We study new properties and characterizations of rc-Lindelöf sets and almost rc-Lindelöf sets; a special interest is given to the mapping properties of such sets. We also obtain some product theorems concerning rc-Lindelöf spaces.

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1. Introduction and preliminaries

A subset $A$ of a space $X$ is called regular open if $A = \text{Int} A$, and regular closed if $X \setminus A$ is regular open, or equivalently, if $A = \text{Int} A$. $A$ is called semiopen [16] (resp., preopen [17], semi-preopen [3], $b$-open [4]) if $A \subset \text{Int} A$ (resp., $A \subset \text{Int} A$, $A \subset \text{Int} A$, $A \subset \text{Int} A \cup \text{Int} A$).

The concept of a preopen set was introduced in [6] where the term locally dense was used and the concept of a semi-preopen set was introduced in [1] under the name $\beta$-open. It was pointed out in [3] that $A$ is semi-preopen if and only if $P \subset A \subset P$ for some preopen set $P$. Clearly, every open set is both semiopen and preopen, semiopen sets as well as preopen sets are $b$-open, and $b$-open sets are semi-preopen. $A$ is called semiclosed (resp., preclosed, semi-preclosed, $b$-closed) if $X \setminus A$ is semiopen (resp., preopen, semi-preopen, $b$-open). $A$ is called semiregular [8] if it is both semiopen and semiclosed, or equivalently, if there exists a regular open set $U$ such that $U \subset A \subset \overline{U}$.

Clearly, every regular closed (regular open) set is semiregular. The semiclosure (resp., preclosure, semi-preclosure, $b$-closure) denoted by $\text{scl} A$ (resp., $\text{pcl} A$, $\text{spcl} A$, $\text{bcl} A$) is the intersection of all semiclosed (resp., preclosed, semi-preclosed, $b$-closed) subsets of $X$ containing $A$, or equivalently, is the smallest semiclosed (resp., preclosed, semi-preclosed, $b$-closed) set containing $A$. Dually, the semi-interior (resp., preinterior, semi-preinterior, $b$-interior) denoted by $\text{sint} A$ (resp., $\text{pint} A$, $\text{spint} A$, $\text{bint} A$) is the union of all semiopen (resp., preopen, semi-preopen, $b$-open) subsets of $X$ contained in $A$, or equivalently, is the largest semiopen (resp., preopen, semi-preopen, $b$-open) set contained in $A$.

A function $f$ from a space $X$ into a space $Y$ is called almost open [20] if $f^{-1}(U) \subset f^{-1}(\overline{U})$ whenever $U$ is open in $Y$, semicontinuous [16] if the inverse image of each
open set is semiopen, $\beta$-continuous [1] if the inverse image of each open set is $\beta$-open, weakly $\theta$-irresolute [13] if the inverse image of each regular closed set is semiopen, rc-continuous [14] if the inverse image of each regular closed set is regular closed, and wrc-continuous [2] if the inverse image of each regular closed set is semi-preopen. We will use the term semiprecontinuous to indicate $\beta$-continuous. Clearly, every semicontinuous function is semi-precontinuous, every rc-continuous function is weakly $\theta$-irresolute, and every weakly $\theta$-irresolute function is wrc-continuous. It is also easy to see that a function that is both semicontinuous (resp., semi-precontinuous) and almost open is weakly $\theta$-irresolute (resp., wrc-continuous).

A function $f$ from a space $X$ into a space $Y$ is called somewhat continuous [12] if for each nonempty open set $V$ in $Y$, $\text{int} f^{-1}(V) \neq \emptyset$.

A space $X$ is called a weak $P$-space [18] if for each countable family $\{U_n : n \in \mathbb{N}\}$ of open subsets of $X$, $\bigcup U_n = \bigcup U_n$. Clearly, $X$ is a weak $P$-space if and only if the countable union of regular closed subsets of $X$ is regular closed (closed).

A space $X$ is called rc-Lindelöf [15] (resp., nearly Lindelöf [5]) if every regular closed (resp., regular open) cover of $X$ has a countable subcover, and called almost rc-Lindelöf [10] if every regular closed cover of $X$ has a countable subfamily whose union is dense in $X$.

A subset $A$ of a space $X$ is called an S-set in $X$ [7] if every cover of $A$ by regular closed subsets of $X$ has a finite subcover, and called an rc-Lindelöf set in $X$ (resp., an almost rc-Lindelöf set in $X$) [9] if every cover of $A$ by regular closed subsets of $X$ admits a countable subfamily that covers $A$ (resp., the closure of the union of whose members contains $A$). Obviously, every S-set is an rc-Lindelöf set and every rc-Lindelöf set is an almost rc-Lindelöf set; it is also clear that a subset $A$ of a weak $P$-space $X$ is rc-Lindelöf in $X$ if and only if it is almost rc-Lindelöf in $X$.

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers. For the concepts not defined here, we refer the reader to Engelking [11].

In concluding this section, we recall the following facts for their importance in the material of our paper.

**Theorem 1.1** [9]. If $A$ is an rc-Lindelöf (resp., almost rc-Lindelöf) set in a space $X$ and $B$ is a regular open subset of $X$, then $A \cap B$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$. In particular, a regular open subset $A$ of an rc-Lindelöf (resp., almost rc-Lindelöf) space $X$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$.

**Theorem 1.2** [9]. Let $A$ be a preopen subset of a space $X$ and $B \subset A$. Then $B$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$ if and only if $B$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $A$. In particular, a preopen subset $A$ of a space $X$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$ if and only if $A$ is an rc-Lindelöf (resp., almost rc-Lindelöf) subspace.

**Proposition 1.3** [19]. If $A$ is an almost rc-Lindelöf set in a space $X$ and $A \subset B \subset \overline{A}$, then $B$ is almost rc-Lindelöf in $X$.

**Proposition 1.4** [9]. The countable union of rc-Lindelöf (resp., almost rc-Lindelöf) sets in a space $X$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$. 
Proposition 1.5 [9]. A subset $A$ of a space $X$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$ if and only if every cover of $A$ by semiopen subsets of $X$ admits a countable subfamily the union of whose members (resp., the closure of the union of whose members) contains $A$.

Proposition 1.6 [19]. Let $A$ be a preopen, almost rc-Lindelöf set in a space $X$ and $B$ a regular closed subset of $X$, then $A \cap B$ is almost rc-Lindelöf in $X$. In particular, a regular closed subset $A$ of an almost rc-Lindelöf space $X$ is almost rc-Lindelöf in $X$.

Lemma 1.7. If $A$ is a preopen subset of a space $X$ and $U$ is open in $X$, then $A \cap U \cap A = U \cap A$.

2. Further properties

This section is devoted to study new properties concerning rc-Lindelöf sets and almost rc-Lindelöf sets. We obtain several characterizations of rc-Lindelöf sets and almost rc-Lindelöf sets.

The following proposition is an improvement of Proposition 1.6 and the fact of Theorem 1.1 that a regular open subset of an almost rc-Lindelöf space $X$ is almost rc-Lindelöf in $X$.

Proposition 2.1. Let $A$ be a preopen, almost rc-Lindelöf set in a space $X$ and $B$ a semiregular subset of $X$, then $A \cap B$ is almost rc-Lindelöf in $X$. In particular, a semiregular subset $A$ of an almost rc-Lindelöf space $X$ is almost rc-Lindelöf in $X$.

Proof. Since $B$ is a semiregular subset of $X$, there exists a regular open subset $U$ of $X$ such that $U \subset B \subset \overline{U}$, thus by Lemma 1.7, it follows that $A \cap U \subset A \cap B \subset \overline{U} \cap A \subset \overline{A \cap U}$. Since $A$ is almost rc-Lindelöf set in $X$, it follows from Theorem 1.1 that $A \cap U$ is almost rc-Lindelöf set in $X$. The result yields from Proposition 1.3. □

Proposition 2.2 [19]. If $A$ is a regular closed subset of a space $X$ such that $A$ is almost rc-Lindelöf in $X$, then $A$ is an almost rc-Lindelöf.

The following proposition includes an improvement of Proposition 2.2.

Proposition 2.3. Let $A$ be a semiopen subset of a space $X$ and $B \subset A$. If $B$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$, then $B$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $A$. In particular, if $A$ is a semiopen subset of a space $X$ such that $A$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$, then $A$ is an rc-Lindelöf (resp., almost rc-Lindelöf) subspace.

Proof. Follows from Proposition 1.5 and the fact that if $A$ is a semiopen subset of a space $X$ and $B$ is semiopen in $A$, then $B$ is semiopen in $X$. □

Corollary 2.4 [2]. Let $X$ be an rc-Lindelöf weak $P$-space. If $U \subset A \subset \overline{U}$, where $U$ is a regular open subset of $X$, then $A$ is an rc-Lindelöf subspace.

Proof. By Theorem 1.1, $U$ is an rc-Lindelöf set in $X$ and thus almost rc-Lindelöf in $X$. By Proposition 1.3, $A$ is almost rc-Lindelöf in $X$, but $X$ is a weak $P$-space, so $A$ is rc-Lindelöf in $X$. Finally, since $A$ is semiopen (it is moreover semiregular), it follows from Proposition 2.3 that $A$ is an rc-Lindelöf subspace. □
The following theorem includes new characterizations of rc-Lindelöf sets and almost rc-Lindelöf sets.

**Theorem 2.5.** Let $A$ be a subset of a space $X$. Then the following are equivalent.

(i) $A$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$.

(ii) Every cover of $A$ by semi-preopen subsets of $X$ admits a countable subfamily the union of the closures of whose members (resp., the closure of the union of whose members) contains $A$.

(iii) Every cover of $A$ by $b$-open subsets of $X$ admits a countable subfamily the union of the closures of whose members (resp., the closure of the union of whose members) contains $A$.

(iv) Every cover of $A$ by semiopen subsets of $X$ admits a countable subfamily the union of the closures of whose members (resp., the closure of the union of whose members) contains $A$.

(v) Every cover of $A$ by semiregular subsets of $X$ admits a countable subfamily the union of the closures of whose members (resp., the closure of the union of whose members) contains $A$.

Proof. (i)⇒(ii): follows since the closure of a semi-preopen set is regular closed.

(ii)⇒(iii)⇒(iv)⇒(v)⇒(i): follows from the following implications: regular closed⇒semiregular⇒semiopen⇒$b$-open⇒semi-preopen.

The following theorem also characterizes rc-Lindelöf sets and almost rc-Lindelöf sets, it is a direct consequence of Theorem 2.5 and the definition of rc-Lindelöf (almost rc-Lindelöf) sets.

**Theorem 2.6.** Let $A$ be a subset of a space $X$. Then the following are equivalent.

(i) $A$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$.

(ii) If $U_\alpha = \{U_\alpha : \alpha \in \Lambda\}$ is a family of regular open subsets of $X$ satisfying that for any countable subcollection $U_\alpha^+$ of $U_\alpha$, $A \cap (\cap U_\alpha^+) \neq \emptyset$ (resp., $A \cap \text{int}(\cap U_\alpha^+) \neq \emptyset$), then $A \cap (\cap U_\alpha) \neq \emptyset$.

(iii) If $U_\alpha = \{U_\alpha : \alpha \in \Lambda\}$ is a family of semi-preclosed subsets of $X$ satisfying that for any countable subcollection $U_\alpha^+$ of $U_\alpha$, $A \cap (\cap \{\text{int} U : U \in U_\alpha^+\}) \neq \emptyset$ (resp., $A \cap \text{int}(\cap U_\alpha^+) \neq \emptyset$), then $A \cap (\cap U_\alpha) \neq \emptyset$.

(iv) If $U_\alpha = \{U_\alpha : \alpha \in \Lambda\}$ is a family of $b$-closed subsets of $X$ satisfying that for any countable subcollection $U_\alpha^+$ of $U_\alpha$, $A \cap (\cap \{\text{int} U : U \in U_\alpha^+\}) \neq \emptyset$ (resp., $A \cap \text{int}(\cap U_\alpha^+) \neq \emptyset$), then $A \cap (\cap U_\alpha) \neq \emptyset$.

(v) If $U_\alpha = \{U_\alpha : \alpha \in \Lambda\}$ is a family of semiclosed subsets of $X$ satisfying that for any countable subcollection $U_\alpha^+$ of $U_\alpha$, $A \cap (\cap \{\text{int} U : U \in U_\alpha^+\}) \neq \emptyset$ (resp., $A \cap \text{int}(\cap U_\alpha^+) \neq \emptyset$), then $A \cap (\cap U_\alpha) \neq \emptyset$.

(vi) If $U_\alpha = \{U_\alpha : \alpha \in \Lambda\}$ is a family of semiregular subsets of $X$ satisfying that for any countable subcollection $U_\alpha^+$ of $U_\alpha$, $A \cap (\cap \{\text{int} U : U \in U_\alpha^+\}) \neq \emptyset$ (resp., $A \cap \text{int}(\cap U_\alpha^+) \neq \emptyset$), then $A \cap (\cap U_\alpha) \neq \emptyset$.

3. Invariance properties

In this section, we mainly study several types of functions that preserve the property of being an rc-Lindelöf (almost rc-Lindelöf) set.
Definition 3.1 [19]. A function $f$ from a space $X$ into a space $Y$ is said to be slightly continuous if $f(U) \subset \overline{f(U)}$ whenever $U$ is open in $X$.

In [19], it was shown that if a function $f : X \to Y$ is slightly continuous and weakly $\theta$-irresolute, then $f(A)$ is almost rc-Lindelöf in $Y$ whenever $A$ is almost rc-Lindelöf set in $X$. The following theorem is analogous to this result; it has a similar proof that we will mention for the convenience of the reader.

Theorem 3.2. Let $f : X \to Y$ be a slightly continuous and weakly $\theta$-irresolute function. If $A$ is rc-Lindelöf set in $X$, then $f(A)$ is rc-Lindelöf in $Y$.

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of $f(A)$ by regular closed subsets of $X$. Then $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a cover of $A$ by semiopen subsets of $X$ (as $f$ is weakly $\theta$-irresolute). Since $A$ is rc-Lindelöf in $X$, it follows from Proposition 1.5 that there exist $\alpha_1, \alpha_2, \ldots \in \Lambda$ such that $A \subset \bigcup_{i=1}^\infty f^{-1}(U_{\alpha_i})$. For each $i \in \mathbb{N}$, there is an open subset $V_i$ of $X$ such that $V_i \subset f^{-1}(U_{\alpha_i}) \subset \overline{V_i}$ and thus $\bigcup_{i=1}^\infty f^{-1}(U_{\alpha_i}) = \bigcup_{i=1}^\infty V_i$. Since $f$ is slightly continuous, it follows that $f(A) \subset \bigcup_{i=1}^\infty \overline{f(V_i)} \subset \bigcup_{i=1}^\infty \overline{U_{\alpha_i}} = \bigcup_{i=1}^\infty U_{\alpha_i}$. Hence $f(A)$ is rc-Lindelöf in $Y$. \qed

Corollary 3.3. Let $f : X \to Y$ be a slightly continuous, semicontinuous, and almost open function. If $A$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $X$, then $f(A)$ is rc-Lindelöf (resp., almost rc-Lindelöf) in $Y$.

Corollary 3.4. Let $f : X \to Y$ be a surjective, slightly continuous, semicontinuous, and almost open function. If $X$ is rc-Lindelöf, then $Y$ is rc-Lindelöf.

It will be seen later that the condition slightly continuous of Corollary 3.4 is not essential for preserving the almost rc-Lindelöf property.

Corollary 3.5 [2]. Let $f : X \to Y$ be a surjective, continuous, and almost open function. If $X$ is rc-Lindelöf, then $Y$ is rc-Lindelöf.

Obviously, every continuous function is both semicontinuous and slightly continuous. However, the converse is not true as the following example tells.

Example 3.6. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}\}$, $\tau^* = \{X, \phi, \{a, b\}\}$. Then the identity function from $(X, \tau)$ onto $(X, \tau^*)$ is a semicontinuous, slightly continuous, and almost open surjection. However, it is not continuous.

Proposition 3.7. Let $f : X \to Y$ be a semicontinuous function. If $X$ is extremally disconnected (i.e., every regular closed subset of $X$ is open), then $f$ is slightly continuous.

Proof. Let $U$ be open in $X$. Then $\text{scl}(U) = U \cup \text{int} U = U$ (as $X$ is extremally disconnected). Since $f$ is semicontinuous, it follows that $f(\text{scl}(U)) = f(U) \subset \overline{f(U)}$. Hence $f$ is slightly continuous. \qed

The following corollary is an immediate consequence of Corollary 3.4 and Proposition 3.7.

Corollary 3.8 [2]. Let $f : X \to Y$ be a semicontinuous, almost open surjection, where $X$ is extremally disconnected. If $X$ is rc-Lindelöf, then $Y$ is rc-Lindelöf.
The following example shows that if \( X \) is extremally disconnected and \( f : X \rightarrow Y \) is slightly continuous, almost open surjection, then \( f \) need not be semicontinuous.

**Example 3.9.** Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a, b\}\} \), \( \tau^* = \{X, \phi, \{a\}\} \). Then \((X, \tau)\) is extremally disconnected, also the identity function from \((X, \tau)\) onto \((X, \tau^*)\) is slightly continuous and almost open \(; \) it is, however, not semicontinuous.

**Proposition 3.10** [10]. (i) Let \( f : X \rightarrow Y \) be a somewhat continuous and weakly \( \theta \)-irresolute function. If \( X \) is almost rc-Lindelöf, then \( Y \) is almost rc-Lindelöf.

(ii) Let \( f : X \rightarrow Y \) be a surjective, semicontinuous, and weakly \( \theta \)-irresolute function. If \( X \) is almost rc-Lindelöf, then \( Y \) is almost rc-Lindelöf.

**Corollary 3.11.** Let \( f : X \rightarrow Y \) be a surjective, semicontinuous, and almost open function. If \( X \) is almost rc-Lindelöf, then \( Y \) is almost rc-Lindelöf.

The following corollary is an immediate consequence of Corollary 3.11 and the fact that for a weak \( P \)-space, the concepts of being rc-Lindelöf and almost rc-Lindelöf coincide.

**Corollary 3.12** [2]. Let \( f : X \rightarrow Y \) be a surjective, semicontinuous, and almost open function, where \( Y \) is a weak \( P \)-space. If \( X \) is rc-Lindelöf, then \( Y \) is rc-Lindelöf.

**Definition 3.13.** A function \( f : X \rightarrow Y \) is said to be somewhat precontinuous if for each nonempty open set \( V \) in \( Y \), \( p \text{int} f^{-1}(V) \neq \phi \).

**Remark 3.14.** It was pointed out in [10] that every surjective semicontinuous function is somewhat continuous, a similar result that may be pointed out here asserts that every surjective semi-precontinuous function is somewhat precontinuous. However, the converses of these two facts are not true as the following two examples tell.

**Example 3.15.** Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a, b\}, \{c\}\} \), \( \tau^* = \{X, \phi, \{a, c\}\} \). Then the identity function from \((X, \tau)\) onto \((X, \tau^*)\) is somewhat continuous; it is, however, not semi-continuous.

**Example 3.16.** Let \( X = \{a, b, c, d\} \), \( \tau = \{X, \phi, \{b\}, \{d\}, \{b, d\}, \{a, d\}, \{a, b, d\}\} \), \( \tau^* = \{X, \phi, \{a, b\}\} \). Then the identity function from \((X, \tau)\) onto \((X, \tau^*)\) is even somewhat continuous and thus somewhat precontinuous; it is, however, not semi-precontinuous since \( \{a, b\} \) is not semi-preopen in \((X, \tau)\).

The following result is a slight improvement of Proposition 3.10(i), the similar proof follows from Theorem 2.5 and the fact that if \( A \) is a semiopen subset of a space \( X \), then \( \text{pcl}(A) = \overline{A} \).

**Proposition 3.17.** (i) Let \( f : X \rightarrow Y \) be a somewhat continuous and wrc-continuous function. If \( X \) is almost rc-Lindelöf, then \( Y \) is almost rc-Lindelöf.

(ii) Let \( f : X \rightarrow Y \) be a somewhat precontinuous and weakly \( \theta \)-irresolute function. If \( X \) is almost rc-Lindelöf, then \( Y \) is almost rc-Lindelöf.

**Remark 3.18.** Clearly, every somewhat continuous function is somewhat precontinuous and every weakly \( \theta \)-irresolute function is wrc-continuous. However, the following two examples show that the property of being both somewhat continuous and wrc-continuous
and the property of being both somewhat precontinuous and weakly $\theta$-irresolute are independent.

**Example 3.19.** Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a, b\}\}$, $\tau^* = \{X, \phi, \{a, c\}\}$. Then the identity function from $(X, \tau)$ onto $(X, \tau^*)$ is somewhat precontinuous and weakly $\theta$-irresolute; it is, however, not somewhat continuous.

**Example 3.20.** Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}, \{d\}, \{a, b, c\}, \{b, c, d\}, \{a, d\}\}$, $\tau^* = \{X, \phi, \{a, b\}, \{d\}, \{a, b, d\}\}$. Then the identity function from $(X, \tau)$ onto $(X, \tau^*)$ is somewhat continuous and wrc-continuous; it is, however, not weakly $\theta$-irresolute (observe that $\{d, c\}$ is regular closed in $(X, \tau^*)$ but not semiopen in $(X, \tau)$).

The following result is a slight improvement of Proposition 3.10(ii), it is a direct consequence of Remark 3.14 and Proposition 3.17.

**Corollary 3.21.**
(i) Let $f : X \to Y$ be a surjective, semicontinuous, and wrc-continuous function. If $X$ is almost rc-Lindelöf, then $Y$ is almost rc-Lindelöf.

(ii) Let $f : X \to Y$ be a surjective, semi-precontinuous, and weakly $\theta$-irresolute function. If $X$ is almost rc-Lindelöf, then $Y$ is almost rc-Lindelöf.

**Corollary 3.22 [2].** Let $f : X \to Y$ be a somewhat continuous and wrc-continuous surjection, where $Y$ is a weak $P$-space. If $X$ is rc-Lindelöf, then $Y$ is rc-Lindelöf.

Corollary 3.22 is still true even if the function $f$ is not surjective.

### 4. Product theorems

In this section, we study some types of functions that inversely preserve the property of being an rc-Lindelöf (almost rc-Lindelöf) set. We mainly obtain some product theorems concerning rc-Lindelöf spaces.

**Definition 4.1 [19].** A function $f$ from a space $X$ into a space $Y$ is said to be regular open if it maps regular open subsets onto regular open subsets.

**Definition 4.2 [19].** (i) A subset $A$ of a space $X$ is said to be an rc-$F_\sigma$ subset if $A$ is the countable union of regular closed subsets.

(ii) A function $f$ from a space $X$ into a space $Y$ is said to be weakly almost open if $f^{-1}(A) \subset \overline{f^{-1}(A)}$ whenever $A$ is an rc-$F_\sigma$ subset of $Y$.

In [19], it was shown that every almost open function is weakly almost open, but not conversely.

**Theorem 4.3 [19].** Let $f$ be a weakly almost open and regular open function from a space $X$ onto a space $Y$. Then the following hold.

(i) If for each $y \in Y$, $f^{-1}(y)$ is an $S$-set in $X$, then $X$ is almost rc-Lindelöf whenever $Y$ is almost rc-Lindelöf.

(ii) If for each $y \in Y$, $f^{-1}(y)$ is rc-Lindelöf in $X$, then $X$ is almost rc-Lindelöf whenever $Y$ is almost rc-Lindelöf provided that $X$ is a weak $P$-space.

We point out here that in the result of Theorem 4.3(ii), $X$ being almost rc-Lindelöf may be replaced by rc-Lindelöf since $X$ is a weak $P$-space.
Theorem 4.3 may be improved in the following form.

**Theorem 4.4.** Let \( f \) be a weakly almost open and regular open function from a space \( X \) onto a space \( Y \). Then the following hold.

(i) If for each \( y \in Y \), \( f^{-1}(y) \) is an \( S \)-set in \( X \), then \( f^{-1}(A) \) is almost rc-Lindelöf in \( X \) whenever \( A \) is almost rc-Lindelöf in \( Y \).

(ii) If for each \( y \in Y \), \( f^{-1}(y) \) is rc-Lindelöf in \( X \), then \( f^{-1}(A) \) is rc-Lindelöf in \( X \) whenever \( A \) is rc-Lindelöf in \( Y \) provided that \( X \) is a weak \( P \)-space.

The following theorem shows that the assumption weakly almost open of Theorem 4.4 is not essential for the inverse preservation of the rc-Lindelöf set property.

**Theorem 4.5.** Let \( f \) be a regular open function from a space \( X \) onto a space \( Y \). Then the following hold.

(i) If for each \( y \in Y \), \( f^{-1}(y) \) is an \( S \)-set in \( X \), then \( f^{-1}(A) \) is rc-Lindelöf in \( X \) whenever \( A \) is rc-Lindelöf in \( Y \).

(ii) If for each \( y \in Y \), \( f^{-1}(y) \) is rc-Lindelöf in \( X \), then \( f^{-1}(A) \) is rc-Lindelöf in \( X \) whenever \( A \) is rc-Lindelöf in \( Y \) provided that \( X \) is a weak \( P \)-space.

The proof of the following proposition is straightforward and thus omitted.

**Proposition 4.6.** Let \( X \) be a nearly Lindelöf space and \( Y \) a weak \( P \)-space. Then the projection function \( p : X \times Y \rightarrow Y \) sends regular closed sets onto closed sets.

**Corollary 4.7.** Let \( X, Y \) be two spaces such that \( Y \) is rc-Lindelöf and \( X \times Y \) is extremally disconnected. Then the following hold.

(i) If \( X \) is compact, then \( X \times Y \) is rc-Lindelöf [2].

(ii) If \( X \) is Lindelöf, then \( X \times Y \) is rc-Lindelöf provided that \( X \times Y \) is a weak \( P \)-space.

**Proof.** We will show (ii), the other part is similar. Consider the projection function \( p : X \times Y \rightarrow Y \). Since \( X \times Y \) is a weak \( P \)-space, it follows that \( Y \) is a weak \( P \)-space, but \( X \) is Lindelöf and thus nearly Lindelöf, so by Proposition 4.6, \( p : X \times Y \rightarrow Y \) sends regular closed sets onto closed sets, but \( X \times Y \) is extremally disconnected, so every regular open subset of \( X \times Y \) is regular closed and thus \( p : X \times Y \rightarrow Y \) sends regular open sets onto closed sets, but \( p \) is an open function, so \( p \) is regular open. Also for each \( y \in Y \), \( p^{-1}(y) = X \times \{y\} \) is rc-Lindelöf in \( X \times Y \) (as \( X \) is Lindelöf and \( X \times Y \) is extremally disconnected). Finally, since \( Y \) is rc-Lindelöf, it follows immediately from Theorem 4.5(ii) that \( X \times Y \) is rc-Lindelöf.

The following result is an improvement of Corollary 4.7, it follows from Theorem 1.2, Proposition 1.4, Corollary 4.7, and the fact that the properties of being extremally disconnected (a weak \( P \)-space) are hereditary with respect to open subsets.

**Corollary 4.8.** Let \( X, Y \) be two rc-Lindelöf spaces such that \( X \times Y \) is extremally disconnected. Then the following hold.

(i) If \( X \) is locally compact, that is, for each \( x \in X \), there exists an open set \( U_x \) containing \( x \) such that \( U_x \) is compact, then \( X \times Y \) is rc-Lindelöf.

(ii) If \( X \) is locally Lindelöf, that is, for each \( x \in X \), there exists an open set \( U_x \) containing \( x \) such that \( U_x \) is Lindelöf, then \( X \times Y \) is rc-Lindelöf provided that \( X \times Y \) is a weak \( P \)-space.
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References


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