|
\]

\[
= \sup_{k \geq n+1} \left( |a_{k0} - \alpha_{00} + \sum_{m=1}^{\infty} a_{0m}| + \sum_{m=1}^{\infty} |a_{km} - a_{0m}| \right).
\] (13)

Now, by (9) and (13), we obtain (10). \( \Box \)

Corollary 2. Let \( A \in B(c) \). Then \( A \) is compact if and only if

\[
\lim_{n \to \infty} \left( |a_{n0} - \alpha_{00} + \sum_{m=1}^{\infty} a_{0m}| + \sum_{m=1}^{\infty} |a_{nm} - a_{0m}| \right) = 0.
\] (14)

Let us recall that if \( A \in B(c), y = Ax \), then \( y_0 = x_0 \) for every choice of \( x \) if and only if \( \alpha_{00} = 1 \) and \( a_{01} = a_{02} = \cdots = 0 \) (see, e.g., [6]); in this case \( A \) is called regular. Now, by Corollary 2, we have the next well-known result of Cohen and Dunford [3, Corollary 3].

Corollary 3. Let \( A \in B(c) \) be regular transformation. Then \( A \) is compact if and only if

\[
\lim_{n \to \infty} \left( |a_{n0} - 1| + \sum_{m=1}^{\infty} |a_{nm}| \right) = 0.
\] (15)
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**Corollary 4.** Let $A \in (c, c)$, let $a_{00}$ be as in (3), and let $a_{0m}, n = 1, 2, \ldots$, be as in (4). Then

$$
\frac{1}{2} \limsup_{n \to \infty} \left( |a_{00} - \sum_{m=1}^{\infty} a_{0m}| + \sum_{m=1}^{\infty} |a_{nm} - a_{0m}| \right) \leq \|A\|_q \leq \limsup_{n \to \infty} \left( |a_{00} - \sum_{m=1}^{\infty} a_{0m}| + \sum_{m=1}^{\infty} |a_{nm} - a_{0m}| \right),
$$

(16)

and $A$ is compact if and only if

$$
\lim_{n \to \infty} \left( |a_{00} - \sum_{m=1}^{\infty} a_{0m}| + \sum_{m=1}^{\infty} |a_{nm} - a_{0m}| \right) = 0.
$$

(17)

Let us remark that Corollary 4 implies that compact conservative matrix is conull.

If we recall the characterizations of the sets $(c, c_0)$ and $(c_0, c_0)$ [5, 6], and remark that in this case the projector $P_n(x) = (x_1, x_2, \ldots, x_n, 0, \ldots)$ maps $c_0$ into $c_0$, and $\|I - P_n\| = 1$, then by the proof of Theorem 1, we have the next result.

**Corollary 5.** If $A \in (c, c_0)$ or $A \in (c_0, c_0)$, then

$$
\|A\|_q = \limsup_{n \to \infty} \sum_{m=1}^{\infty} |a_{nm}|,
$$

(18)

and $A$ is compact if and only if

$$
\lim_{n \to \infty} \sum_{m=1}^{\infty} |a_{nm}| = 0.
$$

(19)

**Corollary 6.** If $A \in (c_0, c)$, then

$$
\frac{1}{2} \limsup_{n \to \infty} \sum_{m=1}^{\infty} |a_{nm} - a_{0m}| \leq \|A\|_q \leq \limsup_{n \to \infty} \sum_{m=1}^{\infty} |a_{nm} - a_{0m}|,
$$

(20)

and $A$ is compact if and only if

$$
\lim_{n \to \infty} \sum_{m=1}^{\infty} |a_{nm} - a_{0m}| = 0.
$$

(21)

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References


Bruno De Malafosse: LMAH Université du Havre, IUT du Havre, BP 4006, 76610 Le Havre, France

E-mail address: bdemalaf@wanadoo.fr

Eberhard Malkowsky: Mathematisches Institut, Universität Giessen, Arndtstrasse 6, D-35392 Giessen, Germany

E-mail addresses: eberhard.malkowsky@math.uni-giessen.de; ema@bankerinter.net

Vladimir Rakočević: Department of Mathematics, Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, 18000 Niš, Serbia and Montenegro

E-mail address: v rakoc@bankerinter.net