ON THE BASIS NUMBER OF THE CORONA OF GRAPHS

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Received 5 February 2005; Revised 19 June 2006; Accepted 22 June 2006

The basis number $b(G)$ of a graph $G$ is defined to be the least integer $k$ such that $G$ has a $k$-fold basis for its cycle space. In this note, we determine the basis number of the corona of graphs, in fact we prove that $b(v \circ T) = 2$ for any tree and any vertex $v$ not in $T$, $b(v \circ H) \leq b(H) + 2$, where $H$ is any graph and $v$ is not a vertex of $H$, also we prove that if $G = G_1 \circ G_2$ is the corona of two graphs $G_1$ and $G_2$, then $b(G_1) \leq b(G) \leq \max\{b(G_1), b(G_2) + 2\}$, moreover we prove that if $G$ is a Hamiltonian graph, then $b(v \circ G) \leq b(G) + 1$, where $v$ is any vertex not in $G$, and finally we give a sequence of remarks which gives the basis number of the corona of some of special graphs.

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1. Introduction

In this note, we consider only finite, undirected, simple graphs. Our terminology and notation will be standard except as indicated. For undefined terms, see [7]. Let $G$ be a $(p, q)$ graph (i.e., $G$ has $p$ vertices and $q$ edges), and let $e_1, e_2, \ldots, e_q$ be an ordering of its edges. Then any subset $E$ of edges in $G$ corresponds to $(0, 1)$-vector $(v_1, \ldots, v_q)$ with $v_i = 1$ if $e_i \in E$ and $v_i = 0$ if $e_i \notin E$. The vectors form a $q$-dimensional vector space over the field of two elements $\mathbb{Z}_2$ and is denoted by $(\mathbb{Z}_2)^q$. The vectors in $(\mathbb{Z}_2)^q$ which correspond to the cycles in $G$ generate a subspace called the cycle space of $G$ and is denoted by $C(G)$, we will say that the cycles themselves, instead of saying the vectors corresponding to the cycles, generate $C(G)$. It is well known (see [7, page 39]) that if $G$ is a $(p, q)$ graph with $k$ components, then $\dim C(G) = \chi(G) = q - p + k$, where $\chi(G)$ is the cyclomatic number of $G$. A basis for $C(G)$ is called a $k$-fold basis if each edge of $G$ occurs in at most $k$ of the cycles in the basis. The basis number of $G$ denoted by $b(G)$ is the smallest integer $k$ such that $C(G)$ has a $k$-fold basis. The corona (see [7, page 167]) of two graphs $G_1$ and $G_2$, denoted by $G_1 \circ G_2$, is defined to be the graph $G$ obtained by taking one copy of $G_1$ (which has $p_1$ vertices) and $p_1$ copies of $G_2$, and then joining the vertex of $G_1$ to every vertex in the $i$th copy of $G_2$. If $G_1$ is a $(p_1, q_1)$ graph and $G_2$ is a $(p_2, q_2)$ graph, then it follows from the definition of the corona that $G_1 \circ G_2$ has $p_1(1 + p_2)$ vertices and $q_1 + p_1q_2 + p_1p_2$ edges.
(see [7, page 168]). It is clear that if $G_1$ is connected, then $G_1 \circ G_2$ is connected, and in general $G_1 \circ G_2$ is not isomorphic to $G_2 \circ G_1$.

In the rest of this note, $P_n$, $C_n$, $S_n$, and $W_n$ stand for the path, the cycle, the star, and the wheel of $n$ vertices. A theta graph $\theta_n$ is defined to be a cycle $C_n$ with $n$ vertices, respectively, to which we add a new edge that joins two nonadjacent vertices of $C_n$.

MacLane [8] proved that a graph $G$ is planar if and only if $b(G) \leq 2$. Schmeichel [9] proved that for $n \geq 5$, $b(K_n) = 3$ and for $m, n \geq 5$, $b(K_{m,n}) = 4$ except for $K_{6,10}$, $K_{5,5}$, and $K_{6,6}$, where $n = 5, 6, 7$, and 8. Banks and Schmeichel [6] proved that for $n \geq 7$, $b(Q_n) = 7$, where $Q_n$ is the $n$-cube. Ali [1] proved that $b(K_{n,n,...,n}) \leq 9$, $b(K_{n,n,n}) = 3$ for all $n \geq 3$, and $b(K_{i,i,n}) \leq 4$.

Moreover, Ali [2] proved that $b(C_m \wedge P_n) = 2$, and $b(C_m \wedge C_n) = 3$. Al-Rhayyel [4] proved that $b(P_2 \times \theta_n) = 2$ and $b(\theta_n \times \theta_n) = 3$ for all $n, m \geq 4$. Al-Rhayyel [5] proved that $b(P_2 \wedge W_n) = 2$ and $b(P_m \wedge W_n) = 3$ for all $m \geq 3$, $n \geq 4$, and $n$ is even where $\wedge$ and $\times$ are the direct and the cartesian products of graphs, respectively. Next we restate [3, Theorem 2.3].

**Theorem 1.1.** Let $G'$ be a graph obtained from $G$ by deleting an edge $e$ of at most 2-fold in a basis $B$ for $C(G)$. Then $b(G) - 1 \leq b(G') \leq b(G)$.

The purpose of this note is to investigate the basis number of the corona of graphs, in fact we prove that for any two graphs $G_1$ and $G_2$, if $G = G_1 \circ G_2$, then $b(G_1) \leq b(G) \leq \max\{b(G_1), b(G_2) + 2\}$ and we give the exact basis number of the corona of some special graphs.

### 2. Main results

This section is devoted for proving the main results of this note, and this is done by writing a sequence of theorems and remarks.

**Remark 2.1.** We note that if $v$ is not a vertex of $G$, then $b(v \circ G) = 2$, where $G$ is any one of the following graphs: $P_n$, $C_n$ or $S_n$ and $b(v \circ G) = 3$ if $G$ is either $W_n$ or $K_n$ ($n \geq 4$).

**Lemma 2.2.** Let $T$ be a tree with $p$ vertices ($p \geq 3$) if $v$ is any point which is not a vertex of $T$, and if $G = v \circ T$, then $b(G) = 2$, and hence $G$ is planar.

**Proof.** Assume that $G$ is not planar. Then, by Kuratowski’s theorem, $G$ contains a subdivision of $K_5$ or $K_{3,3}$. Then $G - x$ cannot be an acyclic graph for any $x \in V(G)$, while $G - v$ is a tree. This is a contradiction, and hence $G$ is planar. Therefore, $b(G) \leq 2$. If $b(G) = 1$, then $G$ has a 1-fold basis, which implies that $\dim C(G) \leq |E(G)|/3$ since each cycle contains at least three edges. Since $|E(G)| = 2p - 1$ and $\dim C(G) = p - 1$, we have $p - 1 \leq (2p - 1)/3$, which implies that $p \leq 2$. This is a contradiction. Therefore, $b(G) = 2$. 

**Lemma 2.3.** Let $H$ be any connected $(p,q)$ graph and let $v$ be any vertex which is not a vertex of $H$. If $G = v \circ H$, then $b(G) \leq b(H) + 2$.

**Proof.** Let $u_1, u_2, \ldots, u_p$ be the vertices of $H$. Since $\dim C(G) = q$ and $\dim C(H) = q - p + 1$, $\dim C(G) - \dim C(H) = p - 1$. Let $T$ be a spanning tree of $H$. Then $b(v \circ T) = 2$, $\dim C(v \circ T) = p - 1$, and each cycle in $v \circ T$ must contain an edge of the form $vu_i$ for some $i \in \{1, 2, \ldots, p\}$. Thus the cycles in $v \circ T$ are independent from the cycles in $H$. Let $B_1$ be a $(H)$-fold basis for $C(H)$, and let $B_2$ be a 2-fold basis for $v \circ T$. Then clearly...
B = B_1 ∪ B_2 is an independent set of cycles with |B| = \dim C(G), hence B is a basis for C(G). Note that if e is an edge of G, then either e is an edge of H or e = vu_i for some i \in \{1,2,\ldots,p\}. If e = vu_i, then f_B(e) ≤ f_{B_1}(e) ≤ 2, and if e is an edge of H, then clearly f_B(e) ≤ b(H) + 2. Thus, b(G) ≤ b(H) + 2.

\[\square\]

**Theorem 2.4.** Let G_1 and G_2 be two connected graphs. If G = G_1 ∘ G_2, then \(b(G_1) \leq b(G) \leq \max\{b(G_1), b(G_2) + 2\}\).

**Proof.** Clearly \(b(G_1) \leq b(G)\). Let \(v_1,\ldots,v_n\) be the vertices of \(G_1\) and let \(H_k = v_k \circ G_2\) and let \(B_k\) be the basis of \(H_k\), for each \(k = 1,\ldots,n\). Clearly \(E(H_i) \cap E(H_j) = \emptyset\), for all \(i \neq j, j \in \{1,\ldots,n\}\). Therefore, \(\bigcup_{k=1}^{n} B_k\) is linearly independent. Let \(B = (\bigcup_{k=1}^{n} B_k) \cup B(G_1)\), where \(B(G_1)\) is a \(b(G_1)\)-fold basis for \(G_1\). Since \(E(G_1) \cap E((\bigcup_{k=1}^{n} H_k)) = \emptyset\), as a result \(B\) is linearly independent. Since \(|E(B)| = \dim C(G)\), \(B\) is a basis of \(C(G)\). By Lemma 2.3, \(b(H_k) \leq b(G_2) + 2\) for each \(k \in \{1,\ldots,n\}\). Therefore, \(b(G_1) \leq b(G) \leq \max\{b(G_1), b(G_2) + 2\}\). \(\square\)

**Lemma 2.5.** Let \(H\) be a Hamiltonian graph, and let \(v\) be any point which is not a vertex of \(H\). If \(G = v \circ H\), then \(b(G) \leq b(H) + 1\).

**Proof.** Let \(C = u_1u_2,\ldots,u_nu_1\) be a spanning cycle of \(H\), then \(G\) is obtained from \(H\) by joining every vertex \(v_i\) of \(H\) to the vertex \(v\). Let \(B = \{vu_iu_{i+1} : i = 1,2,\ldots,n - 1\} \cup B(H)\), where \(B(H)\) is a \(b(H)\)-fold basis of \(C(H)\). Then, clearly that \(B\) is a basis of \(C(G)\) and given any edge of \(H\), then it occurs in at most one of these cycles, hence \(b(G) \leq b(H) + 1\). \(\square\)

**Corollary 2.6.** If \(G_1\) and \(G_2\) are two graphs such that \(b(G_1) \geq b(G_2) + 2\), then \(b(G_1 \circ G_2) = b(G_1)\). Moreover, if \(G_2\) is Hamiltonian, and \(b(G_1) \geq b(G_2) + 1\), then \(b(G_1 \circ G_2) = b(G_1)\).

**References**


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