ON AN EXTENSION OF SINGULAR INTEGRALS ALONG MANIFOLDS OF FINITE TYPE

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We extend the L^p -boundedness of a class of singular integral operators under the H^1 kernel condition on a compact manifold from the homogeneous Sobolev space $\dot{L}_\alpha^p(\mathbb{R}^n)$ to the Lebesgue space $L^p(\mathbb{R}^n)$.

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1. Introduction

Let $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$, $n \geq 2$, with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $\Omega(x')$ be a homogeneous function of degree 0, with $\Omega \in L^1(\mathbf{S}^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega(x') d\sigma(x') = 0, \quad (1.1)$$

where $x' = x/|x|$ for any $x \neq 0$.

Suppose that h is an $L^\infty(\mathbb{R}^+)$ function; the singular integral operator $SI_{\Omega,h}$ is defined by

$$SI_{\Omega,h}(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} h(|y|) \frac{\Omega(y')}{|y|^n} f(x-y) dy \quad (1.2)$$

for all test functions f , where $y' = y/|y| \in \mathbf{S}^{n-1}$.

We denote $SI_{\Omega,h}(f)$ by $SI_\Omega(f)$ if $h = 1$. The operator SI_Ω was first studied by Calderón and Zygmund in their well-known papers (see [1, 2]). They proved that SI_Ω is $L^p(\mathbb{R}^n)$ bounded, $1 < p < \infty$, provided that $\Omega \in L\text{Log}^+ L(\mathbf{S}^{n-1})$ satisfying (1.1). They also showed that the space $L\text{Log}^+ L(\mathbf{S}^{n-1})$ cannot be replaced by any Orlicz space $L^\phi(\mathbf{S}^{n-1})$ with a monotonically increasing function ϕ satisfying $\phi(t) = o(t \log t)$, $t \rightarrow \infty$, that is, $L(\text{Log}^+ L)^{1-\varepsilon}(\mathbf{S}^{n-1})$, $0 < \varepsilon \leq 1$. The idea of their proof was as follows.

Suppose that $\Omega \in L^1(\mathbf{S}^{n-1})$ is an odd function, then one can easily show that

$$SI_\Omega(f)(x) = \frac{1}{2} \int_{\mathbf{S}^{n-1}} \Omega(y') \left\{ \int_{-\infty}^{\infty} f(x - ty') t^{-1} dt \right\} d\sigma(y'). \quad (1.3)$$

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By the method of rotation and the well-known L^p -boundedness of the Hilbert transform, one then obtains the L^p -boundedness of SI_Ω under the weak condition $\Omega \in L^1(\mathbf{S}^{n-1})$.

For even kernels, the condition $\Omega \in L^1(\mathbf{S}^{n-1})$ is insufficient. It turns out that the right condition is $\Omega \in L\log^+ L(\mathbf{S}^{n-1})$ (as far as the size of Ω is concerned). The idea of Calderón and Zygmund is to compose the operator SI_Ω with the Riesz transforms R_j , $1 \leq j \leq n$, and to show that $R_j(SI_\Omega)$ is a singular integral operator with an appropriate odd kernel. Thus

$$\|R_j(SI_\Omega)(f)\|_p \leq C_p \|f\|_p \quad (1.4)$$

for all test functions $f \in \mathcal{S}$. Furthermore, one can obtain

$$\begin{aligned} \|SI_\Omega(f)\|_p &= \left\| \left(\sum_{j=1}^n R_j^2 \right) SI_\Omega(f) \right\|_p \leq \sum_{j=1}^n \|R_j(R_j SI_\Omega(f))\|_p \\ &\leq nC \sum_{j=1}^n \|R_j SI_\Omega(f)\|_p \leq n^2 CC_p \|f\|_p \end{aligned} \quad (1.5)$$

for all test functions $f \in \mathcal{S}$, since $-\sum_{j=1}^n R_j^2$ is the identity map. Using the above method, Connett [7] and Ricci and Weiss [15] independently obtained the same L^p -boundedness of SI_Ω under the weak condition $\Omega \in H^1(\mathbf{S}^{n-1})$, where $H^1(\mathbf{S}^{n-1})$ is the Hardy space which contains $L\log^+ L(\mathbf{S}^{n-1})$ as a proper subspace.

In [12], Fefferman generalized this Calderón-Zygmund singular integral by replacing the kernel $\Omega(x')|x|^{-n}$ by $h(|x|)\Omega(x')|x|^{-n}$, where h is an arbitrary L^∞ function. This allows the kernel to be rough not only on the sphere but also in the radial direction. For the singular integral operator $SI_{\Omega,h}$ with the kernel $K(x) = h(|x|)(\Omega(x')/|x|^n)$, the formula (1.3) now is

$$SI_{\Omega,h}(f)(x) = \int_{\mathbf{S}^{n-1}} \Omega(y') \left\{ \int_0^\infty f(x - ty') h(t) t^{-1} dt \right\} d\sigma(y'). \quad (1.6)$$

Clearly, the method of Calderón and Zygmund can no longer be used to estimate the above integral in (1.6) even if Ω is odd, since the integral in parentheses cannot be reduced to the Hilbert transform for an arbitrary $h(t)$. Thus, one needs to find a new approach.

Using a method which is different from Calderón and Zygmund, Fefferman showed in [12] that if Ω satisfies a Lipschitz condition, then $SI_{\Omega,h}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Later in [8], using Littlewood-Paley theory and Fourier transform methods, Duoandikoetxea and Rubio de Francia improved Fefferman's results by assuming a roughness condition $\Omega \in L^q(\mathbf{S}^{n-1})$ (see also [3, 13, 14]). By modifying the method in [8], recently, Fan and Pan [11] have improved the above results on $SI_{\Omega,h}$ by assuming a roughness condition $\Omega \in H^1(\mathbf{S}^{n-1})$.

Noting that $\mathbf{S}^{n-1}$ is an $(n-1)$ -dimensional compact manifold in $\mathbb{R}^{n-1}$, Duoandikoetxea and Rubio de Francia [8] introduced the following extension of the operator $\text{SI}_{\Omega,h}$.

Let $m, n \in \mathbb{N}$, $m \leq n-1$, and let $\mathcal{M}$ be a compact, smooth, m -dimensional manifold in $\mathbb{R}^n$. Suppose that $\mathcal{M} \cap \{rv : r > 0\}$ contains at most one point for any $v \in \mathbf{S}^{n-1}$. Let $\mathcal{C}(\mathcal{M})$ denote the cone $\{r\theta : r > 0, \theta \in \mathcal{M}\}$ equipped with the measure $ds(r\theta) = r^m dr d\sigma(\theta)$, where $d\sigma$ represents the induced Lebesgue measure on $\mathcal{M}$. For a locally integrable function in $\mathcal{C}(\mathcal{M})$ of the form

$$K(r\theta) = r^{-m-1}h(r)\Omega(\theta), \quad (1.7)$$

where Ω satisfies

$$\int_{\mathcal{M}} \Omega(\theta) d\sigma(\theta) = 0, \quad (1.8)$$

they defined the corresponding singular integral operator $\text{SI}_{\mathcal{M},\Omega,h}$ on $\mathbb{R}^n$ by

$$\begin{aligned} (\text{SI}_{\mathcal{M},\Omega,h} f)(x) &= \text{p.v.} \int_{\mathcal{C}(\mathcal{M})} f(x-y) K(y) ds(y) \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^{\infty} \int_{\mathcal{M}} f(x-r\theta) \Omega(\theta) h(r) r^{-1} d\sigma(\theta) dr \end{aligned} \quad (1.9)$$

initially for $f \in \mathcal{S}(\mathbb{R}^n)$.

In [8], Duoandikoetxea and Rubio de Francia obtained the following results regarding $\text{SI}_{\mathcal{M},\Omega,h}$.

THEOREM 1.1. *Let $\text{SI}_{\mathcal{M},\Omega,h}$ be given as in (1.7)–(1.9). Suppose that*

- (i) $\Omega \in L^q(\mathcal{M})$,
- (ii) $\sup_{R>0} ((1/R) \int_0^R |h(r)|^2 dr) < \infty$,
- (iii) $\mathcal{M}$ has a contact of finite order with every hyperplane.

Then $\text{SI}_{\mathcal{M},\Omega,h}$ extends to a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Inspired by the earlier result of Fan and Pan regarding $\Omega \in H^1(\mathbf{S}^{n-1})$, Cheng and Pan [5] established the following.

THEOREM 1.2. *Let $\text{SI}_{\mathcal{M},\Omega,h}$ be given as in Theorem 1.1, and let h and $\mathcal{M}$ satisfy (ii) and (iii), respectively. If $\Omega \in H^1(\mathcal{M})$, then $\text{SI}_{\mathcal{M},\Omega,h}$ extends to a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.*

The main purpose of this paper is to extend Theorem 1.2 to the case $\Omega \in H^r(\mathcal{M})$ with $0 < r < 1$. The space $H^r(\mathcal{M})$ is a distribution space when $0 < r < 1$. The definition of $H^r(\mathcal{M})$ can be found in Section 2, but here we must define the operator in the sense of distribution.

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Let $\langle \Omega, \phi \rangle$ be the pairing between $\Omega \in H^r(\mathcal{M})$ and a C^∞ function ϕ on $\mathcal{M}$. For $0 \leq \alpha$, we define the singular integral operator $SI_{\mathcal{M}, \Omega, h, \alpha} f(x)$ by

$$SI_{\mathcal{M}, \Omega, h, \alpha} f(x) = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^{\infty} \langle f(x - r \cdot), \Omega(\cdot) \rangle h(r) r^{-1-\alpha} dr, \quad (1.10)$$

where $f \in \mathcal{S}(\mathbb{R}^n)$, h, Ω satisfy (ii) and (iii) in Theorem 1.1, respectively, and $\Omega \in H^r(\mathcal{M})$ satisfies

$$\langle \Omega, P_m|_{\mathcal{M}} \rangle = 0 \quad (1.11)$$

for all polynomials on $\mathbb{R}^n$ with degree $m \leq [\alpha]$ and $r = m/m + \alpha$.

When $\mathcal{M} = \mathbb{S}^{n-1}$, the operator $SI_{\mathbb{S}^{n-1}, \Omega, h, \alpha}$ was studied in [4]. It is not difficult to check that (1.10) is well defined and it is finite for all $x \in \mathbb{R}^n$.

When $\alpha = 0$, the operator $SI_{\mathbb{S}^{n-1}, \Omega, h, 0}$ is exactly the operator $SI_{\mathcal{M}, \Omega, h}$.

The main result of this paper is as follows.

THEOREM 1.3. *Let $SI_{\mathcal{M}, \Omega, h, \alpha}$ be given as in (1.10), and let $h, \mathcal{M}$ satisfy (ii) and (iii) as in Theorem 1.1, respectively. If $\Omega \in H^r(\mathcal{M})$ satisfies (1.11), then $SI_{\mathcal{M}, \Omega, h, \alpha}$ extends to a bounded operator from the homogeneous Sobolev space $\dot{L}_\alpha^p(\mathbb{R}^n)$ to the Lebesgue space $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.*

2. Definitions and lemmas

Let $\mathcal{M}$ be a compact, smooth, m -dimensional manifold in $\mathbb{R}^n$, $m \leq n - 1$. The Hardy spaces $H^p(\mathcal{M})$ can be defined by using the maximal operator

$$\mathcal{A} : f \longrightarrow (\mathcal{A}f)(x) = \sup_{t>0} |u(t, x)|, \quad (2.1)$$

where $u(t, x)$ is the solution of the boundary value problem

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_x \right) u &= 0, \quad (t, x) \in \mathbb{R}^+ \times \mathcal{M}, \\ u(0, x) &= f(x), \quad x \in \mathcal{M}. \end{aligned} \quad (2.2)$$

Here Δ_x denotes the Laplace-Beltrami operator of $\mathcal{M}$.

Definition 2.1. Define

$$H^p(\mathcal{M}) = \{f \in \mathcal{S}'(\mathcal{M}) : \|\mathcal{A}f\|_{L^p(\mathcal{M})} < \infty\}. \quad (2.3)$$

For $f \in H^p(\mathcal{M})$, we set $\|f\|_{H^p(\mathcal{M})} = \|\mathcal{A}f\|_{L^p(\mathcal{M})}$.

It is well known that since $\mathcal{M}$ is compact,

$$H^p(\mathcal{M}) = L^p(\mathcal{M}) \subset L \log^+ L(\mathcal{M}) \subset H^1(\mathcal{M}) \subset H^r(\mathcal{M}), \quad 0 < r < 1 < p, \quad (2.4)$$

and all the inclusions are proper.

Let $B_n(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$. To give the atomic characterization of H^r , we need to define atoms on $\mathcal{M}$.

Definition 2.2. A function $a(\cdot)$ on $\mathcal{M}$ is called an H^r atom if there are $\rho > 0$ and $\theta_0 \in \mathcal{M}$ such that

- (1) $\text{supp}(a) \subseteq B_n(\theta_0, \rho) \cap \mathcal{M}$,
- (2) $\|a\|_\infty \leq \rho^{-m/r}$,
- (3) $\int_{\mathcal{M}} a(\theta) P_k|_{\mathcal{M}}(\theta) d\sigma(\theta) = 0$,

for all polynomials P_k on $\mathbb{R}^n$, with degrees $k \leq [m(1/r - 1)]$.

If $\Omega \in H^r(\mathcal{M})$, then there exist H^r atoms $\{a_j\}$ and complex numbers $\{c_j\}$ such that

$$\Omega = \sum c_j a_j, \quad \sum |c_j|^r \cong \|\Omega\|_{H^r(\mathcal{M})}^r \quad (\text{see [6]}). \quad (2.5)$$

Definition 2.3. A smooth mapping ϕ from an open set U in $\mathbb{R}^m$ into $\mathbb{R}^n$ is said to be of finite type at $u_0 \in U$ if, for every $\eta \in \mathbb{S}^{n-1}$, there exists a nonzero multi-index $\omega = \omega(\eta)$ such that

$$\left. \frac{\partial^\omega [\eta \cdot \phi(u)]}{\partial u^\omega} \right|_{u=u_0} \neq 0. \quad (2.6)$$

By the smoothness and compactness of $\mathcal{M}$, we may assume that there is a smooth mapping ϕ from a neighborhood of $\overline{B_m(0, 1)}$ into $\mathbb{R}^n$ such that

- (i) $\theta_0 \in \phi(B_m(0, 1/2))$ and $\mathcal{M} \cap B_n(\theta_0, \rho) \subset \phi(B_m(0, 1)) \subset \mathcal{M}$;
- (ii) the vectors $\partial\phi/\partial u_1, \dots, \partial\phi/\partial u_m$ are linearly independent for each $u \in \overline{B_m(0, 1)}$;
- (iii) ϕ is of finite type at every point in $\overline{B_m(0, 1)}$ (see [16, page 350]).

Thus there is a smooth function $J(u)$ such that

$$\int_{\phi(B_m(0, 1))} F d\sigma = \int_{B_m(0, 1)} F(\phi(u)) J(u) du, \quad (2.7)$$

for any integrable function F on $\mathcal{M}$. Since $\mathcal{M}$ is compact, we may assume that all ϕ raised from atoms a satisfy $|\phi(u) - \phi(u_0)| \leq |u - u_0|$.

Now given $\Omega \in H^r(\mathcal{M})$, then for each H^r atom, $a(\theta)$ supported in $\mathcal{M} \cap B_n(\theta_0, \rho)$, write $b(u) = a(\phi(u))J(u)\chi_{B_m(0, 1)}$. Let $u_0 = \phi^{-1}(\theta_0)$. It follows from (i)–(iii) that

$$\begin{aligned} \text{supp}(b) &\subset B_m(u_0, \rho), \\ \|b\|_\infty &\leq C\rho^{-m/r}, \quad \text{we may assume that } C = 1, \\ \int_{\mathbb{R}^m} b(u) (\phi(u) - \phi(u_0))^k du &= 0, \end{aligned} \quad (2.8)$$

for all $|k| \leq [\alpha]$, where $k = (k_1, k_2, \dots, k_m)$ is a multi-index and $k = \sum_{i=1}^m k_i$.

We will need the following result (see [8]).

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LEMMA 2.4. Let $\{a_k\}$ be a lacunary sequence of positive numbers such that $a_k > 0$ and $\inf_{k \in \mathbb{Z}} |a_{k+1}/a_k| = \tau > 1$. Let τ_k be a sequence of Borel measures in $\mathbb{R}^n$. Suppose that $\|\tau_k\| \leq 1$ and

$$(1) |\hat{\tau}_k| \leq C|a_{k+1}\xi|^\gamma,$$

$$(2) |\hat{\tau}_k| \leq C|a_k\xi|^{-\gamma},$$

for all $k \in \mathbb{Z}$, and suppose also that for some $q > 1$,

$$(3) \|\tau^*(f)\| \leq C\|f\|_q,$$

where τ^* is the maximal operator: $\tau^*(f) = \sup_k \|\tau_k * f\|$. Then

$$Tf(x) = \sum_{k=-\infty}^{\infty} \tau_k * f(x) \quad (2.9)$$

is a bounded operator on $L^p(\mathbb{R}^n)$ for $|1/p - 1/2| < 1/2q$.

We will also need the following result (see [8, 9, 11]).

LEMMA 2.5. Let $l, n \in \mathbb{N}$, and $\{\tau_{s,k} : 0 \leq s \leq l, \text{ and } k \in \mathbb{Z}\}$ be a family of measures on $\mathbb{R}^n$ with $\tau_{0,k} = 0$ for every $k \in \mathbb{Z}$. Let $\{\alpha_{sj} : 1 \leq s \leq l, \text{ and } j = 1, 2\} \subset \mathbb{R}^+$, $\{\eta_s : 1 \leq s \leq l\} \subset \mathbb{R}^+ \setminus \{1\}$, $\{M_s : 1 \leq s \leq l\} \subset \mathbb{N}$, and $L_s : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be linear transformations for $1 \leq s \leq l$. Suppose that

$$(i) \|\tau_{s,k}\| \leq 1 \text{ for } k \in \mathbb{Z} \text{ and } 0 \leq s \leq l;$$

$$(ii) \|\hat{\tau}_{s,k}(\xi)\| \leq C(\eta_s^k |L_s \xi|)^{-\alpha_{s2}} \text{ for } \xi \in \mathbb{R}^m, k \in \mathbb{Z}, \text{ and } 0 \leq s \leq l;$$

$$(iii) \|\hat{\tau}_{s,k}(\xi) - \hat{\tau}_{s-1,k}(\xi)\| \leq C(\eta_s^k |L_s \xi|)^{\alpha_{s1}} \text{ for } \xi \in \mathbb{R}^m, k \in \mathbb{Z}, \text{ and } 0 \leq s \leq l;$$

$$(iv) \text{ for some } p_0 > 2, \text{ there exists a } C > 0 \text{ such that}$$

$$\left\| \sum_{k \in \mathbb{Z}} \left(|\tau_{s,k} * g_k|^2 \right)^{1/2} \right\|_{L^{p_0}(\mathbb{R}^n)} \leq C \left\| \sum_{k \in \mathbb{Z}} \left(|g_k|^2 \right)^{1/2} \right\|_{L^{p_0}(\mathbb{R}^n)}, \quad (2.10)$$

for all $\{g_k\} \in L^{p_0}(\mathbb{R}^n, l^2)$ and $1 \leq s \leq l$.

Then for every $p \in (p'_0, p_0)$, there exists a positive constant C_p such that

$$\left\| \sum_{k \in \mathbb{Z}} \tau_{l,k} * f \right\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}, \quad (2.11)$$

$$\left\| \left(\sum_{k \in \mathbb{Z}} |\tau_{l,k} * f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}$$

hold for all $f \in L^p(\mathbb{R}^n)$. The constant C_p is independent of the linear transformations $\{L_s\}_{s=1}^l$.

3. Proof of theorem

We will prove the theorem in three different cases: $0 < \alpha < 1$, $\alpha = 1, 2, 3, \dots$, and $\alpha > 1$, $\alpha \notin \mathbb{Z}$. Without loss of generality, we may assume that $\Omega(\theta) = a(\theta)$ is an H^r atom as defined in Definition 2.2, the details can be found in [4].

Case 1 ($0 < \alpha < 1$). Using the “lift” property of the Riesz potential and the definition of the space $\dot{L}_\alpha^p(\mathbb{R}^n)$, it is known that for any $\alpha > 0$ and $f \in \dot{L}_\alpha^p(\mathbb{R}^n)$, one can write $f = G_\alpha * f_\alpha$ with $|\hat{G}_\alpha(\xi)| \approx |\xi|^{-\alpha}$, $|G_\alpha(y)| \approx |y|^{-n+\alpha}$, and $\|f_\alpha\|_p \approx \|f\|_{\dot{L}_\alpha^p}$.

We write

$$(\text{SI}_{\mathcal{M}, \Omega, h, \alpha} f)(x) = \sum_k \mu_{k, \alpha} * f_\alpha(x), \quad (3.1)$$

where

$$\mu_{k, \alpha}(x) = \int_{2^k}^{2^{k+1}} \int_{\mathcal{M}} G_\alpha(x - r\theta) \Omega(\theta) h(r) r^{-1-\alpha} d\sigma(\theta) dr. \quad (3.2)$$

In light of Lemma 2.4, in order to show that $\|\text{SI}_{\mathcal{M}, \Omega, \alpha} f\|_{L^p} \leq C\|f\|_{\dot{L}_\alpha^p}$, it suffices to show that

- (i) $\|\mu_{k, \alpha}\|_{L^1(\mathbb{R}^n)} \leq C$,
- (ii) $|\hat{\mu}_{k, \alpha}(\xi)| \leq C|2^k \xi \rho|^{1-\alpha}$,
- (iii) $|\hat{\mu}_{k, \alpha}(\xi)| \leq C|2^k \xi \rho|^{-\alpha}$,
- (iv) $\|\sup_{k \in \mathbb{Z}} |\mu_{k, \alpha} * f|\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^q(\mathbb{R}^n)}$, for all $q \in (1, \infty)$.

Now, by the cancellation condition of $b(u) = \Omega(\phi(u))f(u)\chi_{B_m(0,1)}(u)$, we have

$$\begin{aligned} \|\mu_{k, \alpha}\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left| \int_{2^k}^{2^{k+1}} \left[\int_{B_m(0,1)} (G_\alpha(x - r\phi(u)) \right. \right. \\ &\quad \left. \left. - G_\alpha(x - r\phi(u_0))) b(u) du \right] |h(r)| r^{-1-\alpha} dr \right| dx \\ &\leq \int_{2^k}^{2^{k+1}} r^{-1-\alpha} \int_{B_m(0,1)} |b(u)| \\ &\quad \times \int_{\mathbb{R}^n} |G_\alpha(x - r\phi(u)) - G_\alpha(x - r\phi(u_0))| dx |h(r)| du dr. \end{aligned} \quad (3.3)$$

Letting $y = x - r\phi(u_0)$, we have

$$\int_{\mathbb{R}^n} |G_\alpha(x - r\phi(u)) - G_\alpha(x - r\phi(u_0))| dx = \int_{\mathbb{R}^n} |G_\alpha(y + r(\phi(u) - \phi(u_0))) - G_\alpha(y)| dy. \quad (3.4)$$

As we mentioned before, $|\phi(u) - \phi(u_0)| \leq |u - u_0| \leq \rho$, for $u \in \text{supp}(b)$.

We write

$$\begin{aligned} &\int_{\mathbb{R}^n} |G_\alpha(y + r(\phi(u) - \phi(u_0))) - G_\alpha(y)| dy \\ &= \int_{|y| \geq 3r\rho} |G_\alpha(y + r(\phi(u) - \phi(u_0))) - G_\alpha(y)| dy \\ &\quad + \int_{|y| < 3r\rho} |G_\alpha(y + r(\phi(u) - \phi(u_0))) - G_\alpha(y)| dy \\ &= I_1 + I_2, \quad \text{where } u \text{ is in the support of } b(u). \end{aligned} \quad (3.5)$$

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By the definition of $G_\alpha(x)$, we have, if $y \geq 3r\rho \geq 3r|\phi(u) - \phi(u_0)|$,

$$|G_\alpha(y + r(\phi(u) - \phi(u_0))) - G_\alpha(y)| \leq C \frac{r\rho}{|y|^{n-\alpha+1}}. \quad (3.6)$$

Thus,

$$I_1 \leq C \int_{|y| \geq 3r\rho} \frac{r\rho}{|y|^{n-\alpha+1}} dy \approx (r\rho)^\alpha. \quad (3.7)$$

It is easy to see that

$$I_2 \leq 2 \int_{|y| \leq 5r\rho} |G_\alpha(y)| dy \leq C \int_{|y| \leq 5r\rho} \frac{dy}{|y|^{n-\alpha}} \leq C(r\rho)^\alpha. \quad (3.8)$$

Thus,

$$\begin{aligned} \|\mu_{k,\alpha}\|_{L^1(\mathbb{R}^n)} &\leq \int_{2^k}^{2^{k+1}} r^{-1-\alpha} \int_{B_m(0,1)} |b(u)| \\ &\quad \times \int_{\mathbb{R}^n} |G_\alpha(x - r\phi(u)) - G_\alpha(x - r\phi(u_0))| dx |h(r)| du dr \\ &\leq \int_{2^k}^{2^{k+1}} r^{-1-\alpha} \int_{B_m(0,1)} |b(u)| (r\rho)^\alpha |h(r)| du dr \leq C. \end{aligned} \quad (3.9)$$

To prove (ii), we write

$$|\widehat{\mu}_{k,\alpha}(\xi)| = |(\widehat{\sigma_{k,\alpha} * G_\alpha})(\xi)| = |\widehat{\sigma_{k,\alpha}}(\xi)| |\widehat{G_\alpha}(\xi)| \leq C|\xi|^{-\alpha} |\widehat{\sigma_{k,\alpha}}(\xi)|. \quad (3.10)$$

Thus,

$$\begin{aligned} |\widehat{\mu}_{k,\alpha}(\xi)| &\leq C|\xi|^{-\alpha} \left| \int_{2^k}^{2^{k+1}} \left(\int_{B_m(0,1)} e^{-ir\xi \cdot \phi(u)} b(u) du \right) r^{-1-\alpha} h(r) dr \right| \\ &\leq C|\xi|^{-\alpha} 2^{-k\alpha} \int_{2^k}^{2^{k+1}} \left| \int_{B_m(0,1)} (e^{-ir\xi \cdot \phi(u)} - e^{-ir\xi \cdot \phi(u_0)}) b(u) du \right| r^{-1} |h(r)| dr \\ &\leq C|\xi|^{-\alpha} 2^{-k\alpha} |2^k \xi| \int_{B_m(0,1)} |\phi(u) - \phi(u_0)| |b(u)| du \leq C |2^k \xi \rho|^{1-\alpha}, \end{aligned} \quad (3.11)$$

which proves (ii).

On the other hand,

$$|\widehat{\mu}_{k,\alpha}(\xi)| \leq C|\xi|^{-\alpha} 2^{-k\alpha} \int_{2^k}^{2^{k+1}} \int_{B_m(0,1)} |b(u)| du r^{-1} |h(r)| dr = C |2^k \xi \rho|^{-\alpha}, \quad (3.12)$$

which proves (iii).

It remains to show that

$$\left\| \sup_{k \in \mathbb{Z}} |\mu_{k,\alpha}| * f \right\|_p \leq C \|f\|_p. \quad (3.13)$$

Without loss of generality, assume that $h(r) \geq 0$. Then

$$\begin{aligned} & \left\| \sup_{k \in \mathbb{Z}} |\mu_{k,\alpha}| * f \right\|_{L^q(\mathbb{R}^n)} \\ & \leq C \sup_{k \in \mathbb{Z}} 2^{-k-k\alpha} \int_{2^k}^{2^{k+1}} h(r) \int_{B_m(0,1)} |b(u)| \int_{\mathbb{R}^n} |f(x-z)| |G_\alpha(z-r\phi(u)) \\ & \quad - G_\alpha(z-r\phi(u_0))| dz du dr. \end{aligned} \quad (3.14)$$

In the above integral, we write

$$\begin{aligned} & \int_{\mathbb{R}^n} |f(x-z)| |G_\alpha(z-r\phi(u)) - G_\alpha(z-r\phi(u_0))| dz \\ & = \int_{|z-r\phi(u_0)| > 3r\rho} |f(x-z)| |G_\alpha(z-r\phi(u)) - G_\alpha(z-r\phi(u_0))| dz \\ & \quad + \int_{|z-r\phi(u_0)| \leq 3r\rho} |f(x-z)| |G_\alpha(z-r\phi(u)) - G_\alpha(z-r\phi(u_0))| dz \\ & = I_1(f)(x) + I_2(f)(x), \end{aligned} \quad (3.15)$$

where $u \in B_n(u_0, \rho) \cap \mathcal{M}$.

In the integral $I_1(f)$, we change variables $z-r\phi(u_0) \rightarrow y$ and again write y as z , then

$$I_1(f)(x) = C \int_{|z| > 3r\rho} |f(x-z+r\phi(u_0))| |G_\alpha(z+r\phi(u_0)-r\phi(u)) - G_\alpha(z)| dz. \quad (3.16)$$

Note that $|r\phi(u_0) - r\phi(u)| \leq r\rho < |z|/2$. By the mean value theorem,

$$\begin{aligned} I_1(f)(x) & \leq C \int_{|z| > 3r\rho} r\rho |f(x-z+r\phi(u_0))| |z|^{\alpha-1-n} dz \\ & \cong \int_{S^{n-1}} \int_{3r\rho}^\infty r\rho s^{\alpha-2} |f(x-sz'+r\phi(u_0))| ds d\sigma(z'). \end{aligned} \quad (3.17)$$

Using integration by parts, it is easy to see that

$$\begin{aligned} I_1(f)(x) & \leq C \int_{S^{n-1}} (r\rho)^\alpha (r\rho)^{-1} \int_0^{3r\rho} |f(x-tz'+r\phi(u_0))| dt d\sigma(z') \\ & \quad + C \int_{S^{n-1}} \int_{3r\rho}^\infty r\rho s^{\alpha-3} \int_0^s |f(x-tz'+r\phi(u_0))| ds dt d\sigma(z'). \end{aligned} \quad (3.18)$$

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Let $M_z f(x)$ be the maximal function

$$M_z f(x) = \sup_{t>0} t^{-1} \int_0^t |f(x - rz)| dr. \quad (3.19)$$

It is known in [16, page 477] that there is a constant C independent of z such that

$$\|M_z(f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}. \quad (3.20)$$

Thus we have

$$I_1(f)(x) \leq C(rp)^\alpha \int_{\mathbb{S}^{n-1}} M_{z'} f(x + r\phi(u_0)) d\sigma(z'). \quad (3.21)$$

For the second integral $I_2(f)(x)$, we have $I_2(f)(x) \leq J_1(f)(x) + J_2(f)(x)$, where

$$\begin{aligned} J_1(f)(x) &= \int_{|z - r\phi(u_0)| < 3rp} |f(x - z) G_\alpha(z - r\phi(u))| dz, \\ J_2(f)(x) &= \int_{|z| < 3rp} |f(x - z + r\phi(u_0)) G_\alpha(z)| dz. \end{aligned} \quad (3.22)$$

Let $w = z - r\phi(u)$. Then, in $J_1(f)(x)$, we have

$$|w| \leq |z - r\phi(u_0)| + |r\phi(u) - r\phi(u_0)| \leq 4rp. \quad (3.23)$$

This gives (again write z instead of w)

$$\begin{aligned} J_1(f)(x) &\leq C \int_{|z| < 4rp} |f(x - z - r\phi(u))| |z|^{\alpha-n} dz \\ &= C \int_{\mathbb{S}^{n-1}} \int_0^{4rp} t^{\alpha-1} |f(x - tz' - r\phi(u))| dt d\sigma(z'). \end{aligned} \quad (3.24)$$

Using integration by parts, we obtain

$$J_1(f)(x) \leq C \int_{\mathbb{S}^{n-1}} (rp)^\alpha M_{z'}(f(x - r\phi(u))) d\sigma(z'). \quad (3.25)$$

Similarly, we can have the same estimate on $J_2(f)(x)$ so that

$$J_2(f)(x) \leq C \int_{\mathbb{S}^{n-1}} (rp)^\alpha \{M_{z'} f(x + r\phi(u_0)) + M_{z'}(f(x - r\phi(u)))\} d\sigma(z'). \quad (3.26)$$

Thus

$$\begin{aligned} &\int_{\mathbb{R}^n} |f(x - z)| |G_\alpha(z - r\phi(u)) - G_\alpha(z - r\phi(u_0))| dz \\ &\leq C(rp)^\alpha \int_{\mathbb{S}^{n-1}} \{M_{z'} f(x + r\phi(u_0)) + M_{z'}(f(x - r\phi(u)))\} d\sigma(z'). \end{aligned} \quad (3.27)$$

Therefore, we have

$$\begin{aligned} & \left\| \sup_{k \in \mathbb{Z}} |\mu_{k,\alpha}| * f \right\|_{L^q(\mathbb{R}^n)} \\ & \leq C \int_{B_m(0,1) \times \mathbb{S}^{n-1}} |b(u)| \rho^\alpha \left\{ \|M_{\phi(u_0)} M_{z'}(f)\|_{L^q(\mathbb{R}^n)} + \|M_{\phi(u)} M_{z'} f\|_{L^q(\mathbb{R}^n)} \right\} d\sigma(z') du. \end{aligned} \quad (3.28)$$

Since b is an (r, ∞) atom supported in $B_m(u_0, \rho) \cap \mathcal{M}$ with $r = m/(m + \alpha)$, it is easy to see that

$$\int_{B_m(0,1)} |b(u)| \rho^\alpha du \leq C \quad (3.29)$$

uniformly for b and ρ . Thus

$$\left\| \sup_{k \in \mathbb{Z}} |\mu_{k,\alpha}| * f \right\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^q(\mathbb{R}^n)}. \quad (3.30)$$

By Lemma 2.4, Case 1 is established.

Case 2 ($\alpha = 1, 2, 3, \dots$). Using Taylor's expansion about θ_0 , we have, for $j = (j_1, \dots, j_m)$,

$$\begin{aligned} (\text{SI}_{\mathcal{M}, \Omega, h, \alpha} f)(x) &= \sum_{|j|=\alpha} C_j \int_0^1 (1-t)^{\alpha-1} \int_0^\infty \int_{B_m(0,1)} \mathcal{B}(u) r^{-1} h(r) \\ &\quad \times D^j f(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u))) du dr dt, \end{aligned} \quad (3.31)$$

where C_j 's are constants and $\mathcal{B}(u) = b(u)(\phi(u) - \phi(u_0))^j$. Clearly, $\mathcal{B}(u)$ is an H^1 atom with the same support as b .

For each j , $|j| = \alpha$, define the measures $\{\sigma_{\phi, \mathcal{B}, h, k, \alpha} | k \in \mathbb{Z}\}$ on $\mathbb{R}^n$ by

$$\begin{aligned} & \int_{\mathbb{R}^n} F(x) d\sigma_{\phi, \mathcal{B}, h, k, \alpha} \\ &= \int_0^1 (1-t)^{\alpha-1} \int_{2^k}^{2^{k+1}} \int_{B_m(0,1)} F(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u))) \mathcal{B}(u) r^{-1} h(r) du dr dt. \end{aligned} \quad (3.32)$$

LEMMA 3.1. Suppose that h satisfies (ii) in Theorem 1.1. Then for $1 < p < \infty$, there exists a constant $C_p > 0$ such that

$$\left\| \left(\sum_{k \in \mathbb{Z}} |\sigma_{\phi, \mathcal{B}, h, k, \alpha} * g_k|^2 \right)^{1/2} \right\|_p \leq C_p \left\| \left(\sum_{k \in \mathbb{Z}} |g_k|^2 \right)^{1/2} \right\|_p \quad (3.33)$$

holds for all continuous mappings ϕ and measurable functions $\{g_k\}$ on $\mathbb{R}^n$.

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Proof. For $\xi \in \mathbb{R}^n$, we define the maximal operator M_ξ on $\mathbb{R}^n$ by

$$(M_\xi f)(x) = \sup_{k \in \mathbb{Z}} \left[2^{-k} \int_{2^k}^{2^{k+1}} |f(x+r\xi)| dr \right]. \quad (3.34)$$

It follows from the L^p -boundedness of the one-dimensional Hardy-Littlewood maximal operator that

$$\|M_\xi f\|_p \leq A_p \|f\|_p, \quad (3.35)$$

for $1 < p \leq \infty$, where A_p is independent of ξ .

By duality, we may assume that $p > 2$, then for $\{g_k\} \in L^p(\mathbb{R}^n, l^2)$, there exists a function $w \in L^{(p/2)'}(\mathbb{R}^n)$ such that $\|w\|_{(p/2)'} = 1$ and

$$\left\| \left(\sum_{k \in \mathbb{Z}} |\sigma_{\phi, \mathcal{B}, h, k, \alpha} * g_k|^2 \right)^{1/2} \right\|_p^2 = \int_{\mathbb{R}^n} \left(\sum_{k \in \mathbb{Z}} |\sigma_{\phi, \mathcal{B}, h, k, \alpha} * g_k|^2 \right) w(x) dx. \quad (3.36)$$

By Hölder's inequality and (3.35),

$$\begin{aligned} & \left\| \left(\sum_{k \in \mathbb{Z}} |\sigma_{\phi, \mathcal{B}, h, k, \alpha} * g_k|^2 \right)^{1/2} \right\|_p^2 \\ & \leq \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} \left| \int_0^1 (1-t)^{\alpha-1} \int_{2^k}^{2^{k+1}} \int_{B_m(0,1)} g_k(xr\phi(u_0) + rt(\phi(u_0) - \phi(u))) \right. \\ & \quad \left. \times \mathcal{B}(u)r^{-1}h(r)dudr dt \right|^2 w(x) dx \\ & \leq C \|\mathcal{B}\|_1 \sum_{k \in \mathbb{Z}} 2^{-k} \int_{\mathbb{R}^n} \int_0^1 \int_{2^k}^{2^{k+1}} \int_{B_m(0,1)} |g_k(x-r\phi(u_0) + rt(\phi(u_0) - \phi(u)))|^2 \\ & \quad \times |\mathcal{B}(u)w(x)| du dr dt dx \\ & = C \|\mathcal{B}\|_1 \int_0^1 \int_{B_m(0,1)} |\mathcal{B}(u)| \\ & \quad \times \left[\sum_{k \in \mathbb{Z}} 2^{-k} \int_{2^k}^{2^{k+1}} \int_{\mathbb{R}^n} |g_k(x)|^2 |w(x+r\phi(u_0) + rt(\phi(u_0) - \phi(u)))| dx dr \right] du dt \\ & \leq C \|\mathcal{B}\|_1 \int_0^1 \int_{B_m(0,1)} \left[\int_{\mathbb{R}^n} \left(\sum_{k \in \mathbb{Z}} |g_k(x)|^2 \right) (M_{\phi(u_0)+t(\phi(u_0)-\phi(u))} w)(x) dx \right] |\mathcal{B}(u)| du dt \\ & \leq C \|\mathcal{B}\|_1^2 \left\| \left(\sum_{k \in \mathbb{Z}} |g_k|^2 \right)^{1/2} \right\|_p^2. \end{aligned} \quad (3.37)$$

We also have the following estimates for $\sigma_{\phi, \mathcal{B}, h, k, \alpha}$. □

LEMMA 3.2. Suppose that ϕ is smooth and of finite type at every point in $\overline{B_m(0,1)}$ and h satisfies (ii) in Theorem 1.1. Then there exists a $\delta > 0$ such that

$$|\hat{\sigma}_{\phi, \mathcal{B}, h, k, \alpha}| \leq C \|\mathcal{B}\|_2 (2^k |\xi|)^{-\delta}. \quad (3.38)$$

Proof.

$$|\hat{\sigma}_{\phi, \mathcal{B}, h, k, \alpha}(\xi)| = \left| \int_0^1 (1-t)^{\alpha-1} \int_{2^k}^{2^{k+1}} h(r) r^{-1} e^{i\xi r \phi(u_0)} e^{-i\xi r t \phi(u_0)} \int_{B_m(0,1)} \mathcal{B}(u) e^{i\xi r t \phi(u)} du dr dt \right|. \quad (3.39)$$

Changing variables ($s = rt$), we have

$$\begin{aligned} |\hat{\sigma}_{\phi, \mathcal{B}, h, k, \alpha}(\xi)| &= \left| \int_0^1 (1-t)^{\alpha-1} \int_{2^k t}^{2^{k+1} t} h\left(\frac{s}{t}\right) s^{-1} e^{i\xi(s/t)\phi(u_0)} e^{-i\xi s \phi(u_0)} \right. \\ &\quad \times \left. \int_{B_m(0,1)} \mathcal{B}(u) e^{i\xi s \phi(u)} du ds dt \right| \\ &\leq \int_0^1 |(1-t)^{\alpha-1}| \int_{2^k t}^{2^{k+1} t} |h(s/t) s^{-1}| \left| \left(\int_{B_m(0,1)} \mathcal{B}(u) e^{i\xi s \phi(u)} du \right) \right| ds dt. \end{aligned} \quad (3.40)$$

The remainder of the proof is similar to the proof of Lemma 3.3 in [5].

The following result is similar to those in [10], see also [5]. $\square$

LEMMA 3.3. Let $\mathcal{B}(\cdot)$ be a function satisfying $\text{supp}(\mathcal{B}) \subset B_m(0, \rho)$ and $\|\mathcal{B}\|_\infty \leq \rho^{-m}$ for some $\rho < 1$. Suppose that h satisfies (ii) in Theorem 1.1. Then there exists a constant $C > 0$ such that

$$\begin{aligned} &\left| \int_0^1 (1-t)^{\alpha-1} \int_{2^k}^{2^{k+1}} h(r) r^{-1} \left(\int_{B_m(0,1)} \mathcal{B}(u) e^{-irt[Q(u) + \sum_{|\beta|=s} d_\beta u^\beta]} du \right) dr dt \right| \\ &\leq C \left(2^k \rho^s \sum_{|\beta|=s} |d_\beta| \right)^{-1/(4s)} \end{aligned} \quad (3.41)$$

holds for all polynomials $Q: \mathbb{R}^m \rightarrow \mathbb{R}$ with $\deg(Q) < s$ and $\{d_\beta\} \subset \mathbb{R}$. The constant C is independent of ρ .

Now, by Lemma 3.2, there exists a $\delta > 0$ such that

$$|\hat{\sigma}_{\phi, \mathcal{B}, h, k, \alpha}(\xi)| \leq C (2^k |\xi|)^{-\delta} \rho^{-m/2}. \quad (3.42)$$

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Let $l = [m/(2\delta)] + 1$. Following the proof of Theorem 3.7 in [5], we define a sequence of mappings $\{\Phi^s\}_{s=0}^{s=l}$ by

$$\begin{aligned} \Phi^l &= \phi = (\phi_1, \dots, \phi_n), \\ \Phi^s(u) &= \left(\sum_{|\beta| \leq s} \frac{1}{\beta!} \frac{\partial^\beta \phi_1(u_0)}{\partial u^\beta} (u - u_0)^\beta, \dots, \sum_{|\beta| \leq s} \frac{1}{\beta!} \frac{\partial^\beta \phi_n(u_0)}{\partial u^\beta} (u - u_0)^\beta \right) \end{aligned} \quad (3.43)$$

for $s = 0, 1, \dots, l-1$.

Let

$$\sigma_{s,k,\alpha} = \sigma_{\Phi^s, \mathcal{B}, h, k, \alpha} \quad (3.44)$$

for $0 \leq s \leq l$ and $k \in \mathbb{Z}$.

In order to show that $\|\text{SI}_{\mathcal{M}, \Omega, h, \alpha} f\|_{L^p} \leq C\|f\|_{L^p_\alpha}$, it suffices to show that the family of measures $\{\sigma_{s,k,\alpha}\}$ satisfies the conditions of Lemma 2.5.

By its definition and Lemma 3.2, the family of measures $\{\sigma_{s,k,\alpha}\}$ satisfies conditions (i) and (iv) in Lemma 2.5, for any $p_0 > 2$.

It is easy to see that

$$\|\sigma_{s,k,\alpha}\| \leq \|\mathcal{B}\|_1 \int_0^1 |(1-t)^{\alpha-1}| \int_{2^k}^{2^{k+1}} r^{-1} |h(r)| dr dt \leq C. \quad (3.45)$$

Also we have

$$\sigma_{0,k,\alpha}(x) = 0, \quad \text{by the cancellation condition of } \mathcal{B}(u). \quad (3.46)$$

For $j = 1, \dots, n$, let

$$d_{j,\beta} = \frac{1}{\beta!} \frac{\partial^\beta \phi_j(u_0)}{\partial u^\beta}. \quad (3.47)$$

By (3.42) and Lemma 3.3, we have

$$\begin{aligned} |\hat{\sigma}_{l,k,\alpha}(\xi)| &\leq C(2^k \rho^l |\xi|)^{-\delta}, \\ |\hat{\sigma}_{s,k,\alpha}(\xi)| &\leq C \left(2^k \rho^s \sum_{|\beta|=s} \left| \sum_{j=1}^n d_{j\beta} \xi_j \right| \right)^{-1/(4s)} \end{aligned} \quad (3.48)$$

for $1 \leq s \leq l-1, k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$. We also have,

$$\begin{aligned} &|\hat{\sigma}_{l,k,\alpha}(\xi) - \hat{\sigma}_{l-1,k,\alpha}(\xi)| \\ &\leq \left| \int_0^1 |(1-t)^{\alpha-1}| \int_{2^k}^{2^{k+1}} |h(r)| r^{-1} \int_{B_m(0,1)} |\mathcal{B}(u)| |e^{i\xi r t \phi(u)} - e^{i\xi r t \phi^{l-1}(u)}| du dr dt \right| \\ &\leq C|\xi| 2^k \int_{B_m(0,1)} |\mathcal{B}(u)| |(\phi(u) - \phi^{l-1}(u))| du \leq C(2^k |\xi| \rho^l). \end{aligned} \quad (3.49)$$

Similarly,

$$\begin{aligned} |\hat{\sigma}_{s,k,\alpha}(\xi) - \hat{\sigma}_{s-1,k,\alpha}(\xi)| &\leq C2^k \int_{B_m(0,1)} |\mathcal{B}(u)| |\xi \cdot (\phi^s(u) - \phi^{s-1}(u))| du \\ &\leq C2^k \rho^s \sum_{|\beta|=s} \left| \sum_{j=1}^n d_{j\beta} \xi_j \right| \end{aligned} \quad (3.50)$$

for $1 \leq s \leq l-1$, $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$.

Invoking Lemma 2.5, Case 2 is established.

Case 3 ($\alpha > 1$, $\alpha \notin \mathbb{Z}$). Write $\alpha = [\alpha] + \gamma$, $\gamma \in (0, 1)$.

Similar to the case $\alpha = 1, 2, 3, \dots$, by Taylor's expansion, we have

$$\begin{aligned} (\text{SI}_{\mathcal{M}, \Omega, h, \alpha} f)(x) &= \sum_{|j|=\alpha} C_j \int_0^1 (1-t)^{\alpha-1} \int_0^\infty r^{-1-\gamma} h(r) \int_{B_m(0,1)} \mathcal{B}(u) \\ &\quad \times D^j f(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u))) du dt dr, \end{aligned} \quad (3.51)$$

where $\mathcal{B}(u) = b(u)(\phi(u) - \phi(u_0))^j$. Clearly, $\mathcal{B}(u)$ is an H^r atom, where $r = m/(m + \gamma)$.

Similar to Case 1, again using the “lift” property of the Riesz potential and the definition of the space $\dot{L}_\alpha^p(\mathbb{R}^n)$, it is known that for any $\gamma > 0$ and $f \in \dot{L}_\alpha^p(\mathbb{R}^n)$, one can write $f = G_\gamma * f_\gamma$ with $|\hat{G}_\gamma(\xi)| \approx |\xi|^{-\gamma}$, $|G_\gamma(y)| \approx |y|^{-n+\gamma}$, and $\|f_\gamma\|_p \approx \|f\|_{\dot{L}_\alpha^p}$.

We write

$$(\text{SI}_{\mathcal{M}, \Omega, h, \alpha, k} f)(x) = \sum_k \sigma_{k,\gamma} * f_\gamma, \quad (3.52)$$

where

$$\begin{aligned} \sigma_{k,\gamma} &= \int_0^1 (1-t)^{\alpha-1} \int_{2^k}^{2^{k+1}} r^{-1-\gamma} h(r) \int_{B_m(0,1)} \mathcal{B}(u) G_\gamma(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u))) du dr dt \\ &= \int_0^1 (1-t)^{\alpha-1} \int_{2^k}^{2^{k+1}} r^{-1-\gamma} h(r) \int_{B_m(0,1)} \mathcal{B}(u) \\ &\quad \times [G_\gamma(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u))) - G_\gamma(x - r\phi(u_0))] du dr dt. \end{aligned} \quad (3.53)$$

Again, by Lemma 2.4, in order to show that $\|\text{SI}_{\mathcal{M}, \Omega, h, \alpha, k} f\|_{L^p} \leq C\|f\|_{\dot{L}_\alpha^p}$, it suffices to show that

- (i) $\|\sigma_{k,\gamma}\|_{L^1(\mathbb{R}^n)} \leq C$,
- (ii) $|\hat{\sigma}_{k,\gamma}(\xi)| \leq C|2^k \xi \rho|^{1-\gamma}$,
- (iii) $|\hat{\sigma}_{k,\gamma}(\xi)| \leq C|2^k \xi \rho|^{-\gamma}$,
- (iv) $\|\sup_{k \in \mathbb{Z}} |\sigma_{k,\gamma}| * f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^q(\mathbb{R}^n)}$.

The proof is similar to the proof for Case 1. We leave the details to the reader.

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