We introduce the concepts of lifting modules and (quasi-)discrete modules relative to a given left module. We also introduce the notion of SSRS-modules. It is shown that (1) if $M$ is an amply supplemented module and $0 \to N' \to N \to N'' \to 0$ an exact sequence, then $M$ is $N$-lifting if and only if it is $N'$-lifting and $N''$-lifting; (2) if $M$ is a Noetherian module, then $M$ is lifting if and only if $M$ is $R$-lifting if and only if $M$ is an amply supplemented SSRS-module; and (3) let $M$ be an amply supplemented SSRS-module such that Rad$(M)$ is finitely generated, then $M = K \oplus K'$, where $K$ is a radical module and $K'$ is a lifting module.

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1. Introduction and preliminaries

Extending modules and their generalizations have been studied by many authors (see [2, 3, 8, 7]). The motivation of the present discussion is from [2, 8], where the concepts of extending modules and (quasi-)continuous modules with respect to a given module and CESS-modules were studied, respectively. In this paper, we introduce the concepts of lifting modules and (quasi-)discrete modules relative to a given module and SSRS-modules. It is shown that (1) if $0 \to N' \to N \to N'' \to 0$ is an exact sequence and $M$ an amply supplemented module, then $M$ is $N$-lifting if and only if it is both $N'$-lifting and $N''$-lifting; (2) if $M$ is a Noetherian module, then $M$ is lifting if and only if $M$ is $R$-lifting if and only if $M$ is an amply supplemented SSRS-module; and (3) let $M$ be an amply supplemented SSRS-module such that Rad$(M)$ is finitely generated, then $M = K \oplus K'$, where $K$ is a radical module and $K'$ is a lifting module.

Throughout this paper, $R$ is an associative ring with identity and all modules are unital left $R$-modules. We use $N \leq M$ to indicate that $N$ is a submodule of $M$. As usual, Rad$(M)$ and Soc$(M)$ stand for the Jacobson radical and the socle of a module $M$, respectively.

Let $M$ be a module and $S \leq M$. $S$ is called small in $M$ (notation $S \ll M$) if $M \neq S + T$ for any proper submodule $T$ of $M$. Let $N$ and $L$ be submodules of $M$, $N$ is called a supplement of $L$ in $M$ if $N + L = M$, and $N$ is minimal with respect to this property. Equivalently,
$M = N + L$ and $N \cap L \ll N$. $N$ is called a supplement submodule if $N$ is a supplement of some submodule of $M$. $M$ is called an amply supplemented module if for any two submodules $A$ and $B$ of $M$ with $A + B = M$, $B$ contains a supplement of $A$. $M$ is called a weakly supplemented module (see [5]) if for each submodule $A$ of $M$ there exists a submodule $B$ of $M$ such that $M = A + B$ and $A \cap B \ll M$. Let $B \leq A \leq M$. If $A/B \ll M/B$, then $B$ is called a coessential submodule of $A$ and $A$ is called a coessential extension of $B$ in $M$. A submodule $A$ of $M$ is called coclosed if $A$ has no proper coessential submodules in $M$. Following [5], $B$ is called an $s$-closure of $A$ in $M$ if $B$ is a coessential submodule of $A$ and $B$ is coclosed in $M$.

Let $B \leq A \leq M$. If $A/B \ll M/B$, then $B$ is called a coessential submodule of $A$ and $A$ is called a coessential extension of $B$ in $M$. A submodule $A$ of $M$ is called coclosed if $A$ has no proper coessential submodules in $M$. Following [5], $B$ is called an $s$-closure of $A$ in $M$ if $B$ is a coessential submodule of $A$ and $B$ is coclosed in $M$.

Let $M$ be a module. $M$ is called a lifting module (or satisfies $(D_1)$) (see [9]) if for every submodule $A$ of $M$, there exists a direct summand $K$ of $M$ such that $K \leq A$ and $A/K \ll M/K$, equivalently, $M$ is amply supplemented and every supplement submodule of $M$ is a direct summand. $M$ is called discrete if $M$ is lifting and has the following condition.

$(D_2)$ If $A \leq M$ such that $M/A$ is isomorphic to a direct summand of $M$, then $A$ is a summand of $M$.

$M$ is called quasidiscrete if $M$ is lifting and has the following condition:

$(D_3)$ For each pair of direct summands $A$ and $B$ of $M$ with $A + B = M$, $A \cap B$ is a direct summand of $M$. For more details on these concepts, see [9].

**Lemma 1.1** (see [12, 19.3]). Let $M$ be a module and $K \leq L \leq M$.

1. $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.
2. If $M'$ is a module and $\phi : M \to M'$ a homomorphism, then $\phi(L) \ll M'$ whenever $L \ll M$.

**Lemma 1.2** (see Lemma 1.1 in [5]). Let $M$ be a weakly supplemented module and $N \leq M$. Then the following statements are equivalent.

1. $N$ is a supplement submodule of $M$.
2. $N$ is coclosed in $M$.
3. For all $X \leq N$, $X \ll M$ implies $X \ll N$.

**Lemma 1.3** (see Proposition 1.5 in [5]). Let $M$ be an amply supplemented module. Then every submodule of $M$ has an $s$-closure.

**Lemma 1.4** (see [12, 41.7]). Let $M$ be an amply supplemented module. Then every coclosed submodule of $M$ is amply supplemented.

### 2. Relative lifting modules

To define the concepts of relative lifting and (quasi-)discrete modules, we dualize the concepts of relative extending and (quasi-)continuous modules introduced in [8] in this section. We start with the following.

Let $N$ and $M$ be modules. We define the family

$$
\$(N,M) = \left\{ A \leq M \mid \exists X \leq N, \exists f \in \text{Hom}(X,M), \exists \frac{A}{f(X)} \ll \frac{M}{f(X)} \right\}.
$$

(2.1)
Proposition 2.1. $(N,M)$ is closed under small submodules, isomorphic images, and coessential extensions.

Proof. We only show that $(N,M)$ is closed under coessential extensions. Let $A \in (N,M)$, $A \leq A' \leq M$, and $A'/A \ll M/A$. There exist $X \leq N$ and $f \in \text{Hom}(X,M)$ such that $f(X) \leq A$ and $A/f(X) \ll M/f(X)$ since $A \in (N,M)$. Note that $A'/A \ll M/A$, so $A'/f(X) \ll M/f(X)$ by Lemma 1.1(1). Thus $A' \in (N,M)$. $\square$

Lemma 2.2. Let $A \in (N,M)$ and $A$ be coclosed in $M$. Then $B \in (N,M)$ for any submodule $B$ of $A$.

Proof. There exist $X \leq N$ and $f \in \text{Hom}(X,M)$ such that $f(X) \leq A$ and $A/f(X) \ll M/f(X)$ by hypothesis. Since $A$ is coclosed in $M$, $f(X) = A$. Let $B$ be any submodule of $A$ and $Y = f^{-1}(B) \leq X \leq N$. Then $f|_Y : Y \to M$ is a homomorphism such that $f|_Y(Y) = B$ for $f(X) = A$. Clearly $B/f|_Y(Y) \ll M/f(Y)$. Therefore $B \in (N,M)$. $\square$

Lemma 2.3. Let $C \leq A \leq B \leq M$ and $A$ be a coessential submodule of $B$. If $C$ is an $s$-closure of $A$, then it is also an $s$-closure of $B$.

Proof. It is clear by Lemma 1.1(1). $\square$

Proposition 2.4. Let $M$ be an amply supplemented module. Then every $A$ in $(N,M)$ has an $s$-closure $\bar{A}$ in $(N,M)$.

Proof. Since $A \in (N,M)$, there exist $X \leq N$ and $f \in \text{Hom}(X,M)$ such that $A/f(X) \ll M/f(X)$. Note that $M$ is amply supplemented, and so $f(X)$ has an $s$-closure $\bar{A}$ in $M$ by Lemma 1.3. Thus $\bar{A}$ is also an $s$-closure of $A$ by Lemma 2.3. The verification for $\bar{A} \in (N,M)$ is analogous to that for $B \in (N,M)$ in Lemma 2.2. $\square$

Let $N$ be a module. Consider the following conditions for a module $M$.

$(N,M)$-D$_1$ For every submodule $A \in (N,M)$, there exists a direct summand $K$ of $M$ such that $K \leq A$ and $A/K \ll M/K$.

$(N,M)$-D$_2$ If $A \in (N,M)$ such that $M/A$ is isomorphic to a direct summand of $M$, then $A$ is a direct summand of $M$.

$(N,M)$-D$_3$ If $A$ and $L$ are direct summands of $M$ with $A \in (N,M)$ and $A + L = M$, then $A \cap L$ is a direct summand of $M$.

Definition 2.5. Let $N$ be a module. A module $M$ is said to be $N$-lifting, $N$-discrete, or $N$-quasidiscrete if $M$ satisfies $(N,M)$-D$_1$, $(N,M)$-D$_1$ and $(N,M)$-D$_2$ or $(N,M)$-D$_1$ and $(N,M)$-D$_3$, respectively.

One easily obtains the hierarchy: $M$ is $N$-discrete $\Rightarrow M$ is $N$-quasidiscrete $\Rightarrow M$ is $N$-lifting. Clearly, the notion of relative discreteness generalizes the concept of discreteness. For any module $N$, lifting modules are $N$-lifting. But the converse is not true as shown in the following examples.

Example 2.6. Since, for any module $M$, $(0,M) = \{A \mid A \ll M\}$ and $0$ is a direct summand of $M$ such that $A/0 \ll M/0$ for any $A \in (0,M)$, all modules are $0$-lifting. However, the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ is not lifting since the supplement submodule $\langle (1,2) \rangle$
((1, 2)) is a supplement of ((1, 1)) and is not a direct summand of it though it is amply supplemented.

**Example 2.7.** Let $M$ be a module with zero socle and $S$ a simple module. Then $M$ is $S$-lifting since $\langle S, M \rangle$ is a family only containing all small submodules of $M$. So all torsion-free $\mathbb{Z}$-modules are $S$-lifting for any simple $\mathbb{Z}$-module $S$ (see [12, Exercise 21.17]). In particular, $\mathbb{Z} \mathbb{Z}$ and $\mathbb{Z} \mathbb{Q}$ are $S$-lifting for any simple $\mathbb{Z}$-module, but each one is not a lifting module.

**Lemma 2.8.** Let $M$ be a module. Then $\langle M, M \rangle = \{ A \mid A \leq M \} = \bigcup_{N \in R-\text{Mod}} \langle N, M \rangle$, where $R-\text{Mod}$ denotes the category of left $R$-module.

**Proof.** It is straightforward. □

**Proposition 2.9.** Let $M$ be a module. Then $M$ is lifting or (quasi-)discrete if and only if $M$ is $M$-lifting or $M$-(quasi-)discrete if and only if $M$ is $N$-lifting or $N$-(quasi)-discrete for any module $N$.

**Proof.** It is clear by Lemma 2.8. □

**Proposition 2.10.** Let $M$ be an amply supplemented module. Then the condition $\langle N, M \rangle -D_1$ is inherited by coclosed submodules of $M$.

**Proof.** Let $M$ satisfy $\langle N, M \rangle -D_1$ and $H$ be a coclosed submodule of $M$. $H$ is amply supplemented by Lemma 1.4. For any $A \in \langle N, H \rangle$, $A$ has an $s$-closure $\overline{A} \in \langle N, H \rangle$ in $H$ by Proposition 2.4. Since $\overline{A} \in \langle N, H \rangle \subseteq \langle N, M \rangle$ and $M$ satisfies $\langle N, M \rangle -D_1$, there is a direct summand $K$ of $M$ such that $K \subseteq \overline{A}$ and $\overline{A}/K \leq M/K$. By Lemma 1.2, $\overline{A}/K \leq H/K$. Now $\overline{A} = K$ since $\overline{A}$ is coclosed in $H$. Thus $H$ satisfies $\langle N, H \rangle -D_1$. □

**Corollary 2.11.** Let $M$ be an amply supplemented module. Then the condition $\langle N, M \rangle -D_1$ is inherited by direct summands of $M$.

**Proposition 2.12.** Let $M$ be an amply supplemented module. Then $\langle N, M \rangle -D_1$ is inherited by direct summands of $M$.

**Proof.** (1) Let $M$ satisfy $\langle N, M \rangle -D_2$ and $H$ be a direct summand of $M$. We will show that $H$ satisfies $\langle N, H \rangle -D_2$.

Let $A \in \langle N, H \rangle \subseteq \langle N, M \rangle$ and $H/A$ is isomorphic to a direct summand of $H$. Since $H$ is a direct summand of $M$, there exists $H' \leq M$ such that $M = H \oplus H'$. Thus $M/A = (H \oplus H')/A \simeq (H/A) \oplus H'$, and so $M/A$ is isomorphic to a direct summand of $M$. $A$ is a direct summand of $M$ since $M$ satisfies $\langle N, M \rangle -D_2$, and hence $A$ is a direct summand of $H$.

(2) Let $A \in \langle N, H \rangle \subseteq \langle N, M \rangle$ and $A$, $L$ be direct summands of $H$ with $A + L = H$. We will show that $A \cap L$ is a direct summand of $H$. Since $H$ is a direct summand of $M$, there exists $H' \leq M$ such that $M = H \oplus H'$. Thus $M = (A + L) \oplus H' = A + (L \oplus H')$. Now $A \cap (L \oplus H')$ is a direct summand of $M$ since $M$ satisfies $\langle N, M \rangle -D_3$. Note that $A \cap (L \oplus H') = A \cap L$, so $A \cap L$ is a direct summand of $H$. □

**Theorem 2.13.** Let $M$ be an amply supplemented module and $A \in \langle N, M \rangle$ a direct summand of $M$. If $M$ is $N$-(quasi-)discrete, then $A$ is (quasi-)discrete.
Corollary. For all submodules \( N \),

\[ M = N'' - \text{lifting} \quad \text{and} \quad N' - \text{lifting}. \]

**Proof.** Without loss of generality we can assume that \( N' \leq N \) and \( N'' = N/N' \). By definition, \( N' \leq N \) implies \( (N', M) \subseteq (N, M) \). Next, let \( A_2 \in (N'', M) \). Then there exist \( X \leq N = N/N' \) and \( f \in \text{Hom}(X, M) \) such that \( A_2/f(X) \ll M/f(X) \). Write \( X = Y/N' \), \( Y \leq N \) and let \( \delta : Y \to Y/N' \) be the canonical homomorphism. It is clear that \( g = f\delta \in \text{Hom}(Y, M) \) and \( g(Y) = f(X) \), hence \( A_2/g(Y) \ll M/g(Y) \). Thus \( A_2 \in (N, M) \). Therefore \( (N', M) \cup (N'', M) \subseteq (N, M) \). The rest is obvious. \( \square \)

Dual to [8, Proposition 2.7], we have the following.

**Theorem 2.15.** Let \( 0 \to N' \to N \to N'' \to 0 \) be an exact sequence and \( M \) an amply supplemented module. Then \( M \) is \( N' \)-lifting if and only if it is both \( N' \)-lifting and \( N'' \)-lifting.

**Proof.** Let \( M \) be \( N' \)-lifting. Then it is both \( N' \)-lifting and \( N'' \)-lifting by Proposition 2.14. Conversely suppose that \( M \) is both \( N' \)-lifting and \( N'' \)-lifting. For any submodule \( A \subseteq (N, M) \), \( A \) has an \( s \)-closure \( \overline{A} \subseteq (N, M) \) by Proposition 2.4. Since \( \overline{A} \subseteq (N, M) \), there exist \( X \leq N \) and \( f \in \text{Hom}(X, M) \) such that \( \overline{A}/f(X) \ll M/f(X) \). Since \( \overline{A} \) is coclosed in \( M \), \( f(X) = \overline{A} \). Write \( Y = X \cap N' \leq N' \) and \( f|_Y : Y \to M \) is a homomorphism, then \( f(Y) \leq f(X) = \overline{A} \). Let \( \overline{f(Y)} \) be an \( s \)-closure of \( f(Y) \) in \( \overline{A} \) (for \( \overline{A} \) is amply supplemented). Thus we conclude that \( f(Y)/\overline{f(Y)} \ll M/\overline{f(Y)} \) and \( \overline{f(Y)} \in (N', M) \). Since \( M \) is \( N' \)-lifting, there exists a direct summand \( K \) of \( M \) such that \( f(Y)/K \ll M/K \). It is easy to see \( \overline{f(Y)} \) is coclosed in \( M \), hence \( \overline{f(Y)} = K \) is a direct summand of \( M \). Write \( M = \overline{f(Y)} \oplus K', K' \leq M \) and \( \overline{A} = \overline{A} \cap M = \overline{f(Y)} \oplus (\overline{A} \cap K') \). Define \( h: W = (X + N')/N' \to M \) by \( h(x + N') = \pi f(x) \), where \( \pi: \overline{A} \to \overline{A} \cap K' \) denotes the canonical projection. It is clear that \( h(W) = \overline{A} \cap K' \), thus \( (\overline{A} \cap K')/h(W) \ll M/h(W) \), and hence \( (\overline{A} \cap K') \in (N'', M) \). Since \( M \) is \( N'' \)-lifting, there exists a direct summand \( K'' \) of \( M \) such that \( (\overline{A} \cap K')/K'' \ll M/K'' \). Since \( \overline{A} \cap K' \) is coclosed in \( M \), \( \overline{A} \cap K' = K'' \). Now \( \overline{A} \cap K' \) is a direct summand of \( K' \). Thus \( \overline{A} \) is a direct summand of \( M \). It follows that \( M \) is \( N' \)-lifting. \( \square \)

**Corollary 2.16.** Let \( M \) be an amply supplemented module. If \( M \) is \( N_i \)-lifting for \( i = 1, 2, \ldots, n \) and \( N = \bigoplus_i N_i \), then \( M \) is \( N \)-lifting.

**Corollary 2.17.** Let \( M \) be an amply supplemented module. Then \( M \) is lifting if and only if \( M \) is \( N' \)-lifting and \( M/N \)-lifting for every submodule \( N \) of \( M \) if and only if \( M \) is \( N' \)-lifting and \( M/N \)-lifting for some submodule \( N \) of \( M \).

Recall that a module \( M \) is said to be **distributive** if \( N \cap (K + L) = (N \cap K) + (N \cap L) \) for all submodules \( N, K, L \) of \( M \). A module \( M \) has **SSP** (see [4]) if the sum of any pair of direct summands of \( M \) is a direct summand of \( M \).

**Corollary 2.18.** Let \( 0 \to N' \to N \to N'' \to 0 \) be an exact sequence and let \( M \) be a distributive and amply supplemented module with **SSP**. If \( M \) is both \( N' \)-quasidiscrete and \( N'' \)-quasidiscrete, then \( M \) is \( N' \)-quasidiscrete.
6 Generalized lifting modules

Proof. We only need to show that $M$ satisfies $(N, M)-D_3$ when $M$ satisfies $(N', M)-D_3$ and $(N'', M)-D_3$ by Theorem 2.15. Let $A \in (N, M)$ and $A, H$ be direct summands of $M$ with $A + H = M$. We know that $A = A_1 \oplus A_2$, where $A_1 \in (N', M)$, $A_2 \in (N'', M)$ from the proof of Theorem 2.15. Since $M$ is a distributive module with SSP, $A_1 \cap H$ and $A_2 \cap H$ are direct summands of $M$. This implies that $A \cap H$ is a direct summand of $M$. Thus $M$ satisfies $(N, M)-D_3$. □

3. SSRS-modules

In [2], a module is called a CESS-module if every complement with essential socle is a direct summand. As a dual of CESS-modules, the concept of SSRS-modules is given in this section. It is proven that: (1) let $M$ be an amply supplemented SSRS-module such that $\text{Rad}(M)$ is finitely generated, then $M = K \oplus K'$, where $K$ is a radical module and $K'$ is a lifting module; (2) let $M$ be a finitely generated amply supplemented module, then $M$ is an SSRS-module if and only if $M/K$ is a lifting module for every coclosed submodule $K$ of $M$.

Definition 3.1. A module is called an SSRS-module if every supplement with small radical is a direct summand.

Lifting modules are SSRS-modules, but the converse is not true. For example, $\mathbb{Z}\mathbb{Z}$ is an SSRS-module which is not a lifting module.

Proposition 3.2. Let $M$ be an SSRS-module. Then any direct summand of $M$ is an SSRS-module.

Proof. Let $K$ be a direct summand of $M$ and $N$ a supplement submodule of $K$ such that $\text{Rad}(N) \ll N$. Let $N$ be a supplement of $L$ in $K$, that is, $N + L = K$ and $N \cap L \ll N$. Since $K$ is a direct summand of $M$, there exists $K' \leq M$ such that $M = K \oplus K'$. Note that $M = N + (L \oplus K')$ and $N \cap (L \oplus K') = N \cap L \ll N$. Therefore $N$ is a supplement of $L \oplus K'$ in $M$. Thus $N$ is a direct summand of $M$ since $M$ is an SSRS-module. So $N$ is a direct summand of $K$. The proof is complete. □

Proposition 3.3. Let $M$ be a weakly supplemented SSRS-module and $K$ a coclosed submodule of $M$. Then $K$ is an SSRS-module.

Proof. It follows from the assumption and [4, Lemma 2.6(3)]. □

Proposition 3.4. Let $M$ be an amply supplemented module. Then $M$ is an SSRS-module if and only if for every submodule $N$ with small radical, there exists a direct summand $K$ of $M$ such that $K \leq N$ and $N/K \ll M/K$.

Proof. “$\Rightarrow$.” Let $N$ be a supplement submodule with small radical. By assumption, there exists a direct summand $K$ of $M$ such that $K \leq N$ and $N/K \ll M/K$. Since $N$ is coclosed in $M$, $N = K$. Thus $N$ is a direct summand of $M$.

“$\Leftarrow$.” Let $N \leq M$ with $\text{Rad}(N) \ll N$. There exists an $s$-closure $\overline{N}$ of $N$ since $M$ is amply supplemented. Since $\text{Rad}(N) \ll M$ (for $\text{Rad}(N) \ll N$) and $\text{Rad}(\overline{N}) \leq \text{Rad}(N)$,
Rad(\(N\)) \ll N and \(\overline{N}\) is a supplement submodule by Lemma 1.2. Therefore \(\overline{N}\) is a direct summand of \(M\) by assumption. This completes the proof.

**Corollary 3.5.** Let \(M\) be an amply supplemented SSRS-module. Then every simple submodule of \(M\) is either a direct summand or a small submodule of \(M\).

**Proposition 3.6.** Let \(M\) be an amply supplemented module. Then \(M\) is an SSRS-module if and only if for every submodule \(N\) of \(M\), every \(s\)-closure of \(N\) with small radical is a lifting module and a direct summand of \(M\).

**Proof.** It is straight forward. \(\square\)

**Proposition 3.7.** Let \(M\) be an amply supplemented SSRS-module. Then \(M = K \oplus K'\), where \(K\) is semisimple and \(K'\) has small socle.

**Proof.** For \(\text{Soc}(M)\), there exists a direct summand \(K\) of \(M\) such that \(\text{Soc}(M)/K \ll M/K\) by Proposition 3.4. It is easy to see that \(K\) is semisimple. Since \(K\) is a direct summand of \(M\), there exists \(K' \leq M\) such that \(M = K \oplus K'\). Note that \(\text{Soc}(M) = \text{Soc}(K) \oplus \text{Soc}(K')\). So \(\text{Soc}(M)/K = (K \oplus \text{Soc}(K'))/K \ll M/K = (K \oplus K')/K\). Thus \(\text{Soc}(K') \ll K'\). \(\square\)

Recall that a module \(M\) is called a radical module if \(\text{Rad}(M) = M\). Dual to [2, Theorem 2.6], we have the following.

**Theorem 3.8.** Let \(M\) be an amply supplemented SSRS-module such that \(\text{Rad}(M)\) is finitely generated. Then \(M = K \oplus K'\), where \(K\) is a radical module and \(K'\) is a lifting module.

**Proof.** \(\text{Rad}(\text{Rad}(M)) \ll \text{Rad}(M)\) since \(\text{Rad}(M)\) is finitely generated. There exists a direct summand \(K\) of \(M\) such that \(\text{Rad}(M)/K \ll M/K\) by Proposition 3.4. Since \(K\) is a direct summand of \(M\), there exists \(K' \leq M\) such that \(M = K \oplus K'\). Note that \(\text{Rad}(M) = \text{Rad}(K) \oplus \text{Rad}(K')\). Therefore \(K = K \cap \text{Rad}(M) = \text{Rad}(K)\) and \(\text{Rad}(M)/K = (\text{Rad}(K) \oplus \text{Rad}(K'))/K \ll M/K = (K \oplus K')/K\). Thus \(\text{Rad}(K) = K\) and \(\text{Rad}(K') \ll K'\).

Next, we show that \(K'\) is a lifting module. \(K'\) is amply supplemented since it is a direct summand of \(M\). So we only prove that every supplement submodule of \(K'\) is a direct summand of \(K'\). Let \(N\) be a supplement submodule of \(K'\). By Lemma 1.2 and \(\text{Rad}(K') \ll K'\), we know that \(\text{Rad}(N) \ll N\). \(N\) is a direct summand of \(K'\) since \(K'\) is an SSRS-module by Proposition 3.2. The proof is complete. \(\square\)

**Corollary 3.9.** Let \(M\) be an amply supplemented module with small radical. Then \(M\) is an SSRS-module if and only if \(M\) is a lifting module.

**Theorem 3.10.** Let \(M\) be a finitely generated amply supplemented module. Then the following statements are equivalent.

1. \(M\) is an SSRS-module.
2. \(M\) is a lifting module.
3. \(M/K\) is a lifting module for every coclosed submodule \(K\) of \(M\).

**Proof.** (1) \(\iff\) (2) follows from Corollary 3.9.

(3) \(\Rightarrow\) (1) is clear.

(1) \(\Rightarrow\) (3) we only prove that any supplement submodule of \(M/K\) is a direct summand. Let \(A/K\) be a supplement submodule of \(M/K\). \(A\) is coclosed in \(M\) since \(A/K\) is coclosed in...
M/K and K is coclosed in M. Rad(A) ≪ A since M is finitely generated and A is coclosed in M. A is a direct summand of M by assumption. Thus A/K is a direct summand of M/K.

Lemma 3.11. Let M be a module. Then the following statements are equivalent.

1. For every cyclic submodule N of M, there exists a direct summand K of M such that K ≤ N and N/K ≪ M/K.
2. For every finitely generated submodule N of M, there exists a direct summand K of M such that K ≤ N and N/K ≪ M/K.

Proof. See [12, 41.13].

Corollary 3.12. Let M be a Noetherian module. Then the following statements are equivalent.

1. M is R-lifting.
2. M is F-lifting, for any free module F.
3. M is lifting.
4. M is an amply supplemented SSRS-module.

Proof. It is easy to see that $(R, M)$ and $(F, M)$ are closed under cyclic submodules. The rest follows immediately from Theorem 3.10 and Lemma 3.11.

Corollary 3.13. Let R be a left perfect (semiperfect) ring. Then every SSRS-module (finitely generated SSRS-module) is a lifting module.

Proof. It follows from the fact that every module over a left perfect ring has small radical, [11, Theorems 1.6 and 1.7] and Corollary 3.9.

A module M is uniserial (see [6]) if its submodules are linearly ordered by inclusion and it is serial if it is a direct sum of uniserial submodules. A ring R is right (left) serial if the right (left) R-module $R_R$ is serial and it is serial if it is both right and left serial.

Corollary 3.14. The following statements are equivalent for a ring R with radical J.

1. R is an artinian serial ring and $J^2 = 0$.
2. R is a left semiperfect ring and every finitely generated module is an SSRS-module.
3. R is a left perfect ring and every module is an SSRS-module.

Proof. It holds by [6, Theorem 3.15], [10, Theorem 1 and Proposition 2.13], and Corollary 3.13.

References


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