ON SOME EXPONENTIAL MEANS. PART II

JÓZSEF SÁNDOR AND GHEORGHE TOADER

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We prove some new inequalities involving an exponential mean, its complementary, and some means derived from known means by applying the exp-log method.

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1. Introduction

All the means that appear in this paper are functions $M : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ with the property that

$$\min(a, b) \leq M(a, b) \leq \max(a, b) \quad \forall a, b > 0. \quad (1.1)$$

Of course $M(a, a) = a$, for all $a > 0$. As usual A, G, L, I, A_p denote the arithmetic, geometric, logarithmic, identric, respectively, power means of two positive numbers, defined by

$$\begin{aligned} A &= A(a, b) = \frac{a+b}{2}, & G &= G(a, b) = \sqrt{ab}, \\ L &= L(a, b) = \frac{b-a}{\log b - \log a}, & I &= I(a, b) = \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{1/(b-a)}, \\ A_p &= A_p(a, b) = \left(\frac{a^p + b^p}{2} \right)^{1/p}, & p &\neq 0. \end{aligned} \quad (1.2)$$

In [16], the first part of this paper, we have studied the exponential mean

$$E = E(a, b) = \frac{be^b - ae^a}{e^b - e^a} - 1 \quad (1.3)$$

introduced in [23]. Another exponential mean was defined in [19] by

$$\bar{E} = \bar{E}(a, b) = \frac{ae^b - be^a}{e^b - e^a} + 1. \quad (1.4)$$

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It is the complementary of E , according to a definition from [4], that is,

$$\bar{E} = 2A - E. \quad (1.5)$$

A basic inequality proved in [23] is

$$E > A, \quad (1.6)$$

which gives the new inequality

$$\bar{E} < A. \quad (1.7)$$

More general means have been studied in [14, 17, 19]. For example, letting $f(x) = e^x$ in [14, Formula (5)], we recapture (1.6). We note that by selecting $f(x) = \log x$ in [14, Formula (8)], and then $f(x) = 1/x$, we get the standard inequalities

$$G < L < I < A \quad (1.8)$$

(for history, see, e.g., [7]).

In what follows, for any mean M , we will denote by $\mathcal{M}$ the new mean given by

$$\mathcal{M}(x, y) = \log M(e^x, e^y), \quad x, y > 0. \quad (1.9)$$

As we put $a = e^x$, $b = e^y$ and then take logarithms, we call this procedure the exp-log method. The method will be applied also to some inequalities for deriving new inequalities. For example, in [16] we proved that

$$E = \mathcal{J}, \quad (1.10)$$

and so (1.8) becomes

$$A < \mathcal{L} < E < \mathcal{A}. \quad (1.11)$$

In [16], it was also shown that

$$\begin{aligned} A + \mathcal{A} - \mathcal{L} &< E < 2\mathcal{L} - A, \\ \mathcal{A}_{2/3} &< E < \mathcal{A}_{\log 2} \end{aligned} \quad (1.12)$$

(see also [6, 22]). In [9], the first author improved the inequality (1.6) by

$$E > \frac{A + 2\mathcal{A}}{3} > A. \quad (1.13)$$

This is based on the following identity proved there:

$$(E - A)(a, b) = \frac{A(e^a, e^b)}{L(e^a, e^b)} - 1. \quad (1.14)$$

We get the same result using the known result

$$I > \frac{2A + G}{3} > (A^2 G)^{1/3} \quad (1.15)$$

and the exp-log method.

The aim of this paper is to obtain other inequalities related to the above means.

2. Main results

(1) After some computations, the inequality (1.6) becomes

$$\frac{e^b - e^a}{b - a} < \frac{e^a + e^b}{2}. \quad (2.1)$$

This follows at once from the Hadamard inequality

$$\frac{1}{b - a} \int_a^b f(t) dt < \frac{f(a) + f(b)}{2}, \quad (2.2)$$

applied to the strictly convex function $f(t) = e^t$. We note that by the second Hadamard inequality, namely

$$\frac{1}{b - a} \int_a^b f(t) dt > f\left(\frac{a + b}{2}\right), \quad (2.3)$$

for the same function, one obtains

$$\frac{e^b - e^a}{b - a} > e^{(a+b)/2}, \quad (2.4)$$

which has been proposed as a problem in [3].

The relation (1.11) improves the inequality (2.1), which means that $\mathcal{A} > \mathcal{L}$, and improves (2.4), which means that $\mathcal{L} > A$. In fact, by the above remarks, one can say that

$$E > A \iff \mathcal{A} > \mathcal{L}. \quad (2.5)$$

(2) In [23], it was proven that E is not comparable with A_λ for $\lambda > 5/3$. Then in [17], we have shown, among others, that

$$A(a, b) < E(a, b) < A(a, b) \cdot e^{|b-a|/2}. \quad (2.6)$$

Now, if $|b - a|$ becomes small, clearly $e^{|b-a|/2}$ approaches to 1, that is, the conjecture $E > A_\lambda$ of [23] cannot be true for any $1 < \lambda \leq 5/3$.

We get another double inequality from (1.5) and (1.6):

$$A < E < 2A. \quad (2.7)$$

These inequalities cannot be improved. Indeed, for $1 < \lambda < 2$, we have

$$\lim_{x \rightarrow \infty} [E(1, x) - \lambda A(1, x)] = \infty, \quad (2.8)$$

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but

$$E(1,1) - \lambda A(1,1) = 1 - \lambda < 0, \quad (2.9)$$

thus E is not comparable with λA .

On the other hand,

$$\bar{E}(a,b) = \frac{e^b(a+1) - e^a(b+1)}{e^b - e^a} = (a+1)(b+1) \cdot \frac{f(b) - f(a)}{e^b - e^a}, \quad (2.10)$$

where $f(x) = e^x/(x+1)$. By Cauchy's mean value theorem,

$$\frac{f(b) - f(a)}{e^b - e^a} = \frac{f'(c)}{e^c}, \quad c \in (a,b). \quad (2.11)$$

Since

$$\frac{f'(c)}{e^c} = \frac{c}{(c+1)^2} \leq \frac{1}{4}, \quad (2.12)$$

we get

$$0 < 2A - E \leq \frac{(a+1)(b+1)}{4}. \quad (2.13)$$

(3) By using the series representation

$$\log \frac{I}{G} = \sum_{k=1}^{\infty} \frac{1}{2k+1} \left(\frac{b-a}{b+a} \right)^{2k}, \quad (2.14)$$

(see [9, 21]), we can deduce the following series representation:

$$(E - A)(a,b) = \sum_{k=1}^{\infty} \frac{1}{2k+1} \left(\frac{e^b - e^a}{e^b + e^a} \right)^{2k}. \quad (2.15)$$

By (2.1), $|e^b - e^a|/(e^b + e^a) < |b - a|/2$, thus we get the estimate

$$(E - A)(a,b) < \sum_{k=1}^{\infty} \frac{1}{2k+1} \left(\frac{|b-a|}{2} \right)^{2k}. \quad (2.16)$$

The series is convergent at least for $|b - a| < 2$. Writing

$$\frac{A(e^a, e^b)}{L(e^a, e^b)} = e^{\mathcal{A}(a,b) - \mathcal{L}(a,b)}, \quad (2.17)$$

the identity (1.14) implies the relation

$$E - A = e^{\mathcal{A} - \mathcal{L}} - 1. \quad (2.18)$$

This gives again the equivalence (2.5). But one can obtain also a stronger relation by writing $e^x > 1 + x + x^2/2$, for $x > 0$. Thus (2.18) gives

$$E - A > \mathcal{A} - \mathcal{L} + \frac{1}{2}(\mathcal{A} - \mathcal{L})^2. \quad (2.19)$$

(4) Consider the inequality proved in [10]:

$$\frac{2}{e}A < I < A. \quad (2.20)$$

By the exp-log method, we deduce

$$\log 2 - 1 + \mathcal{A} < E < \mathcal{A}. \quad (2.21)$$

From the inequality

$$I < \frac{2}{e}(A + G) = \frac{4}{e} \left(\frac{\sqrt{a} + \sqrt{b}}{2} \right)^2, \quad (2.22)$$

given in [5], we have, by the same method,

$$E(x, y) < 2\log 2 - 1 + 2\mathcal{A} \left(\frac{x}{2}, \frac{y}{2} \right). \quad (2.23)$$

Relation (2.23) may be compared with the left-hand side of (2.21). Take now the relation

$$L < L(A, G) = \frac{A - G}{\log(A/G)} \quad (2.24)$$

from [5]. Since $A - G = 1/2(\sqrt{a} - \sqrt{b})^2$, one obtains

$$\mathcal{A} - A < \frac{1}{2e^{\mathcal{L}}} (e^{x/2} - e^{y/2})^2. \quad (2.25)$$

The relation

$$L^3 > \left(\frac{A + G}{2} \right)^2 G, \quad (2.26)$$

from [13], gives similarly

$$3\mathcal{L}(x, y) > A(x, y) + 4\mathcal{A} \left(\frac{x}{2}, \frac{y}{2} \right), \quad (2.27)$$

while the inequality

$$\log \frac{I}{L} > 1 - \frac{G}{L}, \quad (2.28)$$

from [7], offers the relation

$$E - \mathcal{L} > 1 - e^{A - \mathcal{L}}. \quad (2.29)$$

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(5) The exp-log method applied to the inequality

$$L > \sqrt{GI}, \quad (2.30)$$

given in [2, 11], implies that

$$\mathcal{L} > \frac{A+E}{2} > \frac{2A+\mathcal{A}}{3}. \quad (2.31)$$

On the other side, the inequality

$$I > \sqrt{AL}, \quad (2.32)$$

proven in [11], gives on the same way the inequality

$$E > \frac{\mathcal{A}+\mathcal{L}}{2}. \quad (2.33)$$

After all, we have the double inequality

$$\frac{\mathcal{A}+\mathcal{L}}{2} < E < 2\mathcal{L} - A. \quad (2.34)$$

(6) Consider now the inequality

$$3I^2 < 2A^2 + G^2, \quad (2.35)$$

from [20]. It gives

$$\log 3 + 2E < \log (e^{2A} + 2e^{2\mathcal{A}}). \quad (2.36)$$

Similarly

$$I > \frac{2A+G}{3}, \quad (2.37)$$

given in [8], implies that

$$\log 3 + E > \log (2e^A + e^{\mathcal{A}}). \quad (2.38)$$

In fact, the relation

$$I > \frac{A+L}{2}, \quad (2.39)$$

from [7], gives

$$\log 2 + E > \log (e^{\mathcal{L}} + e^{\mathcal{A}}), \quad (2.40)$$

but this is weaker than (2.38), as follows from [8]. The inequalities (2.33) and (2.40) can be combined as

$$E > \log \left(\frac{e^{\mathcal{L}} + e^{\mathcal{A}}}{2} \right) > \frac{\mathcal{L} + \mathcal{A}}{2}, \quad (2.41)$$

where the second inequality is a consequence of the concavity of the logarithmic function. We notice also that by

$$L + I < A + G, \quad (2.42)$$

given in [1], one can write

$$e^{\mathcal{L}} + e^E < e^{\mathcal{A}} + e^A. \quad (2.43)$$

(7) In [9] Sándor proved the inequality

$$I(a^2, b^2) < \frac{A^4(a, b)}{I^2(a, b)}. \quad (2.44)$$

By the exp-log method, we get

$$E(2x, 2y) < 4\mathcal{A}(x, y) - 2E(x, y). \quad (2.45)$$

It is interesting to note that by the equality

$$\log \frac{I^2(\sqrt{a}, \sqrt{b})}{I(a, b)} = \frac{G(a, b)}{L(a, b)} - 1, \quad (2.46)$$

given in [7], we have the identity

$$2E\left(\frac{x}{2}, \frac{y}{2}\right) - E(x, y) = e^{A(x, y) - \mathcal{L}(x, y)} - 1. \quad (2.47)$$

Putting $x \rightarrow x/2$, $y \rightarrow y/2$ in (2.45), and taking into account (2.47), we can write

$$2E(x, y) + e^{A(x, y) - \mathcal{L}(x, y)} - 1 < 4\mathcal{A}\left(\frac{x}{2}, \frac{y}{2}\right). \quad (2.48)$$

This may be compared to (2.23).

(8) We consider now applications of the special Gini mean

$$S = S(a, b) = (a^a b^b)^{1/(a+b)} \quad (2.49)$$

(see [15]). Its attached mean (by the exp-log method)

$$\mathcal{G}(x, y) = \frac{xe^x + ye^y}{e^x + e^y} = \log S(e^x, e^y) \quad (2.50)$$

is a special case of

$$M_f(x, y) = \frac{xf(x) + yf(y)}{f(x) + f(y)} \quad (2.51)$$

which was defined in [18]. Using the inequality

$$\left(\frac{S}{A}\right)^2 < \left(\frac{I}{G}\right)^3 \quad (2.52)$$

from [15], we get

$$2\mathcal{S} - 2\mathcal{A} < 3E - 3A. \quad (2.53)$$

The inequalities

$$\frac{A^2}{I} < S < \frac{A^4}{I^3} < \frac{A^2}{G} \quad (2.54)$$

given in [15] imply that

$$2\mathcal{A} - E < \mathcal{S} < 4\mathcal{A} - 3E < 2\mathcal{A} - A. \quad (2.55)$$

These offer connections between the exponential means E and $\mathcal{S}$.

Let now the mean

$$U = U(a, b) = \frac{1}{3} \sqrt{(2a + b)(a + 2b)}. \quad (2.56)$$

In [12], it is proved that

$$G < \sqrt[4]{U^3 G} < I < \frac{U^2}{A} < U < A. \quad (2.57)$$

By the exp-log method, we get

$$A < \frac{1}{4}(3\mathcal{U} + A) < E < 2\mathcal{U} - \mathcal{A} < \mathcal{U} < \mathcal{A}. \quad (2.58)$$

These relations offer a connection between the means E and $\mathcal{U}$.

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József Sándor: Department of Mathematics, Babes-Bolyai University,
3400 Cluj-Napoca, Romania
E-mail address: jsandor@math.ubbcluj.ro

Gheorghe Toader: Department of Mathematics, Technical University of Cluj-Napoca,
3400 Cluj-Napoca, Romania
E-mail address: gheorghe.toader@math.utcluj.ro