PRODUCTS OF DERIVATIONS WHICH ACT AS LIE DERIVATIONS ON COMMUTATORS OF RIGHT IDEALS

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Let R be a prime ring of characteristic different from 2, I a nonzero right ideal of R , d and δ nonzero derivations of R , and $s_4(x_1, x_2, x_3, x_4)$ the standard identity of degree 4. If the composition $(d\delta)$ is a Lie derivation of $[I, I]$ into R , then either $s_4(I, I, I, I)I = 0$ or $\delta(I)I = d(I)I = 0$.

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Throughout this note, R will be always a prime ring of characteristic different from 2 with center $Z(R)$, extended centroid C , and two-sided Martindale quotient ring Q . Let $f : R \rightarrow R$ be additive mapping of R into itself. It is said to be a derivation of R if $f(xy) = f(x)y + xf(y)$, for all $x, y \in R$. Let $S \subseteq R$ be any subset of R . If for any $x, y \in S$, $f([x, y]) = [f(x), y] + [x, f(y)]$, then the mapping f is called a Lie derivation on S . Obviously any derivation of R is a Lie derivation on any arbitrary subset S of R .

A typical example of a Lie derivation is an additive mapping which is the sum of a derivation and a central map sending commutators to zero.

The well-known Posner first theorem states that if δ and d are two nonzero derivations of R , then the composition $(d\delta)$ cannot be a nonzero derivation of R [12, Theorem 1]. An analog of Posner's result for Lie derivations was proved by Lanski [8]. More precisely, Lanski showed that if δ and d are two nonzero derivations of R and L is a Lie ideal of R , then $(d\delta)$ cannot be a Lie derivation of L into R unless $\text{char}(R) = 2$ and either R satisfies $s_4(x_1, \dots, x_4)$, the standard identity of degree 4, or $d = \alpha\delta$, for $\alpha \in C$.

This note is motivated by the previous cited results. Our main theorem gives a generalization of Lanski's result to the case when $(d\delta)$ is a Lie derivation of the subset $[I, I]$ into R , where I is a nonzero right ideal of R and the characteristic of R is different from 2. The statement of our result is the following.

THEOREM 1. *Let R be a prime ring of characteristic different from 2, I a nonzero right ideal of R , d and δ nonzero derivations of R , and $s_4(x_1, \dots, x_4)$ the standard identity of degree 4. If the composition $(d\delta)$ is a Lie derivation of $[I, I]$ into R , then either $s_4(I, I, I, I)I = 0$ or $\delta(I)I = d(I)I = 0$.*

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Remark 2. Notice that for all $u, v \in [I, I]$, we obviously have that

$$(d\delta)([u, v]) = [(d\delta)(u), v] + [u, (d\delta)(v)] + [\delta(u), d(v)] + [d(u), \delta(v)]. \quad (1)$$

Hence, since we suppose that $(d\delta)$ is a Lie derivation on $[I, I]$, we will always assume as a main hypothesis that $[\delta(u), d(v)] + [d(u), \delta(v)] = 0$, for any $u, v \in [I, I]$.

Remark 3. The assumption $S_4(I, I, I, I)I \neq 0$ is essential to the main result. For example, consider $R = M_3(F)$, for F a field of characteristic 3, and let e_{ij} be the usual matrix unit in R . Let $I = (e_{11} + e_{22})R$, δ the inner derivation induced by the element e_{13} , d the inner derivation induced by the element e_{12} , that is, $\delta(x) = [e_{13}, x] = e_{13}x - xe_{13}$, and $d(x) = [e_{12}, x] = e_{12}x - xe_{12}$, for all $x \in R$. In this case, notice that $S_4(x_1, x_2, x_3, x_4)x_5$ is an identity for I , moreover

$$\begin{aligned} & [\delta([(e_{11} + e_{22})x_1, (e_{11} + e_{22})x_2]), d([(e_{11} + e_{22})y_1, (e_{11} + e_{22})y_2])] \\ & + [d([(e_{11} + e_{22})x_1, (e_{11} + e_{22})x_2]), \delta([(e_{11} + e_{22})y_1, (e_{11} + e_{22})y_2])] \\ & = (d([(e_{11} + e_{22})x_1, (e_{11} + e_{22})x_2])[(e_{11} + e_{22})y_1, (e_{11} + e_{22})y_2])e_{13} \\ & - (d([(e_{11} + e_{22})y_1, (e_{11} + e_{22})y_2])[(e_{11} + e_{22})x_1, (e_{11} + e_{22})x_2])e_{13} = 0 \end{aligned} \quad (2)$$

for any $x_1, x_2, y_1, y_2 \in R$, but clearly $d(I)I = [e_{12}, I]I \neq 0$.

In the particular case $I = R$ and both d, δ are inner derivations, induced, respectively, by some elements $a, b \in R$, our theorem has the following flavor.

LEMMA 4. *Let R be a prime ring of characteristic different from 2, $a, b \in R$ such that $[[a, v], [b, u]] + [[b, v], [a, u]] = 0$, for all $v, u \in [R, R]$. Then either a is a central element of R or b is a central one.*

The proof is a clear special case of [8, Theorem 6].

We first fix some notations and recall some useful facts.

Remark 5. Denote by $T = Q *_C C\{X\}$ the free product over C of the C -algebra Q and the free C -algebra $C\{X\}$, with X a countable set consisting of noncommuting indeterminates $\{x_1, \dots, x_n\}$. The elements of T are called generalized polynomials with coefficients in Q . I, IR , and IQ satisfy the same generalized polynomial identities with coefficients in Q . For more details about these objects, we refer the reader to [1, 2, 4].

Remark 6. Any derivation of R can be uniquely extended to a derivation of Q , and so any derivation of R can be defined on the whole of Q [2, Proposition 2.5.1]. Moreover Q is a prime ring as well as R and the extended centroid C of R coincides with the center of Q [2, Proposition 2.1.7, Remark 2.3.1].

Remark 7. Let $f(x_1, \dots, x_n, d(x_1), \dots, d(x_n))$ be a differential identity of R . One of the following holds (see [7]):

- (1) either d is an inner derivation in Q , in the sense that there exists $q \in Q$ such that $d(x) = [q, x]$, for all $x \in Q$ and Q satisfies the generalized polynomial identity $f(x_1, \dots, x_n, [q, x_1], \dots, [q, x_n])$;

(2) or R satisfies the generalized polynomial identity

$$f(x_1, \dots, x_n, y_1, \dots, y_n). \quad (3)$$

Moreover I , IR , and IQ satisfy the same differential identities with coefficients in Q (see [9]).

Finally, as a consequence of [11, Theorem 2], we have the following.

Remark 8. Let R be a prime ring and $\sum_{i=1}^m a_i X b_i + \sum_{j=1}^n c_j X d_j = 0$, for all $X \in R$, where $a_i, b_i, c_j, d_j \in RC$. If $\{a_1, \dots, a_m\}$ are linearly C -independent, then each b_i is C -dependent on $d_1, \dots, d_n$. Analogously, if $\{b_1, \dots, b_m\}$ are linearly C -independent, then each a_i is C -dependent on $c_1, \dots, c_n$.

For the remainder of the note we will assume that the hypothesis of the theorem holds but that the conclusion is false.

Thus, we will always suppose that there exist $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9 \in I$ such that $s_4(c_1, c_2, c_3, c_4)c_5 \neq 0$, and either $\delta(c_6)c_7 \neq 0$ or $d(c_8)c_9 \neq 0$.

We begin with the following.

LEMMA 9. *Let δ and d both be Q -inner derivations such that either $\delta(I)I \neq 0$ or $d(I)I \neq 0$. Then R is a ring satisfying a nontrivial generalized polynomial identity.*

Proof. By Remark 2, we assume that $[\delta(u), d(v)] + [d(u), \delta(v)] = 0$, for any $u, v \in [I, I]$. Let $a, b \in Q$ such that $\delta(x) = [a, x]$ and $d(x) = [b, x]$, for all $x \in R$.

Without loss of generality, we may assume in this context that $\delta(I)I \neq 0$. Notice that if $\{y, ay\}$ are linearly C -dependent for all $y \in I$, then there exists $\alpha \in C$, such that $(a - \alpha)I = 0$ (see [10, Lemma 3]). If we replace a by $a - \alpha$, since they induce the same inner derivation, it follows that $\delta(I)I = [a - \alpha, I]I = 0$, a contradiction. Thus there exists $x \in I$ such that $\{x, ax\}$ are linearly C -independent.

Let $x \in I$ such that $\{x, ax\}$ are linearly C -independent and r_1, r_2, r_3, r_4 are any elements of R . Then

$$[[a, [xr_1, xr_2]], [b, [xr_3, xr_4]]] + [[b, [xr_1, xr_2]], [a, [xr_3, xr_4]]] = 0. \quad (4)$$

Denote

$$\begin{aligned} F_1 &= (r_1 x r_2 - r_2 x r_1) b [x r_3, x r_4] - (r_1 x r_2 - r_2 x r_1) [x r_3, x r_4] b \\ &\quad - (r_3 x r_4 - r_4 x r_3) b [x r_1, x r_2] + (r_3 x r_4 - r_4 x r_3) [x r_1, x r_2] b, \\ F_2 &= -(r_3 x r_4 - r_4 x r_3) a [x r_1, x r_2] + (r_3 x r_4 - r_4 x r_3) [x r_1, x r_2] a \\ &\quad + (r_1 x r_2 - r_2 x r_1) a [x r_3, x r_4] - (r_1 x r_2 - r_2 x r_1) [x r_3, x r_4] a, \\ F_3 &= (r_3 x r_4 - r_4 x r_3) b a [x r_1, x r_2] - (r_1 x r_2 - r_2 x r_1) b a [x r_3, x r_4] \\ &\quad + (r_1 x r_2 - r_2 x r_1) a [x r_3, x r_4] b - (r_3 x r_4 - r_4 x r_3) b [x r_1, x r_2] a \\ &\quad - (r_1 x r_2 - r_2 x r_1) b a [x r_3, x r_4] + (r_1 x r_2 - r_2 x r_1) b [x r_3, x r_4] a \\ &\quad + (r_3 x r_4 - r_4 x r_3) a b [x r_1, x r_2] - (r_3 x r_4 - r_4 x r_3) a [x r_1, x r_2] b. \end{aligned} \quad (5)$$

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Hence (4) is $axF_1 + bxF_2 + xF_3 = 0$. If $\{ax, bx, x\}$ are linearly C -independent, then (4) is a nontrivial generalized polynomial identity for R , since $F_1 \neq 0$ in T , using $b \notin C$. On the other hand, if there exist $\alpha_1, \alpha_2 \in C$ such that $bx = \alpha_1x + \alpha_2ax$, it follows that R satisfies

$$axF_1 + \alpha_1xF_2 + \alpha_2axF_2 + xF_3 = 0, \quad (6)$$

that is, again a nontrivial GPI, because $\{x, ax\}$ are linearly C -independent, by the choice of x and since $F_1 + F_2 \neq 0$ in T , using $a, b \notin C$.

The same argument shows that if $d(I)I \neq 0$, then there exists $x \in I$ such that $\{x, bx\}$ are linearly C -independent and R satisfies in any case a nontrivial GPI. $\square$

At this point, we need a result that will be useful in the continuation of the note.

Remark 10. Let $R = M_n(F)$ be the ring of $n \times n$ matrices over the field F , denote by e_{ij} the usual matrix unit with 1 in the (i, j) -entry and zero elsewhere. Since there exists a set of matrix units that contains the idempotent generator of a given minimal right ideal, we observe that any minimal right ideal is part of a direct sum of minimal right ideals adding to R . In light of this and applying [6, Proposition 5, page 52], we may assume that any minimal right ideal of R is a direct sum of minimal right ideals, each of the form $e_{ii}R$.

LEMMA 11. *Let $R = M_n(F)$ be the ring of $n \times n$ matrices over the field F of characteristic different from 2 and $n \geq 2$. Let d be a nonzero inner derivation of R , and I a nonzero right ideal of R . If a is a nonzero element of I such that $(d([x_1, x_2])[x_3, x_4] - d([x_3, x_4])[x_1, x_2])a = 0$, for all $x_1, x_2, x_3, x_4 \in I$, then either $s_4(I, I, I, I)I = 0$ or d is induced by an element $b \in R$ such that $(b - \beta)I = 0$, for a suitable $\beta \in Z(R)$.*

Proof. Let b be an element of R which induces the derivation d , that is, $d(x) = [b, x]$, for all $x \in R$. As above, let e_{ij} be the usual matrix unit with 1 in the (i, j) -entry and zero elsewhere and write $a = \sum a_{ij}e_{ij}$, $b = \sum b_{ij}e_{ij}$, with a_{ij} and b_{ij} elements of F .

We know that I has a number of uniquely determined simple components: they are minimal right ideals of R and I is their direct sum. In light of Remark 10, we may write $I = eR$ for some $e = \sum_{i=1}^t e_{ii}$ and $t \in \{1, 2, \dots, n\}$. Since $s_4(I, I, I, I)I = 0$ in case $t \leq 2$, we may suppose that $t \geq 3$.

First of all, we want to prove that $b_{rs} = 0$ for all $s \leq t$ and $r \neq s$. To do this, suppose by contradiction that there exist $i \neq j$ such that $b_{ij} \neq 0$ ($j \leq t$). Without loss of generality, we replace b by $b_{ij}^{-1}(b - b_{jj}I_n)$, where I_n is the identity matrix in $M_n(F)$ so that we assume $b_{ij} = 1$ and $b_{jj} = 0$. Moreover $a = ex$ for a suitable $x \in R$.

Let now $k \leq t$, $k \neq i, j$, $[x_1, x_2] = e_{ki}$, $[x_3, x_4] = e_{ji}$. In this case, we have

$$0 = ([b, e_{ki}]e_{ji}a - [b, e_{ji}]e_{ki})a \quad (7)$$

and left multiplying by e_{kk} ,

$$e_{ki}be_{ji}a = 0, \quad (8)$$

that is, since $b_{ij} = 1$, $e_{ii}a = 0$.

On the other hand, if we choose $[x_1, x_2] = e_{ki}$ and $[x_3, x_4] = e_{jk}$, we have

$$0 = ([b, e_{ki}]e_{jk} - [b, e_{jk}]e_{ki})a = [b, e_{ki}]e_{jk}a = -b_{ij}e_{kk}a. \quad (9)$$

Therefore $e_{rr}a = 0$ for all $r \neq j$, that is, $a = e_{jj}a$. Finally, consider $[x_1, x_2] = e_{ki}$ and $[x_3, x_4] = e_{kk} - e_{jj}$. Then

$$0 = ([b, e_{ki}](e_{kk} - e_{jj}) - [b, e_{kk} - e_{jj}]e_{ki})a = e_{ki}be_{jj}a, \quad (10)$$

that is, $e_{jj}a = 0$. This implies that $ea = 0$, so that $a = 0$, a contradiction.

This argument says that if $a \neq 0$, then $b_{ij} = 0$ for all $i \neq j$, $j \leq t$.

Suppose that $(b - \beta)I \neq 0$, for $\beta \in F$. In this case, there exist $1 \leq r, s \leq t$, with $r \neq s$, such that $b_{rr} \neq b_{ss}$.

Let f be the F -automorphism of R defined by $f(x) = (1 - e_{rs})x(1 + e_{rs})$. Thus we have that $f(x) \in I$, for all $x \in I$ and

$$([f(b), [x_1, x_2]][x_3, x_4] - [f(b), [x_3, x_4]][x_1, x_2])f(a) = 0 \quad (11)$$

for all $x_1, x_2, x_3, x_4 \in I$. If $a \neq 0$, then $f(a) \neq 0$, and as above, the (r, s) -entry of $f(b)$ is zero. On the other hand,

$$f(b) = (1 - e_{rs})b(1 + e_{rs}) = b + b_{rr}e_{rs} - b_{ss}e_{rs}, \quad (12)$$

that is, $b_{rr} = b_{ss}$, a contradiction. This means that there exists $\beta \in F$ such that $(b - \beta)I = 0$. Denote $b - \beta = p$. Since b and p induce the same inner derivation d , we have that $([p, [x_1, x_2]][x_3, x_4] - [p, [x_3, x_4]][x_1, x_2])a = 0$ with $pI = 0$. $\square$

LEMMA 12. *Let R be a prime ring of characteristic different from 2, d a nonzero inner derivation of R , I a nonzero right ideal of R . If a is a nonzero element of I such that $(d([x_1, x_2])[x_3, x_4] - d([x_3, x_4])[x_1, x_2])a = 0$, for all $x_1, x_2, x_3, x_4 \in I$, then either $s_4(I, I, I, I)I = 0$ or d is induced by an element $b \in R$ such that $(b - \beta)I = 0$, for a suitable $\beta \in Z(R)$.*

Proof. As a reduction of Lemma 9, we have that if R is not a GPI ring, then we are done. Thus consider the only case when R satisfies a nontrivial generalized polynomial identity.

Thus the Martindale quotient ring Q of R is a primitive ring with nonzero socle $H = \text{Soc}(Q)$. H is a simple ring with minimal right ideals. Let D be the associated division ring of H , by [11] D is a simple central algebra finite-dimensional over $C = Z(Q)$. Thus $H \otimes_C F$ is a simple ring with minimal right ideals, with F an algebraic closure of C . Let b be an element of R which induces the derivation d . Moreover $([b, [x_1, x_2]][x_3, x_4] - [b, [x_3, x_4]][x_1, x_2])a = 0$, for all $x_1, x_2, x_3, x_4 \in IH \otimes_C F$ (see, e.g., [4, Theorem 2]). Notice that if C is finite, we choose $F = C$.

Now we claim that for any $c \in IH$, there exists $\beta \in C$ with $(b - \beta)c = 0$. If not, then for some $c \in IH$, $(b - \beta)c \neq 0$ for all $\beta \in C$, so in particular $bc \neq 0$. Since H is regular [5], there exists $g^2 = g \in IH$, such that $c \in gIH$, and $e^2 = e \in H \otimes_C F$, such that

$$g, bg, gb, a, c, bc, cb \in e(H \otimes_C F)e \cong M_n(F), \quad n \geq 3. \quad (13)$$

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Let $x_1, x_2, x_3, x_4 \in ge(H \otimes_C F)e$ and $a = eae \neq 0$, then

$$0 = e([b, [x_1, x_2]][x_3, x_4] - [b, [x_3, x_4]][x_1, x_2])eae. \quad (14)$$

Applying Lemma 11, we have that $e(b - \lambda)ec = 0$ for $\lambda \in C$, so $(b - \beta)c = 0$, contradicting the choice of c .

As in the proof of Lemma 9, by [10, Lemma 3], we conclude that there exists $\beta \in C$ such that $(b - \beta)I = 0$. $\square$

LEMMA 13. *If δ and d are both inner derivations, then the theorem holds.*

Proof. By Remark 2, we assume that $[\delta(u), d(v)] + [d(u), \delta(v)] = 0$, for any $u, v \in [I, I]$. Let $a, b \in Q$ such that $\delta(x) = [a, x]$ and $d(x) = [b, x]$, for all $x \in R$. Since in light of Lemma 9, R satisfies a nontrivial GPI, then without loss of generality, R is simple and equal to its own socle and $IR = I$. In fact, Q has nonzero socle H with nonzero right ideal $J = IH$ [11]. Note that H is simple, $J = JH$, and J satisfies the same basic conditions as I . Now just replace R by H , I by J , and we are done.

Recall that $s_4(c_1, c_2, c_3, c_4)c_5 \neq 0$ and either $\delta(c_6)c_7 \neq 0$ or $d(c_8)c_9 \neq 0$. By the regularity of R , there exists an element $e^2 = e \in IR$ such that $eR = c_1R + c_2R + c_3R + c_4R + c_5R + c_6R + c_7R + c_8R + c_9R$ and $ec_i = c_i$, for $i = 1, \dots, 9$. We note that $s_4(eRe, eRe, eRe, eRe) \neq 0$ (and $\dim_C(eRe) \geq 9$).

Let $x, y, z \in R$, so

$$[[a, [e, ex(1 - e)]], [b, [ey, ez]]] + [[b, [e, ex(1 - e)]], [a, [ey, ez]]] = 0. \quad (15)$$

Denote $A = (1 - e)ae$, $B = (1 - e)be$. Assume that $A = 0$ but $B \neq 0$. Consider first the case when $\{1 - e, (1 - e)a\}$ are linearly C -independent. Equation (15), multiplied on the left by $(1 - e)$, says that

$$-(1 - e)b[ey, ez]aex(1 - e) + (1 - e)b[ey, ez]ex(1 - e)a = 0. \quad (16)$$

By Remark 8 and since $\{1 - e, (1 - e)a\}$ are linearly C -independent, it follows that there exists $\lambda_1 \in C$ such that $-(1 - e)b[ey, ez]ae = \lambda_1(1 - e)b[ey, ez]e$.

Therefore

$$(1 - e)b[ey, ez]ex\lambda_1(1 - e) + (1 - e)b[ey, ez]ex(1 - e)a = 0, \quad (17)$$

which implies that $(1 - e)b[ey, ez]e = 0$, again since $\{1 - e, (1 - e)a\}$ are linearly C -independent. If we denote $T = eR$, $(1 - e)b[T, T]T = 0$ forces $(1 - e)bT[T, T]T = 0$, so either $(1 - e)bT = 0$ or $[T, T]T = 0$. Thus we have that either $B = (1 - e)be = 0$ or $[x_1, x_2]x_3$ is an identity for eR . In this last case, a fortiori $s_4(x_1, x_2, x_3, x_4)x_5$ is an identity for eR . In both cases, we have a contradiction, since we suppose that $B \neq 0$ and $s_4(c_1, c_2, c_3, c_4)c_5 \neq 0$.

Suppose now that $(1 - e)a = \lambda(1 - e)$, for some $\lambda \in C$. Equation (16) says that

$$-(1 - e)b[ey, ez]aex(1 - e) + \lambda(1 - e)b[ey, ez]ex(1 - e) = 0, \quad (18)$$

and so

$$-(1-e)b[ey, ez]ae + \lambda(1-e)b[ey, ez]e = 0, \quad (19)$$

that is, for $a' = \lambda e - ae$,

$$(1-e)b[ey, ez]a' = 0. \quad (20)$$

Denote $U = [ey, ez]a'$. Since $(1-e)be[UX_1, ex_2]a' = 0$, for all $x_1, x_2 \in R$, it follows that $(1-e)bex_2UX_1a' = 0$, and so either $a' = 0$ or $U = 0$. Again denote $T = eR$. If $U = 0$, we have $[T, T]a' = 0$, so that $[T, T]Ta' = 0$, which implies either $a' = 0$ or $[T, T]T = 0$. Since $[eR, eR]e \neq 0$, we have $ae = \lambda e$ in any case.

All the previous arguments say that $(a - \lambda)e = 0$. Replacing a by $a - \lambda = a''$, since they induce the same inner derivation, we may assume that for all $x, y, z, t \in R$,

$$[[a'', [ex, ey]], [b, [ez, et]]] + [[b, [ex, ey]], [a'', [ez, et]]] = 0. \quad (21)$$

Left multiplying (21) by $(1-e)$, we have

$$(1-e)be[[ez, et], [ex, ey]]a'' = 0, \quad (22)$$

in particular

$$0 = (1-e)be[[ez, et], [ex, ey(1-e)]]a'' = (1-e)be[ez, et]exey(1-e)a'', \quad (23)$$

and by the previous same argument, $(1-e)a'' = 0$, that is, $a'' = ea''$. In light of this, by (22),

$$(eR(1-e)be)[[eze, ete], [exe, eye]](ea''Re) = 0. \quad (24)$$

Let G be the subgroup of eRe generated by the polynomial $[[eze, ete], [exe, eye]]$. It is easy to see that G is a noncentral Lie ideal of eRe . In this condition, it is well known that $[eRe, eRe] \subseteq G$, and so $eR(1-e)be[eRe, eRe]ea''Re = 0$.

Consider now the simple Artinian ring eRe , then we have that

$$eR(1-e)be[ex_1e, ex_2e](ea''Re) = 0 \quad \forall x_1, x_2 \in R. \quad (25)$$

Let $U = [ex_1e, ex_2e](ea''Re)$, so $eR(1-e)beU = 0$. Since

$$(eR(1-e)be)[Uex_1e, ex_2e](ea''Re) = 0, \quad (26)$$

then

$$(eR(1-e)be)x_2Uex_1(ea''Re) = 0. \quad (27)$$

It follows that if $(1-e)be \neq 0$, then $a'' = 0$, that is, $a = \lambda \in C$, a contradiction. Thus the conclusion is that if $A = (1-e)ae = 0$, then $B = (1-e)be = 0$.

Similarly, $A = (1-e)ae = 0$ follows from $B = (1-e)be = 0$.

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Now we assume that neither $A = 0$ nor $B = 0$, and proceed to get contradictions, proving that $A = B = 0$.

Left multiplying (15) by $(1 - e)$ and right multiplying by e , we get

$$(1 - e)ax(1 - e)b[ey, ez]e + (1 - e)b[ey, ez]ex(1 - e)ae \\ + (1 - e)bex(1 - e)a[ey, ez]e + (1 - e)a[ey, ez]ex(1 - e)be = 0. \quad (28)$$

If we denote $A' = A[ye, ze]$, $B' = B[ye, ze]$, it follows that

$$Ax B' + B' x A + B x A' + A' x B = 0. \quad (29)$$

Consider now the case when A, B are linearly C -independent.

In light of Remark 8 and (29), it follows that there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in C such that $B' = \alpha_1 A + \alpha_2 B$, $A' = \alpha_3 A + \alpha_4 B$. So we rewrite (29) as follows:

$$2\alpha_1 Ax A + 2\alpha_4 B x B + (\alpha_2 + \alpha_3)Ax B + (\alpha_2 + \alpha_3)B x A = 0, \quad (30)$$

that is,

$$Ax(2\alpha_1 A + (\alpha_2 + \alpha_3)B) + Bx(2\alpha_4 B + (\alpha_2 + \alpha_3)A) = 0. \quad (31)$$

Since A, B are C -independent, by (31) and again Remark 8, it follows that $2\alpha_1 A + (\alpha_2 + \alpha_3)B = 0$ and $2\alpha_4 B + (\alpha_2 + \alpha_3)A = 0$, so the independence of A and B forces $\alpha_1 = \alpha_4 = \alpha_2 + \alpha_3 = 0$.

Therefore we have that $B[eRe, eRe] \subseteq CB$. Notice that $B[eRe, eRe] \neq 0$. In fact, if $B[eRe, eRe] = 0$, since $[eRe, eRe] \neq (0)$ is a noncentral Lie ideal of the simple Artinian ring eRe , the contradiction $B = 0$ is immediate.

Let $u, v \in [eRe, eRe]$. Hence there exist $\omega_1, \omega_2, 0 \neq \omega \in C$ such that

$$B[u, v] = \omega B \neq 0, \quad Bu = \omega_1 B, \quad Bv = \omega_2 B, \quad (32)$$

and by calculation we get the contradiction

$$0 \neq \omega B = B[u, v] = 0. \quad (33)$$

Hence we may assume that A and B are linearly C -dependent, say $A = \alpha B$, for $0 \neq \alpha \in C$, so also $A' = \alpha B'$. Equation (29) is now $2\alpha B x B' + 2\alpha B' x B = 0$, and it follows that B and B' must be linearly C -dependent, so that $B x B = 0$ and $B = B' = 0$.

Therefore in any case, we have that if $s_4(eR, eR, eR, eR)e \neq 0$, then $(1 - e)be = (1 - e)ae = 0$.

Let $J = eR$, $\bar{J} = J/J \cap I_R(J)$; $\bar{J}$ is a prime C -algebra. Since $d(J) \subseteq J$ and $\delta(J) \subseteq J$, d and δ induce on $\bar{J}$ the following two derivations:

$$\bar{d}: \bar{J} \longrightarrow \bar{J} \quad \text{such that } \bar{d}(\bar{x}) = \overline{d(x)}, \\ \bar{\delta}: \bar{J} \longrightarrow \bar{J} \quad \text{such that } \bar{\delta}(\bar{x}) = \overline{\delta(x)}. \quad (34)$$

Therefore, we have

$$0 = [\overline{\delta([r_1, r_2])}, \overline{d[r_3, r_4]}] + [\overline{d([r_1, r_2])}, \overline{\delta[r_3, r_4]}] \quad (35)$$

for all $\overline{r_1}, \overline{r_2}, \overline{r_3}, \overline{r_4} \in \overline{J}$. By Lemma 4, we have that one of the following holds:

$$\overline{\delta} = \overline{0}, \quad \overline{d} = \overline{0}, \quad \overline{J} \text{ is commutative.} \quad (36)$$

Since $s_4(J, J, J, J)J \neq 0$, the last case cannot occur. On the other hand, now we prove that also the other cases lead us to contradictions.

Suppose that the first case occurs, that is, $\delta(J)J = 0$. By the lemma in [3], there exists an element $q = a - \alpha \in Q$, with $\alpha \in C$, such that $(a - \alpha)J = 0$. Moreover a and q induce the same inner derivation δ , so that we have

$$([b, [x_1, x_2]][x_3, x_4] - [b, [x_3, x_4]][x_1, x_2])q = 0 \quad \forall x_1, x_2, x_3, x_4 \in J. \quad (37)$$

In particular, for any $r \in R$, choose $[x_1, x_2] = [e, er(1 - e)] = er(1 - e)$. From (37), it follows that

$$[b, [x_3, x_4]]eR(1 - e)q = 0. \quad (38)$$

If $(1 - e)q = 0$, we have $q = eq \in J$. Under this condition, by Lemma 12, it follows from (37) that either $q = 0$, which implies the contradiction $a \in C$ and $\delta = 0$, or $(b - \beta)J = 0$ for a suitable $\beta \in C$, that is, $d(eR)eR = 0$. So consider the case when $[b, [x_3, x_4]]e = 0$ for all $x_3, x_4 \in J$, and remember that $be = ebe$. This implies that $[ebe, [y_1, y_2]] = 0$ for all $y_1, y_2 \in eRe$, that is, either eRe is a commutative central simple algebra or $ebe \in Ce$. In the first case, we have the contradiction $0 = s_4(ec_1, ec_2, ec_3, ec_4)ec_5 = s_4(c_1, c_2, c_3, c_4)c_5 \neq 0$. In the second one, we get again $d(eR)eR = 0$. Therefore we conclude that in any case, $\delta(eR)eR = d(eR)eR = 0$, which is again a contradiction because of $\delta(c_6)c_7 \neq 0$ or $d(c_8)c_9 \neq 0$.

Obviously by a similar argument and (36), we are also finished when $d(J)J = 0$. $\square$

For the proof of the main theorem, we need the following results.

LEMMA 14. *Let R be a prime ring of characteristic different from 2 and I a nonzero right ideal of R . If for any $x_1, x_2, x_3, x_4 \in I$, $[[x_1, x_2], [x_3, x_4]] = 0$, then $[I, I]I = 0$.*

Proof. First note that if $[y, [I, I]] = 0$, for some $y \in R$, then, for any $s, t \in I$, we have $0 = [y, [st, t]] = [s, t][y, t]$. In particular, for any $x \in IR$, $0 = [sx, t][y, t] = [s, t]x[y, t]$, that is $[s, t]IR[y, t] = 0$. By the primeness of R , we have that either $[s, t]I = 0$, that is, $[I, I]I = 0$, or $[y, I] = 0$. In this last case, $0 = [y, IR] = I[y, R]$ forcing $y \in Z(R)$.

Therefore, if we assume that $[I, I]I \neq 0$, the assumption $[[I, I], [I, I]] = 0$ forces $0 \neq [I, I] \subseteq Z(R)$. Let $s, t \in I$ be such that $[s, t]I \neq 0$ and $[s, t] \in Z(R)$. Then $2s[s, t] = [s^2, t] \in Z(R)$, so $[s, t] \neq 0$ forces $s \in Z(R)$ and we have the contradiction $[s, I] = 0$. $\square$

LEMMA 15. *Let R be a noncommutative prime ring of characteristic different from 2, q a noncentral element of R , and I a nonzero right ideal of R . If for any $x_1, x_2, x_3, x_4 \in I$, $[[q, [x_1, x_2]], [x_3, x_4]] = 0$, then $[I, I]I = 0$.*

Proof. Suppose that $[I, I]I \neq 0$. As in Lemma 14, first we recall that the condition $[y, [I, I]] = 0$ forces $y \in Z(R)$. This means that $[q, [I, I]] \subseteq Z(R)$, since $[[q, [I, I]], [I, I]] = 0$. Moreover we may assume that $[q, [I, I]] \neq 0$, if not, then we are finished by Lemma 14.

Note that from $[q, [I, I]] \subseteq Z(R)$, it follows that $[q, [[I, I], I]] \subseteq Z(R)$. Expanding this yields $[[I, I], [q, I]] \subseteq Z(R)$. Since for all $x \in I$, we have $[[I, I], [q, xq]] \subseteq Z(R)$, then $[[I, I], [q, I]q] \subseteq Z(R)$. Hence

$$0 = [[I, I], [q, I]q, q] = [q, [q, I]][q, [I, I]]. \quad (39)$$

Since the second factor is nonzero and central, we have $[q, [q, I]] = 0$, which implies that for all $x, y \in I$,

$$0 = [q, [q, xy]] = [q, [q, x]y + x[q, y]] = 2[q, x][q, y]. \quad (40)$$

This means that $[q, I][q, I] = 0$ and a fortiori $[q, [I, I]][q, [I, I]] = 0$ giving the contradiction $[q, [I, I]] = 0$. $\square$

We are ready to prove the following main result.

THEOREM 16. *Let R be a prime ring of characteristic different from 2, I a nonzero right ideal of R , d and δ nonzero derivations of R , $s_4(x_1, \dots, x_4)$ the standard identity of degree 4. If the composition $(d\delta)$ is a Lie derivation of $[I, I]$ into R , then either $s_4(I, I, I, I)I = 0$ or $\delta(I)I = d(I)I = 0$.*

Proof. By Remark 2, we assume that $[\delta(u), d(v)] + [d(u), \delta(v)] = 0$, for any $u, v \in [I, I]$. Suppose by contradiction that there exist $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9$ in I such that $s_4(c_1, c_2, c_3, c_4)c_5 \neq 0$ and either $\delta(c_6)c_7 \neq 0$ or $d(c_8)c_9 \neq 0$.

First suppose that δ and d are C -independent modulo D_{int} .

Let $t_1, t_2, t_3, t_4 \in I$, by assumption, R satisfies

$$\begin{aligned} & [[\delta(t_1x_1), t_2x_2] + [t_1x_1, \delta(t_2x_2)], [d(t_3x_3), t_4x_4] + [t_3x_3, d(t_4x_4)]] \\ & + [[d(t_1x_1), t_2x_2] + [t_1x_1, d(t_2x_2)], [\delta(t_3x_3), t_4x_4] + [t_3x_3, \delta(t_4x_4)]] \\ & = [[\delta(t_1)x_1 + t_1\delta(x_1), t_2x_2] + [t_1x_1, \delta(t_2)x_2 + t_2\delta(x_2)], \\ & \quad [d(t_3)x_3 + t_3d(x_3), t_4x_4] + [t_3x_3, d(t_4)x_4 + t_4d(x_4)]] \\ & + [[d(t_1)x_1 + t_1d(x_1), t_2x_2] + [t_1x_1, d(t_2)x_2 + t_2d(x_2)], \\ & \quad [\delta(t_3)x_3 + t_3\delta(x_3), t_4x_4] + [t_3x_3, \delta(t_4)x_4 + t_4\delta(x_4)]] = 0. \end{aligned} \quad (41)$$

By Kharchenko's theorem [7], R satisfies the generalized polynomial identity

$$\begin{aligned} & [[\delta(t_1)x_1 + t_1y_1, t_2x_2] + [t_1x_1, \delta(t_2)x_2 + t_2y_2], [d(t_3)x_3 + t_3z_3, t_4x_4] + [t_3x_3, d(t_4)x_4 + t_4z_4]] \\ & + [[d(t_1)x_1 + t_1z_1, t_2x_2] + [t_1x_1, d(t_2)x_2 + t_2z_2], [\delta(t_3)x_3 + t_3y_3, t_4x_4] + [t_3x_3, \delta(t_4)x_4 + t_4y_4]] \end{aligned} \quad (42)$$

in particular R satisfies $[[t_1 y_1, t_2 x_2], [t_3 x_3, t_4 z_4]]$, so Q satisfies this as well, and for all $y_1 = x_2 = x_3 = z_4 = 1 \in Q$, it follows that $[[I, I], [I, I]] = 0$. Thus by Lemma 14, we conclude that $[I, I]I = 0$, that is, $s_4(I, I, I, I)I = 0$, which contradicts $s_4(c_1, c_2, c_3, c_4)c_5 \neq 0$.

Now let δ and d be C -dependent modulo D_{int} . There exist $\gamma_1, \gamma_2 \in C$, such that $\gamma_1 \delta + \gamma_2 d \in D_{\text{int}}$, and by Lemma 13, it is clear that at most one of the two derivations can be inner.

Without loss of generality, we may assume that $\gamma_1 \neq 0$, so that $\delta = \alpha d + ad(q)$, for $\alpha \in C$ and $ad(q)$ the inner derivation induced by the element $q \in Q$.

If d is inner, then also δ is inner, and we have that d is an outer derivation of R . Let $t_1, t_2, t_3, t_4 \in I$, R satisfies

$$\begin{aligned}
 & \alpha[[d(t_1 x_1), t_2 x_2] + [t_1 x_1, d(t_2 x_2)], [d(t_3 x_3), t_4 x_4] + [t_3 x_3, d(t_4 x_4)]] \\
 & + [[q, [t_1 x_1, t_2 x_2]], [d(t_3 x_3), t_4 x_4] + [t_3 x_3, d(t_4 x_4)]] \\
 & + \alpha[[d(t_1 x_1), t_2 x_2] + [t_1 x_1, d(t_2 x_2)], [d(t_3 x_3), t_4 x_4] + [t_3 x_3, d(t_4 x_4)]] \\
 & + [[d(t_1 x_1), t_2 x_2] + [t_1 x_1, d(t_2 x_2)], [q, [t_3 x_3, t_4 x_4]]] \\
 & = \alpha[[d(t_1) x_1 + t_1 d(x_1), t_2 x_2] + [t_1 x_1, d(t_2) x_2 + t_2 d(x_2)], \\
 & \quad [d(t_3) x_3 + t_3 d(x_3), t_4 x_4] + [t_3 x_3, d(t_4) x_4 + t_4 d(x_4)]] \\
 & + [[q, [t_1 x_1, t_2 x_2]], [d(t_3) x_3 + t_3 d(x_3), t_4 x_4] + [t_3 x_3, d(t_4) x_4 + t_4 d(x_4)]] \\
 & + \alpha[[d(t_1) x_1 + t_1 d(x_1), t_2 x_2] + [t_1 x_1, d(t_2) x_2 + t_2 d(x_2)], \\
 & \quad [d(t_3) x_3 + t_3 d(x_3), t_4 x_4] + [t_3 x_3, d(t_4) x_4 + t_4 d(x_4)]] \\
 & + [[d(t_1) x_1 + t_1 d(x_1), t_2 x_2] + [t_1 x_1, d(t_2) x_2 + t_2 d(x_2)], [q, [t_3 x_3, t_4 x_4]]],
 \end{aligned} \tag{43}$$

and so the Kharchenko theorem shows that R satisfies

$$\begin{aligned}
 & \alpha[[d(t_1) x_1 + t_1 y_1, t_2 x_2] + [t_1 x_1, d(t_2) x_2 + t_2 y_2], [d(t_3) x_3 + t_3 y_3, t_4 x_4] + [t_3 x_3, d(t_4) x_4 + t_4 y_4]] \\
 & + [[q, [t_1 x_1, t_2 x_2]], [d(t_3) x_3 + t_3 y_3, t_4 x_4] + [t_3 x_3, d(t_4) x_4 + t_4 y_4]] \\
 & + \alpha[[d(t_1) x_1 + t_1 y_1, t_2 x_2] + [t_1 x_1, d(t_2) x_2 + t_2 y_2], [d(t_3) x_3 + t_3 y_3, t_4 x_4] + [t_3 x_3, d(t_4) x_4 + t_4 y_4]] \\
 & + [[d(t_1) x_1 + t_1 y_1, t_2 x_2] + [t_1 x_1, d(t_2) x_2 + t_2 y_2], [q, [t_3 x_3, t_4 x_4]]].
 \end{aligned} \tag{44}$$

In case $\alpha \neq 0$, for $x_1 = x_4 = 0$ in (44), we have that R satisfies

$$2\alpha[[t_1 y_1, t_2 x_2], [t_3 x_3, t_4 y_4]], \tag{45}$$

so Q satisfies this as well and for all $y_1 = x_2 = x_3 = y_4 = 1 \in Q$, it follows that $2\alpha[[I, I], [I, I]] = 0$. Hence, if $\alpha \neq 0$, by Lemma 14, we have the contradiction $[I, I]I = 0$.

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Now let $\alpha = 0$. In this case for $x_4 = 0$ in (44), we have that R satisfies

$$[[q, [t_1x_1, t_2x_2]], [t_3x_3, t_4y_4]]. \quad (46)$$

As above Q satisfies this and, taking $x_1, x_2, x_3, y_4 = 1$, it follows that

$$[[q, [I, I]], [I, I]] = 0. \quad (47)$$

Then, by Lemma 15, we conclude again with the contradiction $[I, I]I = 0$.

Similarly, when $y_2 \neq 0$, then $d = \beta\delta + ad(q)$, for some $\beta \in C$, and mimicking the argument above gives another contradiction. $\square$

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References

- [1] K. I. Beidar, *Rings with generalized identities. III*, Moscow University Mathematics Bulletin **33** (1978), no. 4, 53–58.
- [2] K. I. Beidar, W. S. Martindale III, and A. V. Mikhalev, *Rings with Generalized Identities*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 196, Marcel Dekker, New York, 1996.
- [3] M. Brešar, *One-sided ideals and derivations of prime rings*, Proceedings of the American Mathematical Society **122** (1994), no. 4, 979–983.
- [4] C.-L. Chuang, *GPIs having coefficients in Utumi quotient rings*, Proceedings of the American Mathematical Society **103** (1988), no. 3, 723–728.
- [5] C. Faith and Y. Utumi, *On a new proof of Litoff's theorem*, Acta Mathematica Academiae Scientiarum Hungaricae **14** (1963), no. 3-4, 369–371.
- [6] N. Jacobson, *Structure of Rings*, American Mathematical Society Colloquium Publications, vol. 37, American Mathematical Society, Rhode Island, 1964.
- [7] V. K. Kharchenko, *Differential identities of prime rings*, Algebra and Logic **17** (1978), no. 2, 155–168.
- [8] C. Lanski, *Differential identities, Lie ideals, and Posner's theorems*, Pacific Journal of Mathematics **134** (1988), no. 2, 275–297.
- [9] T.-K. Lee, *Semiprime rings with differential identities*, Bulletin of the Institute of Mathematics. Academia Sinica **20** (1992), no. 1, 27–38.
- [10] ———, *Left annihilators characterized by GPIs*, Transactions of the American Mathematical Society **347** (1995), no. 8, 3159–3165.
- [11] W. S. Martindale III, *Prime rings satisfying a generalized polynomial identity*, Journal of Algebra **12** (1969), no. 4, 576–584.
- [12] E. C. Posner, *Derivations in prime rings*, Proceedings of the American Mathematical Society **8** (1957), no. 6, 1093–1100.

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