We consider the existence of analytic solutions of a certain class of iterative second-order functional differential equation of the form

\[ x''(x[r](z)) = c_0 z^2 + c_1 (x(z))^2 + (c_2 x^{[2]}(z))^2 + \cdots + c_m (x^{[m]}(z))^2, \]

where \( r \) and \( m \) are nonnegative integers, \( c_0, c_1, c_2, \ldots, c_m \) are complex numbers, and \( x^{[j]} \) denotes the \( j \)th iterate of \( x \). In order to obtain analytic solutions of (1.1), we first seek the analytic solutions \( y(z) \) of the following companion equation:

\[ \alpha^2 y''(\alpha^{r+1} z) y' (\alpha^{r+1} z) = \alpha y' (\alpha^{r+1} z) y'' (\alpha^{r+1} z) + \left[ y' (\alpha^{r+1} z) \right]^3 \sum_{j=0}^{m} c_j (y(\alpha^j z))^2 \]  

(1.3)

satisfying the initial value conditions

\[ y(0) = \mu, \quad y'(0) = \eta \neq 0, \]  

(1.4)
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where \( \mu, \eta \) are complex numbers, and \( \alpha \) satisfies one of the following conditions:

(H1) \(|\alpha| > 1\);  
(H2) \(0 < |\alpha| < 1\);  
(H3) \(|\alpha| = 1, \alpha \) is not a root of unity, and \( \log(1/|\alpha^n - 1|) \leq K \log n, n = 2, 3, 4, \ldots \), for some positive constant \( K \). Then we show that (1.2) has an analytic solution of the form

\[
x(z) = y(\alpha y^{-1}(z)),
\]

in a neighborhood of the number \( \mu \), where \( y^{-1}(z) \) is the inverse function of \( y(z) \). Finally, we make use of (1.5) to show how to derive an explicit power series solution of (1.2).

2. Preliminary lemmas

We first obtain the analytic solutions \( y(z) \) of the companion equation (1.3). By (1.3), we have that

\[
\frac{\alpha^2 y''(\alpha^r z) y'(\alpha^r z) - \alpha y'(\alpha^r z) y''(\alpha^r z)}{y'(\alpha^r z)^2} = y'(\alpha^r z) \sum_{j=0}^{m} c_j(y(\alpha^j z))^2, \quad \text{or}
\]

\[
\frac{1}{\alpha^{r-1}} \left( \frac{y'(\alpha^r z)}{y'(\alpha z)} \right)' = y'(\alpha^r z) \sum_{j=0}^{m} c_j(y(\alpha^j z))^2, \quad \text{or}
\]

\[
\frac{1}{\alpha^{r-1}} \left[ \frac{y'(\alpha^r z)}{y'(\alpha z)} - \frac{y'(0)}{y'(\alpha^r z)} \right] = \int_0^z y'(\alpha^r t) \sum_{j=0}^{m} c_j(y(\alpha^j t))^2 dt, \quad \text{or}
\]

\[
\frac{1}{\alpha^{r-1}} \left[ \frac{y'(\alpha^r z)}{y'(\alpha z)} - 1 \right] = \int_0^z y'(\alpha^r t) \sum_{j=0}^{m} c_j(y(\alpha^j t))^2 dt, \quad \text{or}
\]

\[
\frac{1}{\alpha^{r-1}} [y'(\alpha^r z) - y'(\alpha z)] = y'(\alpha^r z) \int_0^z y'(\alpha^r t) \sum_{j=0}^{m} c_j(y(\alpha^j t))^2 dt.
\]

Since \( y(z) \) is an analytic function in a neighborhood of 0, \( y(z) \) can be represented by a power series of the form

\[
y(z) = \sum_{n=0}^{+\infty} b_n z^n,
\]

and we can see easily that \( b_0 = \mu, b_1 = \eta, \) and \( y'(z) = \sum_{n=0}^{+\infty} (n+1)b_{n+1}z^n \). We have

\[
\frac{1}{\alpha^{r-1}} [y'(\alpha^r z) - y'(\alpha^r z)]
\]

\[
= \frac{1}{\alpha^{r-1}} \left[ \sum_{n=0}^{+\infty} (n+1)\alpha^{(r+1)n}z^n - \sum_{n=0}^{+\infty} (n+1)b_{n+1}\alpha^{rn}z^n \right]
\]

\[
= \frac{1}{\alpha^{r-1}} \left[ \sum_{n=0}^{+\infty} (n+1)b_{n+1}(\alpha^n - 1)\alpha^{rn}z^n \right]
\]
\[
\frac{1}{\alpha^r - 1} \left[ \sum_{n=1}^{+\infty} (n+1)b_{n+1}(\alpha^n - 1)\alpha^n z^n \right]
\]
\[
= \frac{1}{\alpha^r - 1} \left[ \sum_{n=0}^{+\infty} (n+2)b_{n+2}(\alpha^{n+1} - 1)\alpha^{(n+1)} z^{n+1} \right]
\]
\[
= \sum_{n=0}^{+\infty} (n+2)(\alpha^{n+1} - 1)\alpha^{n+1}b_{n+2}z^{n+1}.
\]

Therefore,
\[
\frac{1}{\alpha^r - 1} \left[ y'(\alpha^{r+1}z) - y'(\alpha^r z) \right] = \sum_{n=0}^{+\infty} (n+2)(\alpha^{n+1} - 1)\alpha^{n+1}b_{n+2}z^{n+1}.
\]

By means of (2.2), we get that
\[
y^2(z) = \left( \sum_{n=0}^{+\infty} b_n z^n \right)^2 = \sum_{n=0}^{+\infty} \left( \sum_{i=0}^{n} b_i b_{n-i} \right) z^n.
\]

Then
\[
y^2(\alpha^j z) = \sum_{n=0}^{+\infty} \left( \sum_{i=0}^{n} b_i b_{n-i} \right) \alpha^{jn} z^n, \quad j = 0, 1, 2, \ldots, m.
\]

This implies
\[
\int_{z_0}^{z} y'(\alpha^r t) \sum_{j=0}^{m} c_j (y(\alpha^j t))^2 \ dt
\]
\[
= \int_{z_0}^{z} \left( \sum_{n=0}^{+\infty} (n+1)b_{n+1}\alpha^n t^n \right) \left( \sum_{n=0}^{+\infty} \sum_{i=0}^{n} \sum_{j=0}^{m} c_j \alpha^{jn} b_i b_{n-i} t^n \right) dt
\]
\[
= \int_{z_0}^{z} \sum_{n=0}^{+\infty} \sum_{k=0}^{n-k} \sum_{i=0}^{m} c_j \alpha^{(n-k)j+r}(k+1)b_i b_{k+1} b_{n-k-i} dt
\]
\[
= \sum_{n=0}^{+\infty} \left( \sum_{k=0}^{n} \sum_{i=0}^{n-k} \sum_{j=0}^{m} c_j \alpha^{(n-k)j+r}(k+1)b_i b_{k+1} b_{n-k-i} \right) \frac{z^{n+1}}{n+1}.
\]

Therefore,
\[
\int_{z_0}^{z} y'(\alpha^r t) \sum_{j=0}^{m} c_j (y(\alpha^j t))^2 \ dt = \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \sum_{i=0}^{n-k} \sum_{j=0}^{m} c_j \alpha^{(n-k)j+(k+1)r}(k+1)b_i b_{k+1} b_{n-k-i} \frac{z^{n+1}}{n+1}.
\]
Next, we will consider
\[ y'(\alpha z) \int_0^z y'(\alpha t) \sum_{j=0}^m c_j(y(\alpha t))^2 \, dt \]
\[ = \left( \sum_{n=0}^{\infty} (n+1)b_{n+1}\alpha_n z^n \right) \left( \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{i=0}^{n-k} \sum_{j=0}^m c_j \alpha^{(n-k)j+kr} \right) (k+1)b_1b_{k+1}b_{n-k-i}z^{n+1} \]
\[ = \sum_{n=0}^{\infty} \sum_{s=0}^n \sum_{k=0}^n \sum_{i=0}^{n-s-k} \sum_{j=0}^m c_j \alpha^{(n-s-k)j+(k+s)r} \frac{(s+1)(k+1)b_1b_{s+1}b_{k+1}b_{n-s-k-i}z^{n+1}}{n-s+1}. \]

Therefore, we have (2.10)
\[ y'(\alpha z) \int_0^z y'(\alpha t) \sum_{j=0}^m c_j(y(\alpha t))^2 \, dt \]
\[ = \sum_{n=0}^{\infty} \sum_{s=0}^n \sum_{k=0}^n \sum_{i=0}^{n-s-k} \sum_{j=0}^m c_j \alpha^{(n-s-k)j+(k+s)r} \frac{(s+1)(k+1)b_1b_{s+1}b_{k+1}b_{n-s-k-i}z^{n+1}}{n-s+1}. \]

We see that (1.3) is equivalent to the integrodifferential equation (2.1). By (2.1), (2.4), and (2.10), we see that

\[ (n+2)(\alpha^{n+1} - 1) \alpha_{rn+1} b_{n+2} = \frac{\sum_{s=0}^n \sum_{k=0}^n \sum_{i=0}^{n-k-s} \sum_{j=0}^m c_j \alpha^{(n-s-k)j+(k+s)r}}{n-s+1} \times (s+1)(k+1)b_1b_{s+1}b_{n-s-k-i}, \quad n = 0,1,2,\ldots. \]

Therefore,
\[ b_{n+2} = \frac{\sum_{s=0}^n \sum_{k=0}^n \sum_{i=0}^{n-k-s} \sum_{j=0}^m c_j \alpha^{(n-s-k)j+(k+s)r-rn-1}}{(n+2)(n-s+1)(\alpha^{n+1} - 1)} (s+1)(k+1)b_1b_{s+1}b_{n-s-k-i}, \quad n = 0,1,2,\ldots. \]
Proof. For convenience, we let $M = \sum_{j=0}^{m} |c_j|$. By means of (2.12), it follows that for each $n = 0, 1, 2, \ldots,$

$$
|b_{n+2}| \leq \sum_{i=0}^{n} \sum_{k=0}^{n-s} \sum_{t=0}^{n-k-s} \sum_{j=0}^{m} |c_j| |a^{(n-s-k)}(n+2)(n-s+1)(\alpha^{n+1} - 1)|
$$

$$
\times (s+1)(k+1) |b_s| |b_{s+1}| |b_{k+1}| |b_{n-s-k-i}|
$$

(2.13)

Therefore,

$$
|b_{n+2}| \leq \frac{M}{(\alpha^{n+1} - 1)} \sum_{s=0}^{n} \sum_{k=0}^{n-s} \sum_{i=0}^{n-k-s} |b_s| |b_{s+1}| |b_{k+1}| |b_{n-s-k-i}|
$$

(2.14)

where $n = 0, 1, \ldots.$ Let

$$
Q(z, \omega) = \omega^4 - 2|\mu| \omega^3 + |\mu|^2 \omega^2 - \frac{1}{M} (\omega - |\mu| - z)
$$

(2.15)

for $(z, \omega)$ in a neighborhood of $(0, |\mu|)$. We see that $Q(0, |\mu|) = |\mu|^4 - 2|\mu|^4 + |\mu|^4 - (1/M)(|\mu| - |\mu| - 0) = 0$ and $Q'(z, \omega) = 4\omega^3 - 6|\mu| \omega^2 + 2|\mu|^2 \omega - 1/M$, so $Q'(0, |\mu|) = -1/M \neq 0$. Therefore, there exists a unique analytic function $G(z)$ in a neighborhood of 0 such that $G(0) = |\mu|$, $G'(0) = 1$ satisfy the equality $Q(z, G(z)) = 0$. It follows that

$$
G(z) = \sum_{n=0}^{+\infty} C_n z^n,
$$

(2.16)

where $C_0 = |\mu|$, $C_1 = 1$ in a neighborhood of 0. Next, we will show that

$$
C_{n+2} = M \sum_{s=0}^{n} \sum_{k=0}^{n-s} \sum_{i=0}^{n-k-s} C_i C_{s+1} C_{k+1} C_{n-s-k-i}, \quad n = 0, 1, \ldots.
$$

(2.17)

Suppose that (2.17) is true, by (2.16), we will get that

$$
G^3(z) = G(z)G^2(z) = \left(C_0 + \sum_{n=0}^{+\infty} C_n z^n\right) \left(\sum_{n=0}^{+\infty} \left(\sum_{i=0}^{n} C_i C_{n-i}\right) z^n\right)
$$

$$
= C_0 \sum_{n=0}^{+\infty} \left(\sum_{i=0}^{n} C_i C_{n-i}\right) z^n + \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} \sum_{i=0}^{n-k} C_i C_{k+1} C_{n-k-i}\right) z^{n+1},
$$

$$
G^4(z) = G(z)G^3(z) = \left(C_0 + \sum_{n=0}^{+\infty} C_n z^n\right) \left[C_0 \sum_{n=0}^{+\infty} \left(\sum_{i=0}^{n} C_i C_{n-i}\right) z^n + \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} \sum_{i=0}^{n-k} C_i C_{k+1} C_{n-k-i}\right) z^{n+1}\right]
$$
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\[ C_0^2 \sum_{n=0}^{+\infty} \left( \sum_{i=0}^{n} C_i C_{n-i} \right) z^n + 2C_0 \sum_{n=0}^{+\infty} \left( \sum_{k=0}^{n-k} C_i C_{k+1} C_{n-k-1} \right) z^{n+1} \]

\[ + \sum_{n=0}^{+\infty} \left( M \sum_{s=0}^{n} \sum_{k=0}^{n-s} C_i C_{s+1} C_{k+1} C_{n-s-k-1} \right) z^{n+2} \]

\[ = C_0^2 G^2(z) + 2C_0 \left[ G^3(z) - C_0 G^2(z) \right] + \frac{1}{M} \sum_{n=0}^{+\infty} C_{n+2} z^{n+2} \]

\[ = 2C_0 G^3(z) - C_0^2 G^2(z) + \frac{1}{M} \left( G(z) - C_0 - C_1 z \right) \]

\[ = 2|\mu| G^3(z) - |\mu|^2 G^2(z) + \frac{1}{M} \left( G(z) - |\mu| - z \right), \quad (2.18) \]

that is,

\[ G^4(z) = 2|\mu| G^3(z) - |\mu|^2 G^2(z) + \frac{1}{M} \left( G(z) - |\mu| - z \right), \quad (2.19) \]

or

\[ G^4(z) - 2|\mu| G^3(z) + |\mu|^2 G^2(z) - \frac{1}{M} \left( G(z) - |\mu| - z \right) = 0. \quad (2.20) \]

Hence, \( Q(z, G(z)) = 0 \). Furthermore, we see that \( Q(z, G(z)) = 0 \) if and only if (2.17) is true. Therefore, we conclude that (2.17) holds. Now, we know that the power series (2.16) converges in a neighborhood of 0. Therefore, there exists a positive constant \( P \) such that

\[ C_n < P^n \quad (2.21) \]

for \( n = 1, 2, 3, \ldots \). In the following lemma, we show that \( |b_n| \leq C_n d_n \), \( n = 1, 2, \ldots \), where the sequence \( \{d_n\}_{n=1}^{\infty} \) is defined as in Lemma 2.1. Indeed, \( |b_1| = |\eta| \leq 1 = C_1 d_1 \), so it suffices to prove that \( |b_{n+1}| \leq C_{n+1} d_{n+1}, n = 1, 2, \ldots \). Let \( P(n) \) denote the statement that \( |b_{n+1}| \leq C_{n+1} d_{n+1} \). From (2.14) and (2.17), we obtain

\[ |b_2| \leq \left( \sum_{j=0}^{m} |c_j| \right) |\alpha - 1|^{-1} |b_0| |b_1| |b_1| |b_0| \]

\[ \leq M |\alpha - 1|^{-1} C_0 C_1 d_1 C_1 d_1 C_0 \]

\[ = (MC_0 C_1 C_0) \left( |\alpha - 1|^{-1} d_1 d_1 \right) \]

\[ = C_2 |\alpha - 1|^{-1} \max_{n_1 + n_2 = 1, 0 < n_1 \leq n_2} \{d_{n_1} d_{n_2}\} = C_2 d_2. \quad (2.22) \]

Thus, \( P(2) \) is true. Next, suppose that \( P(1), P(2), \ldots, P(n) \) are true, that is, \( |b_{s+1}| \leq C_{s+1} d_{s+1} \), for all \( s = 1, 2, \ldots, n \). By (2.14) and (2.17), we get that

\[ |b_{n+2}| \leq \frac{M}{|\alpha^{n+1} - 1|} \sum_{s=0}^{n-s} \sum_{k=0}^{n-k-s} |b_s| |b_{k+1}| |b_{s+1}| |b_{n-s-k-1}| \]
\[
M \left( \alpha^{n+1} - 1 \right) \sum_{s=0}^{n} \sum_{k=0}^{n-s} \left( |b_0| |b_{k+1}| |b_{s+1}| |b_{n-s-k}| + |b_{n-s-k}| |b_{k+1}| |b_{s+1}| |b_0| \right) + \sum_{i=1}^{n-k-s-1} |b_i| |b_{k+1}| |b_{s+1}| |b_{n-s-k-i}| \\
= M \left( \alpha^{n+1} - 1 \right) \sum_{s=0}^{n} \sum_{k=0}^{n-s} \left( 2 |b_0| |b_{k+1}| |b_{s+1}| |b_{n-s-k}| + \sum_{i=1}^{n-k-s-1} |b_i| |b_{k+1}| |b_{s+1}| |b_{n-s-k-i}| \right) \\
= M \left( \alpha^{n+1} - 1 \right) \left[ \sum_{s=0}^{n} \left( 2 |b_0| |b_{n-s+1}| |b_{s+1}| |b_0| + \sum_{k=0}^{n-s-1} 2 |b_0| |b_{k+1}| |b_{s+1}| |b_{n-s-k}| + \sum_{s=0}^{n} \sum_{k=0}^{n-s-1} |b_i| |b_{k+1}| |b_{s+1}| |b_{n-s-k-i}| \right] \\
\leq M \left( \alpha^{n+1} - 1 \right) \left[ \sum_{s=0}^{n} 2C_0^2 C_{n-s+1} d_{n-s+1} C_{s+1} d_{s+1} + \sum_{s=0}^{n} \sum_{k=0}^{n-s-1} 2C_0 C_{k+1} d_{k+1} C_{s+1} d_{s+1} C_{n-s-k} d_{n-s-k} + \sum_{s=0}^{n} \sum_{k=0}^{n-s-1} \sum_{i=1}^{n-s-k-s-1} C_i d_i C_{k+1} d_{k+1} C_{s+1} d_{s+1} C_{n-s-k-i} d_{n-s-k-i} \right]
\]
Thus, \[ \frac{M}{(\alpha^{n+1} - 1)} \left[ \sum_{s=0}^{n} 2C_0^2 C_{n-s+1} C_{s+1} d_{n-s+1} d_{s+1} ight. \]
\[ + \sum_{s=0}^{n} \sum_{k=0}^{n-s-1} 2C_0 C_{k+1} C_{s+1} C_{n-s-k} d_{k+1} d_{s+1} d_{n-s-k} \]
\[ + \sum_{s=0}^{n} \sum_{k=0}^{n-s-1} \sum_{i=1}^{s} C_i C_{k+1} C_{s+1} C_{n-s-k-i} \left. \right] \]
\[ \leq \frac{M}{(\alpha^{n+1} - 1)} \max_{n_1 + n_2 + \cdots + n_t = n+2} \left\{ d_{n_1} d_{n_2} \cdots d_{n_t} \right\} \]
\[ \times \left[ \sum_{s=0}^{n} 2C_0^2 C_{n-s+1} C_{s+1} + \sum_{s=0}^{n} \sum_{k=0}^{n-s-1} 2C_0 C_{k+1} C_{s+1} C_{n-s-k} \right. \]
\[ + \sum_{s=0}^{n} \sum_{k=0}^{n-s-1} \sum_{i=1}^{s} C_i C_{k+1} C_{s+1} C_{n-s-k-i} \left. \right] \]
\[ = d_{n+2} C_{n+2}. \quad (2.23) \]

Therefore, \( P(n+1) \) is true, we conclude that \( |b_n| \leq C_n d_n \), for all \( n = 1, 2, 3, \ldots \) In view of (2.21) and Lemma 2.1, we see that
\[ |b_n| \leq P^n(2^{5\delta+1})^{n-1} n^{-2\delta}, \quad n = 1, 2, 3, \ldots \quad (2.24) \]

Therefore,
\[ \limsup |b_n|^{1/n} \leq \limsup P(2^{5\delta+1})^{(n-1)/n} n^{-2\delta/n} \]
\[ = \lim P(2^{5\delta+1})^{(n-1)/n} n^{-2\delta/n} = P2^{5\delta+1}. \quad (2.25) \]

Thus, \( 1/\limsup |b_n|^{1/n} \geq 1/P2^{5\delta+1} \), which shows that power series (2.2) converges for \( |z| < 1/P2^{5\delta+1} \). The proof is complete.

**Lemma 2.3.** Suppose that (H1) holds. Then for any \( r \geq m \), (1.3) has an analytic solution of the form (2.2) in a neighborhood of 0.

**Proof.** For \( r \geq m \), \( 0 \leq k + s \leq n \), we have \( s+1 \leq n+1 \), and \( k+1 \leq n-s+1 \), it follows that \((s+1)/(n+1) \leq 1 \) and \((k+1)/(n-s+1) \leq 1 \). Next, we have
\[ (k+s+1)r + j(n-s-k) - rn \]
\[ = (k+s)r + r - (k+s)j + jn - rn \]
\[ = (k+s)(r-j) - n(r-j) + r \]
\[ = (k+s-n)(r-j) + r, \quad \text{so} \]
\[ (k+s+1)r + j(n-s-k) - rn = (k+s-n)(r-j) + r. \quad (2.26) \]
Since $|\alpha| > 1$, $|\alpha|^{(k + s + 1)r + j(n - s - k) - rn} = |\alpha|^{(k + s - n)(r - j) + r} = |\alpha|^{(k + s - n)(r - j)}|\alpha|^r \leq |\alpha|^r$ and the sequence

$$\left\{ \frac{|\alpha|^{r-1} \sum_{j=0}^{m} |c_j|}{|\alpha|^{n+1} - 1} \right\}_{n=1}^{\infty}$$

(2.27)

converges to 0, this sequence is bounded, namely, there exists $M > 0$ such that

$$\frac{|\alpha|^{r-1} \sum_{j=0}^{m} |c_j|}{|\alpha|^{n+1} - 1} \leq M, \quad \forall n = 1, 2, 3, \ldots$$

(2.28)

Therefore,

$$\left| \frac{(s + 1)(k + 1) \sum_{j=0}^{m} |c_j|}{(n + 2)(n - s + 1)(|\alpha|^{n+1} - 1)} \right| \leq \frac{|\alpha|^{r-1} \sum_{j=0}^{m} |c_j|}{|\alpha|^{n+1} - 1} \leq M, \quad \forall n = 1, 2, 3, \ldots$$

(2.29)

In view of (2.10), we get that

$$|b_{n+2}| \leq M \sum_{s=0}^{n} \sum_{k=0}^{n-s} \sum_{i=0}^{n-k-s} |b_1| |b_{s+1}| |b_{k+1}| |b_{n-s-k-i}|, \quad \forall n = 0, 1, 2, \ldots$$

(2.30)

We define a sequence $\{D_n\}_{n=0}^{\infty}$ by $D_0 = |\mu|$, $D_1 = |\eta|$ and

$$D_{n+2} = M \sum_{s=0}^{n} \sum_{k=0}^{n-s} \sum_{i=0}^{n-k-s} D_s D_{s+1} D_{k+1} D_{n-s-k-i}, \quad \forall n = 0, 1, 2, \ldots$$

(2.31)

Next, we will show that $|b_{n+1}| \leq D_{n+1}, n = 1, 2, 3, \ldots$. By definition of $D_n$, we have $|b_0| \leq D_0$, $|b_1| \leq D_1$ and we let $P(n)$ denote the statement that $|b_{n+1}| \leq D_{n+1}$. Then

$$|b_2| \leq M \sum_{s=0}^{0} \sum_{k=0}^{0-s} \sum_{i=0}^{0-k-s} |b_1| |b_{s+1}| |b_{k+1}| |b_{0-s-k-i}|$$

(2.32)

$$= M |b_0| |b_1| |b_1| |b_0| = M |b_0|^2 |b_1|^2 = D_2.$$

Therefore, $P(1)$ is true. Next, suppose that $P(1), P(2), \ldots, P(n)$ are true, so $|b_{t+1}| \leq D_{t+1}$, for $t = 1, 2, 3, \ldots, n$. Therefore,

$$|b_{n+2}| \leq M \sum_{s=0}^{n} \sum_{k=0}^{n-s} \sum_{i=0}^{n-k-s} |b_1| |b_{s+1}| |b_{k+1}| |b_{n-s-k-i}|$$

(2.33)

$$\leq M \sum_{s=0}^{n} \sum_{k=0}^{n-s} \sum_{i=0}^{n-k-s} D_s D_{s+1} D_{k+1} D_{n-s-k-i} = D_{n+2}.$$
Thus,

\[ G(z) = \sum_{n=0}^{+\infty} D_n z^n, \]  

(2.34)

then

\[ G^3(z) = G(z)G^2(z) = \left( D_0 + \sum_{n=0}^{+\infty} D_{n+1}z^{n+1} \right) \left( \sum_{n=0}^{+\infty} \left( \sum_{i=0}^{n} D_i D_{n-i} \right) z^n \right) \]

\[ = D_0 \sum_{n=0}^{+\infty} \left( \sum_{i=0}^{n} D_i D_{n-i} \right) z^n + \sum_{n=0}^{+\infty} \left( \sum_{k=0}^{n} \sum_{i=0}^{n-k} D_i D_{k+1} D_{n-k-i} \right) z^{n+1}, \]

\[ G^4(z) = G(z)G^3(z) = \left( D_0 + \sum_{n=0}^{+\infty} D_{n+1}z^{n+1} \right) \times \left[ D_0 \sum_{n=0}^{+\infty} \left( \sum_{i=0}^{n} D_i D_{n-i} \right) z^n + \sum_{n=0}^{+\infty} \left( \sum_{k=0}^{n} \sum_{i=0}^{n-k} D_i D_{k+1} D_{n-k-i} \right) z^{n+1} \right] \]

\[ [5pt] = D_0^2 + D_0 \left[ G^3(z) - D_0 G^2(z) \right] + \frac{1}{M} \sum_{n=0}^{+\infty} D_{n+2} z^{n+2} \]

\[ = 2D_0 G^3(z) - D_0^2 G^2(z) + \frac{1}{M} (G(z) - D_0 D_1 z) \]

\[ = 2|\mu|G^3(z) - |\mu|^2 G^2(z) + \frac{1}{M} (G(z) - |\mu| - |\eta|z). \]

Thus,

\[ G^4(z) - 2|\mu|G^3(z) + |\mu|^2 G^2(z) - \frac{1}{M} (G(z) - |\mu| - |\eta|z) = 0. \]

(2.36)

Let

\[ R(z, \omega) = \omega^4 - 2|\mu|\omega^3 + |\mu|^2 \omega^2 - \frac{1}{M} (\omega - |\mu| - |\eta|z) \]  

(2.37)

for \((z, \omega)\) in a neighborhood of \((0, |\mu|)\), so we see that \(R(0, \mu) = |\mu|^4 - 2|\mu|^4 + |\mu|^4 - \frac{1}{M}(|\mu| - |\mu| - |\eta|0) = 0\) and \(R'_{\omega}(z, \omega) = 4\omega^3 - 6|\mu|\omega^2 + 2|\mu|^2 \omega - 1/M\), then \(R'_{\omega}(0, |\mu|) = -1/M \neq 0\). Therefore, there exists a unique function \(\omega(z)\) which is analytic in a
neighborhood of 0 such that \(\omega(0) = |\mu|, \omega'(0) = |\eta|\) and satisfies \(R(z, \omega(z)) = 0\). According to (2.34) and (2.36), we have \(G(z) = \omega(z)\). It follows that the power series (2.34) converges in a neighborhood of 0, which implies that the power series (2.2) is also convergent in a neighborhood of 0. The proof is complete.

Lemma 2.4. Suppose that (H2) holds. Then for either \(0 < r \leq m\) and \(c_0 = 0, c_1 = 0, \ldots, c_{r-1} = 0\), or \(r = 0\), (1.3) has an analytic solution of the from (2.2) in a neighborhood of 0.

Proof. By assumption, we get that

\[
\left\{ \frac{|\alpha|^{-1} \sum_{j=0}^{m} |c_j|}{1 - |\alpha|^{n+1}} \right\}_{n=1}^{+\infty}
\]

converges to \(|\alpha|^{-1} \sum_{j=0}^{m} |c_j|\), so it is a bounded sequence which implies that there exists \(M > 0\) such that

\[
\frac{|\alpha|^{-1} \sum_{j=0}^{m} |c_j|}{1 - |\alpha|^{n+1}} \leq M, \quad \forall n = 1, 2, 3, \ldots
\]

There are two cases to consider as follows.

Case 1. \(r = 0\). As \(0 \leq k + s \leq n\), we have

\[
\frac{(s + 1)(k + 1) \sum_{j=0}^{m} c_j \alpha^{(n-s-k)j+(k+s+1)r-rn-1}}{(n+2)(n-s+1)(\alpha^{n+1} - 1)}
\]

\[
= \frac{(s + 1)(k + 1) \sum_{j=0}^{m} c_j \alpha^{(n-s-k)j-1}}{(n+2)(n-s+1)(\alpha^{n+1} - 1)} \leq \frac{|\alpha|^{-1} \sum_{j=0}^{m} |c_j|}{1 - |\alpha|^{n+1}}.
\]

Case 2. \(0 < r \leq m\) and \(c_0 = 0, c_1 = 0, \ldots, c_{r-1} = 0\). We see that \(|\alpha|^{(r-j)(k+s-n)} \leq 1\), where \(r \leq j \leq m\). Then,

\[
|\alpha|^{(n-s-k)j+(k+s+1)r-rn-1} = |\alpha|^{(r-j)(k+s-n)+r-1}
\]

\[
= |\alpha|^{(r-j)(k+s-n)} |\alpha|^{r-1} \leq |\alpha|^{-1}.
\]

Thus,

\[
|\alpha|^{(n-s-k)j+(k+s+1)r-rn-1} \leq |\alpha|^{-1}.
\]

Next, we consider

\[
\frac{(s + 1)(k + 1) \sum_{j=0}^{m} c_j \alpha^{(n-s-k)j+(k+s+1)r-rn-1}}{(n+2)(n-s+1)(\alpha^{n+1} - 1)} \leq \sum_{j=0}^{m} \frac{|c_j| |\alpha|^{(n-s-k)j+(k+s+1)r-rn-1}}{1 - |\alpha|^{n+1}} \leq \frac{|\alpha|^{-1} \sum_{j=0}^{m} |c_j|}{1 - |\alpha|^{n+1}}.
\]
Therefore, by both cases, we have

\[
\left| (s+1)(k+1)\sum_{j=0}^{m} c_j \alpha^{(n-s-k)j+(k+1)r-1}\right| \leq \left| \alpha \right|^{-1} \sum_{j=0}^{m} |c_j| \leq \frac{|\alpha|^{-1} \sum_{j=0}^{m} |c_j|}{1 - |\alpha|^{n+1}}. \tag{2.44}
\]

It follows that

\[
\left| (s+1)(k+1)\sum_{j=0}^{m} c_j \alpha^{(n-s-k)j+(k+1)r-1}\right| \leq M \quad \text{for } n = 1, 2, 3, \ldots. \tag{2.45}
\]

The conclusion of Lemma 2.4 now follows easily from the same argument as in the proof of Lemma 2.3.

\[\square\]

3. Main results

We now state the main result of this paper. Consider the following three hypotheses:

(i) (H3) holds;
(ii) (H1) holds, and \( r \geq m \);
(iii) (H2) holds, and either \( 0 < r \leq m \) and \( c_0 = 0, c_1 = 0, \ldots, c_{r-1} = 0 \), or \( r = 0 \).

**Theorem 3.1.** Suppose one of the conditions (i), (ii), or (iii) is fulfilled. Then, for any \( \mu \), (1.2) has an analytic solution \( x(z) \) in a neighborhood of \( \mu \) satisfying the initial conditions \( x(\mu) = \mu, x'(\mu) = \alpha \). This solution has the form \( x(z) = y(\alpha y^{-1}(z)) \), where \( y(z) \) is an analytic solution of the initial value problem (1.3)-(1.4).

**Proof.** In view of Lemmas 2.2–2.4, we may find a sequence \( \{b_n\}_{n=2}^{\infty} \) such that the function \( y(z) \) of the form (2.2) is an analytic solution of (1.3) in a neighborhood of 0. Since \( y'(0) = \eta \neq 0 \), the function \( y^{-1}(z) \) is analytic in a neighborhood of the \( y(0) = \mu \). If we define \( x(z) \) by means of (1.5), then

\[
x''(x^{[r]}(z)) = x''(y(\alpha^{r}y^{-1}(z)))
= \frac{\alpha^2 y''(\alpha^{r+1}y^{-1}(z)) y'(\alpha^{r}y^{-1}(z)) - \alpha y'(\alpha^{r+1}y^{-1}(z)) y''(\alpha^{r}y^{-1}(z))}{[y'(\alpha^{r}y^{-1}(z))]^3}
= \sum_{j=0}^{m} c_j (y(\alpha^j y^{-1}(z)))^2, \quad \text{by (1.3),}
= \sum_{j=0}^{m} c_j (x^{[j]}(z))^2, \quad \text{as required.} \tag{3.1}
\]

The proof is complete. \( \square \)

We now show how to explicitly construct an analytic solution of (1.2). Since \( x(z) = y(\alpha y^{-1}(z)), x(\mu) = y(\alpha y^{-1}(\mu)) = y(0) = \mu \). By Theorem 3.1, \( x(z) \) is an analytic function in a neighborhood of \( \mu \). Thus \( x(z) \) can be written in a neighborhood of \( \mu \) as

\[
x(z) = \mu + x'(\mu)(z - \mu) + \frac{x''(\mu)(z - \mu)^2}{2!} + \frac{x'''(\mu)(z - \mu)^3}{3!} + \cdots. \tag{3.2}
\]
Next, we will determine the derivatives \( x^{(n)}(\mu) \), \( n = 1, 2, \ldots \). We have \( x(z) = y(\alpha y^{-1}(z)) \), so that \( x'(z) = \alpha y'(\alpha y^{-1}(z))/y'(y^{-1}(z)) \). That is, \( x'(\mu) = \alpha y'(\alpha y^{-1}(\mu))/y'(y^{-1}(\mu)) = \alpha y'(0)/y'(0) = \alpha \). Hence \( x'(\mu) = \alpha \). By means of (1.2), we get that

\[
x''(\mu) = x''(x^{[r]}(\mu)) = \sum_{j=0}^{m} c_j (x^{[j]}(\mu))^2 = \mu^2 \sum_{j=0}^{m} c_j; \tag{3.3}
\]

hence \( x''(\mu) = \mu^2 \sum_{j=0}^{m} c_j \). Next, we have

\[
(x''(x^{[r]}(z)))' = x'''(x^{[r]}(z))(x^{[r]}(z))' = x'''(x^{[r]}(z))x'(x^{[r-1]}(z))x'(x^{[r-2]}(z)) \cdots x'(x(z))x'(z). \tag{3.4}
\]

Therefore, the derivative of \( (x''(x^{[r]}(z))) \) at \( z = \mu \) is

\[
x'''(x^{[r]}(\mu))x'(x^{[r-1]}(\mu))x'(x^{[r-2]}(\mu)) \cdots x'(x(\mu))x'(\mu) = x'''(\mu)[x'(\mu)]^r = x'''(\mu)\alpha^r,
\]

\[
\left( \sum_{j=0}^{m} c_j (x^{[j]}(z))^2 \right)' = \sum_{j=0}^{m} c_j ((x^{[j]}(z))^2)' = 2 \sum_{j=0}^{m} c_j x^{[j]}(z)(x^{[j]}(z))' = 2 \sum_{j=0}^{m} c_j x^{[j]}(z)x'(x^{[j-1]}(z))x'(x^{[j-2]}(z)) \cdots x'(x(z))x'(z). \tag{3.5}
\]

Hence, the first derivative of \( (\sum_{j=0}^{m} c_j (x^{[j]}(z))^2) \) at \( z = \mu \) is \( 2\mu \sum_{j=0}^{m} c_j \alpha^j \). Next, by taking the first derivative of (1.2) at \( z = \mu \), we get that

\[
x'''(\mu)\alpha^r = 2\mu \sum_{j=0}^{m} c_j \alpha^j. \tag{3.6}
\]

Thus,

\[
x'''(\mu) = 2\mu \sum_{j=0}^{m} c_j \alpha^{j-r}. \tag{3.7}
\]

In general, we can show by induction that

\[
(x''(x^{[r]}(z)))^{(n+1)} = ((x^{[r]}(z))')^{n+1}(x^{[r]}(z)) + \sum_{k=1}^{n} P_{k,n+1}((x^{[r]}(z))', (x^{[r]}(z))^\prime, \ldots, (x^{[r]}(z))^{(n+1)}) \left[ x^{(k+2)}(x^{[r]}(z)) \right], \tag{3.8}
\]

for \( n = 1, 2, \ldots \), and

\[
(x^{[j]}(z))^{(l)} = Q_{jl}(x_{10}(z), x_{1,j-1}(z), \ldots, x_{0l}(z), x_{l,j-1}(z)), \tag{3.9}
\]
respectively, where \( x_{ij}(z) = x^t(x^{ij}(z)) \), \( P_{jk} \) and \( Q_{jl} \) are polynomials with nonnegative coefficients. Next, we have

\[
\left( \sum_{j=0}^{m} c_j (x^{[j]}(z))^2 \right)^{n+1} = \sum_{j=0}^{m} c_j (x^{[j]}(z))^2 \\
= 2 \sum_{j=0}^{m} c_j (x^{[j]}(z)) \left( \left( x^{[j]}(z) \right)' \right)^{(n)} \\
= 2 \sum_{j=0}^{m} c_j \left( \sum_{k=0}^{n} C_k^n (x^{[j]}(z))^{(k)} (x^{[j]}(z))^{(n-k+1)} \right) \\
= 2 \sum_{j=0}^{m} \sum_{k=0}^{n} c_j C_k^n (x^{[j]}(z))^{(k)} (x^{[j]}(z))^{(n-k+1)}, \quad n = 1, 2, \ldots.
\]

(3.10)

For convenience, we denote the following notations:

\[
\beta_{jl} = Q_{jl}(\mu), \ldots, x^{[j]}(\mu), \ldots, x^{(j)}(\mu),
\]

(3.11)

where the number of repeats of \( x^{(t)}(\mu) \) is \( l \), for \( t = 1, 2, \ldots, j \). Then, we see that \( \beta_{ij} = (x^{[j]}(\mu))^l \). By differentiating (1.1) for \( n + 1 \) times at \( z = \mu \), we get

\[
\left( \left( x^{[r]}(\mu) \right)' \right)^{n+1} x^{(n+3)}(x^{[r]}(\mu)) \\
+ \sum_{k=1}^{n} \left[ P_{k,n+1} \left( (x^{[r]}(\mu))', (x^{[r]}(\mu))'', \ldots, (x^{[r]}(\mu))^{(n+1)} \right) \right] x^{(k+2)}(x^{[r]}(\mu)) \\
= 2 \sum_{j=0}^{m} \sum_{k=0}^{n} c_j C_k^n (x^{[j]}(\mu))^{(k)} (x^{[j]}(\mu))^{(n-k+1)}. 
\]

(3.12)

Thus,

\[
\alpha_r^{(n+1)} x^{(n+3)}(\mu) + \sum_{k=1}^{n} \left[ P_{k,n+1} (\beta_{1r}, \beta_{2r}, \ldots, \beta_{n+1,r}) \right] x^{(k+2)}(\mu) = 2 \sum_{j=0}^{m} \sum_{k=0}^{n} c_j C_k^n \beta_{kj} \beta_{n-k+1,j}. 
\]

(3.13)

This shows that

\[
x^{(n+3)}(\mu) = \frac{2 \sum_{j=0}^{m} \sum_{k=0}^{n} c_j C_k^n \beta_{kj} \beta_{n-k+1,j} - \sum_{k=1}^{n} \left[ P_{k,n+1} (\beta_{1r}, \beta_{2r}, \ldots, \beta_{n+1,r}) \right] x^{(k+2)}(\mu)}{\alpha_r^{(n+1)}},
\]

(3.14)
where \( n = 1, 2, \ldots \). By means of this formula, it is then easy to write out the explicit form of our solution \( x(z) \) as follows:

\[
x(z) = \mu + \alpha(z - \mu) + \frac{\mu^2}{2!} \sum_{j=0}^{m} c_j(z - \mu)^2 + \frac{2\mu}{3!} \sum_{j=0}^{m} c_j \alpha^{j-r}(z - \mu)^3 + \sum_{n=1}^{+\infty} \frac{1}{(n+3)!} \alpha^{n+3}(\mu)(z - \mu)^{n+3}.
\]  

(3.15)

*Example 3.2.* The following example shows how to construct an analytic solution by using the previous argument. Consider the following functional equation:

\[
x''(x(z)) = x^2(z) + (x^{[2]}(z))^2.
\]  

(3.16)

This is just (1.2) with the choice of \( r = 1, m = 2, c_0 = 1, c_1 = 1, \) and \( c_2 = 1 \). We can easily see that (3.16) satisfies condition (iii) of Theorem 3.1; hence, for any complex numbers \( \mu \) and \( \alpha \) such that \( 0 < |\alpha| < 1 \), (3.16) has an analytic solution \( x(z) \) in a neighborhood of \( \mu \) which satisfies \( x(\mu) = \mu \) and \( x'(\mu) = \alpha \). This analytic solution has the form as in (3.2) in case \( r = 1, m = 2, c_0 = 1, c_1 = 1, \) and \( c_2 = 1 \). We already know that \( x(\mu) = \mu \) and \( x'(\mu) = \alpha \). We will find \( x^{(n)}(\mu), n \geq 2 \). For \( n = 2 \), it follows from (3.16) that

\[
x''(\mu) = x''(x(\mu)) = x^2(\mu) + (x^{[2]}(\mu))^2 = 2\mu^2.
\]  

(3.17)

For \( n = 3 \), it follows from (3.16) that

\[
x'''(x(z)) = (x^2(z))' + \left( (x^{[2]}(z))^2 \right)'.
\]  

(3.18)

Thus,

\[
x'''(x(z))x'(z) = 2x(z)x'(z) + 2x^{[2]}(z)x'(z)x'(z)
\]

\[
= 2x'(z) [x(z) + x^{[2]}(z)x'(z)]
\]  

(3.19)

By putting \( z = \mu \), we obtain

\[
x'''(\mu)\alpha = 2\alpha[\mu + \mu\alpha],
\]  

(3.20)

which gives

\[
x'''(\mu) = 2(1 + \alpha)\mu.
\]  

(3.21)

Similarly, for \( n = 4 \), we obtain

\[
x^{(4)}(\mu) = 2(1 + 2\mu^3 + \alpha^2) + \frac{4(1 + \mu)\mu^3}{\alpha^2}.
\]  

(3.22)
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By continuing this process, we obtain an analytic solution of (3.16) as

$$x(z) = \mu + \alpha(z - \mu) + \mu^2(z - \mu)^2 + \frac{(1 + \alpha)}{3}(z - \mu)^3$$
$$+ \left(\frac{1 + 2\mu^3 + \alpha^2}{12} + \frac{(1 + \mu)\mu^3}{6\alpha^2}\right)(z - \mu)^4 + \cdots. \quad (3.23)$$

Acknowledgments

The second author is supported by Thailand Research Fund (Grant no. RSA4780012) and Faculty of Science, Chiang Mai University. The authors thank the referees for careful readings and helpful comments and suggestions.

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