IDEALS AND GREEN'S RELATIONS IN ORDERED SEMIGROUPS

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Exactly as in semigroups, Green's relations play an important role in the theory of ordered semigroups—especially for decompositions of such semigroups. In this paper we deal with the $\mathcal{J}$ -trivial ordered semigroups which are defined via the Green's relation $\mathcal{J}$, and with the nil and Δ -ordered semigroups. We prove that every nil ordered semigroup is $\mathcal{J}$ -trivial which means that there is no ordered semigroup which is 0-simple and nil at the same time. We show that in nil ordered semigroups which are chains with respect to the divisibility ordering, every complete congruence is a Rees congruence, and that this type of ordered semigroups are Δ -ordered semigroups, that is, ordered semigroups for which the complete congruences form a chain. Moreover, the homomorphic images of Δ -ordered semigroups are Δ -ordered semigroups as well. Finally, we prove that the ideals of a nil ordered semigroup S form a chain under inclusion if and only if S is a chain with respect to the divisibility ordering.

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1. Introduction and prerequisites

In the theory of ordered semigroups—especially for decompositions of ordered semigroups, an important role is played by the Green's relations. In this paper the concepts of $\mathcal{J}$ -trivial, nil ordered semigroups, Δ -ordered semigroups and the concept of complete congruences are introduced. The main results of the paper are the following: if S is an ordered semigroup, A an ideal of S , and B an idempotent ideal of A , then B is an ideal of S . For the corresponding result on semigroups, we refer to [2]. An ideal A of an ordered semigroup S is called idempotent if $A = (A^2)$ [6]. An ordered semigroup S is $\mathcal{J}$ -trivial if and only if the divisibility relation on S is an order on S . The divisibility relation on S is a chain if and only if S is $\mathcal{J}$ -trivial and the principal ideals of S form a chain under inclusion. Moreover the principal ideals of S form a chain with respect to the inclusion relation if and only if the ideals of S do so. For the corresponding result on semigroups, we refer to [12]. The nil ordered semigroups are $\mathcal{J}$ -trivial. As a consequence, an ordered semigroup cannot be 0-simple and nil at the same time.

2 Ideals and Green's relations in ordered semigroups

We introduce the concept of a complete congruence on an ordered semigroup $(S, \cdot, \leq)$ and the concept of the Rees congruence modulo A , where A is an ideal of S . A congruence ρ on $(S, \cdot, \leq)$ is called complete if $a \leq b$ implies $(a, ab) \in \rho$. We prove that if $(S, \cdot, \leq)$ is a nil ordered semigroup which is a chain with respect to the divisibility ordering, then every complete congruence on S is a Rees congruence. Furthermore, the ideals of a nil ordered semigroup S form a chain under inclusion if and only if S is a chain with respect to the divisibility ordering. The concept of a $\triangle$ -ordered semigroup is defined as follows: an ordered semigroup S is called a $\triangle$ -ordered semigroup if the complete congruences on S form a chain under inclusion. We finally prove that the nil ordered semigroups which are chains with respect to the divisibility ordering are $\triangle$ -ordered semigroups and that the homomorphic images of $\triangle$ -ordered semigroups are $\triangle$ -ordered semigroups as well. For the corresponding results on semigroups—without order—we refer to [11].

If $(S, \cdot, \leq)$ is an ordered semigroup and $H \subseteq S$, we denote by $[H]$ the subset of S defined as follows:

$$[H] := \{t \in S \mid t \leq h \text{ for some } h \in H\} \quad [4]. \quad (1.1)$$

A nonempty subset A of an ordered semigroup $(S, \cdot, \leq)$ is called a *left* (resp., *right*) *ideal* of S if the following hold.

- (1) $SA \subseteq A$ (resp., $AS \subseteq A$).
- (2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$ [4].

Equivalent definition.

- (1) $SA \subseteq A$ (resp., $AS \subseteq A$).
- (2) $(A) = A$.

A is called an *ideal* of S if it is both a left and a right ideal of S .

We denote by $L(A)$, $R(A)$, $I(A)$ the left ideal, right ideal, ideal of S , respectively, generated by A ($A \subseteq S$), that is the least, with respect to the inclusion relation, left ideal, right ideal, or ideal of S , respectively, containing A . As usual, $L(A)$ (resp., $R(A)$) coincides with the intersection of all left (resp., right) ideals of S containing A , $I(A)$ with the intersection of all ideals of S containing A . It is easy to check the following:

$$\begin{aligned} L(A) &= (A \cup SA), & R(A) &= (A \cup AS), \\ I(A) &= (A \cup SA \cup AS \cup SAS) \quad \forall A \subseteq S \text{ (cf. also [4])}. \end{aligned} \quad (1.2)$$

For $A = \{a\}$, we write $L(a)$, $R(a)$, $I(a)$ instead of $L(\{a\})$, $R(\{a\})$, $I(\{a\})$, respectively, and we call them the principal left ideal, principal right ideal, or principal ideal of S , respectively, generated by a ($a \in S$).

We have

$$\begin{aligned} L(a) &= \{t \in S \mid t \leq a \text{ or } t \leq ya \text{ for some } y \in S\}, \\ R(a) &= \{t \in S \mid t \leq a \text{ or } t \leq ax \text{ for some } x \in S\}, \\ I(a) &= \{t \in S \mid t \leq a \text{ or } t \leq ya \text{ for some } y \in S \text{ or } t \leq ax \text{ for some } x \in S \\ &\quad \text{or } t \leq xay \text{ for some } x, y \in S\}. \end{aligned} \quad (1.3)$$

It might be noted that if S is a *poe*-semigroup, that is, an ordered semigroup (*po*-semigroup [1]) having a greatest element “ e ” with respect to the order of S , then, for each $a \in S$, the principal ideals $L(a)$, $R(a)$, $I(a)$ are the following:

$$\begin{aligned} L(a) &= \{t \in S \mid t \leq a \text{ or } t \leq ea\}, \\ R(a) &= \{t \in S \mid t \leq a \text{ or } t \leq ae\}, \\ I(a) &= \{t \in S \mid t \leq a \text{ or } t \leq ea \text{ or } t \leq ae \text{ or } t \leq eae\}. \end{aligned} \quad (1.4)$$

The *zero* of an ordered semigroup $(S, \cdot, \leq)$ is an element of S , usually denoted by 0 , such that $0 \leq x$ and $0x = x0 = 0$ for all $x \in S$ [1]. A nonempty subset A of an ordered semigroup $(S, \cdot, \leq)$ is called a *subsemigroup* of S if $(A, \cdot)$ is a semigroup, that is, if $A^2 \subseteq A$.

Remark 1.1. If $(S, \cdot, \leq)$ is an ordered semigroup and A a left (resp., right) ideal of S , then $A^2 \subseteq SA \subseteq A$ (resp., $A^2 \subseteq AS \subseteq A$), so $(A, \cdot)$ is a subsemigroup of S . Then $(A, \cdot)$ is a semigroup, and $(A, \cdot, \leq)$ is an ordered semigroup.

Definition 1.2. Let $(S, \cdot, \leq)$ be an ordered semigroup and A an ideal of S . B is an *ideal* of A , if B is an ideal of the ordered semigroup $(A, \cdot, \leq)$.

Definition 1.3. An equivalence relation ρ on an ordered semigroup S is called a *left* (resp., *right*) *congruence* on S if $(a, b) \in \rho$ implies $(ca, cb) \in \rho$ (resp., $(ac, bc) \in \rho$) for every $c \in S$. An equivalence relation ρ on S is called a *congruence* on S if it is both a left and a right congruence on S .

Definition 1.4. Let $(S, \cdot, \leq)$ be an ordered semigroup. A congruence ρ on S is called *complete* if $a \leq b$ implies $(a, ab) \in \rho$.

The concept of complete semilattice congruences on an ordered semigroup has been introduced in [7]. A semilattice congruence on S is a congruence ρ on S such that $(a^2, a) \in \rho$ and $(ab, ba) \in \rho$ for each $a, b \in S$.

Definition 1.5. In an ordered semigroup S the *Green's relations* $\mathcal{L}$, $\mathcal{R}$, $\mathcal{I}$, $\mathcal{H}$, $\mathcal{D}$ are defined as follows: $a\mathcal{L}b$ if $L(a) = L(b)$, $a\mathcal{R}b$ if $R(a) = R(b)$, $a\mathcal{I}b$ if $I(a) = I(b)$, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$, that is, the smallest equivalence relation of S containing both $\mathcal{L}$ and $\mathcal{R}$.

It is easy to check that $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence on S . This has been proved in case of *poe*-semigroups in [3].

A mapping f of an ordered semigroup $(S, \cdot, \leq)$ into an ordered semigroup $(T, *, \preceq)$ is called a *homomorphism* if $f(ab) = f(a) * f(b)$ and $a \leq b$ implies $f(a) \preceq f(b)$ for each $a, b \in S$ [8].

2. The main results

LEMMA 2.1. *If $(S, \cdot, \leq)$ is an ordered semigroup, A an ideal of S , and B an ideal of A , then $[B] = B$.*

Proof. We have

$$[B] := \{t \in S \mid t \leq b \text{ for some } b \in B\}. \quad (2.1)$$

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If $t \in B$, then, since " $\leq$ " is reflexive on S , we have $t \in (B]$. If $t \in (B]$, then $t \leq b$ for some $b \in B \subseteq A$. Since $S \ni t \leq b \in A$ and A is an ideal of S , we have $t \in A$. Since $A \ni t \leq b \in B$ and B is an ideal of A , we have $t \in B$. Then $(B] \subseteq B$, and thus $(B] = B$. $\square$

LEMMA 2.2 [4]. *Let S be an ordered semigroup. Then the following hold.*

- (1) $A \subseteq (A]$ for all $A \subseteq S$.
- (2) If $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$.
- (3) $(A](B] \subseteq (AB]$ for all $A, B \subseteq S$.
- (4) $((A]) = (A]$ for all $A \subseteq S$.
- (5) If A, B are ideals of S , then $(AB]$, $A \cup B$ are ideals of S .
- (6) In particular, if S is an ordered semigroup, then the set $(SaS]$ is an ideal of S for all $a \in S$.

PROPOSITION 2.3. *Let $(S, \cdot, \leq)$ be an ordered semigroup, A an ideal of S , and B an ideal of A such that $B = (B^2]$. Then B is an ideal of S .*

Proof. First of all, by Lemma 2.1, we have $(B] = B$. Moreover,

$$\begin{aligned} BS &= (B^2]S = (B^2](S] \subseteq (B^2S] = (B(BS)] \\ &\subseteq (B(AS)] \subseteq (BA] \subseteq (B] = B \text{ (cf. also Lemma 2.2).} \end{aligned} \tag{2.2}$$

In a similar way we get $SB \subseteq B$. Since $BS \subseteq B$, $SB \subseteq B$ and $(B] = B$, B is an ideal of S . $\square$

Definition 2.4. An ordered semigroup S is called $\mathcal{F}$ -trivial if the Green's equivalence relation $\mathcal{F}$ on S is the equality relation on S , that is, if $I(a) = I(b)$ implies $a = b$.

Definition 2.5. Let S be an ordered semigroup and $a, b \in S$. Say that b divides a (or a is divisible by b), and write $b \mid a$, if $a \in I(b)$.

Definition 2.6. Let S be an ordered semigroup. A relation " $\preceq$ " is defined on S as follows:

$$\begin{aligned} \preceq &:= \{(a, b) \in S \times S \mid b \mid a\} \\ &= \{(a, b) \in S \times S \mid a \in I(b)\}. \end{aligned} \tag{2.3}$$

The relation " $\preceq$ " is called the *divisibility relation* on S .

PROPOSITION 2.7. *Let $(S, \cdot, \leq)$ be an ordered semigroup. If S is $\mathcal{F}$ -trivial, then the divisibility relation on S is an order on S . Conversely, if the divisibility relation " $\preceq$ " on S is symmetric, then S is $\mathcal{F}$ -trivial.*

Proof. ($\Rightarrow$) If $a \in S$, then $a \in I(a)$, so $(a, a) \in \preceq$. Let $(a, b) \in \preceq$ and $(b, a) \in \preceq$. Since $a \in I(b)$ and $b \in I(a)$, we have $I(a) \subseteq I(b)$ and $I(b) \subseteq I(a)$, that is, $I(a) = I(b)$. Since S is $\mathcal{F}$ -trivial, we have $a = b$. If $(a, b) \in \preceq$, $(b, c) \in \preceq$, then $a \in I(b)$, $b \in I(c)$, so $a \in I(b) \subseteq I(c)$, $a \in I(c)$, and $(a, c) \in \preceq$.

($\Leftarrow$) Let the divisibility relation " $\preceq$ " on S be symmetric and let $a, b \in S$ such that $I(a) = I(b)$. Since $a \in I(b)$ and $b \in I(a)$, we have $(a, b) \in \preceq$ and $(b, a) \in \preceq$, so $a = b$. $\square$

COROLLARY 2.8. *An ordered semigroup S is $\mathcal{F}$ -trivial if and only if the divisibility relation on S is an order on S .*

PROPOSITION 2.9. *Let S be an ordered semigroup. The following statements are equivalent.*

- (i) *S is $\mathcal{F}$ -trivial and the principal ideals of S form a chain under inclusion.*
- (ii) *The divisibility relation “ $\preceq$ ” on S is a chain.*

Proof. (i) $\Rightarrow$ (ii). By Proposition 2.7, the divisibility relation “ $\preceq$ ” is an order on S . Let now $a, b \in S$. By (i), we have $I(a) \subseteq I(b)$ or $I(b) \subseteq I(a)$. If $I(a) \subseteq I(b)$, then $a \in I(b)$, and $(a, b) \in \preceq$. If $I(b) \subseteq I(a)$, then $b \in I(a)$, and $(b, a) \in \preceq$. So “ $\preceq$ ” is a chain.

(ii) $\Rightarrow$ (i). Since “ $\preceq$ ” is an order on S , by Proposition 2.7, S is $\mathcal{F}$ -trivial. Let now $a, b \in S$. Since “ $\preceq$ ” is a chain, we have $(a, b) \in \preceq$ or $(b, a) \in \preceq$. If $(a, b) \in \preceq$, then $a \in I(b)$, so $I(a) \subseteq I(b)$. If $(b, a) \in \preceq$, then $b \in I(a)$, so $I(b) \subseteq I(a)$. Hence the principal ideals of S form a chain. $\square$

PROPOSITION 2.10. *Let S be an ordered semigroup. The principal ideals of S form a chain with respect to the inclusion relation if and only if the ideals of S do so.*

Proof. ($\Rightarrow$) Let A, B be ideals of S such that $A \not\subseteq B$. Then $B \subseteq A$. Indeed, let $b \in B$. We consider an element $a \in A$ such that $a \notin B$. By hypothesis, we have $I(a) \subseteq I(b)$ or $I(b) \subseteq I(a)$. If $I(a) \subseteq I(b)$, then $a \in I(b) \subseteq B$, so $a \in B$ which is impossible. Thus we have $I(b) \subseteq I(a)$, from which $b \in I(a) \subseteq A$.

The implication “ $\Leftarrow$ ” is obvious. $\square$

Definition 2.11 (cf. also [5, 10]). An element a of an ordered semigroup S with zero is called *nilpotent* if there exists an element $n \in \mathbb{N}$ such that $a^n = 0$. An ordered semigroup S having a zero element 0 is called *nil* if every element of S is nilpotent, that is, if for each $a \in S$, there exists $n \in \mathbb{N}$ such that $a^n = 0$. S is called *nilpotent* if there exists $n \in \mathbb{N}$ such that $S^n = \{0\}$. $\mathbb{N} = \{1, 2, \dots\}$ is the set of natural numbers.

If an ordered semigroup S is nilpotent, then it is clearly nil.

Notation 2.12. For convenience, we use the notation $S^1 := S \cup \{1\}$, where 1 is a symbol, such that $1a := a$, $a1 := a$ for each $a \in S$ and $11 := 1$.

THEOREM 2.13. *Every nil ordered semigroup is $\mathcal{F}$ -trivial.*

Proof. Let a, b be two elements of an ordered nil semigroup $(S, \cdot, \leq)$ such that $I(a) = I(b)$. Since $a \in I(b)$, we have $a \leq xby$ for some $x, y \in S^1$. Since $b \in I(a)$, we have $b \leq zat$ for some $z, t \in S^1$. Then

$$\begin{aligned} a &\leq x(zat)y = (xz)a(ty) \leq (xz)xzaty(ty) = (xz)^2a(ty)^2 \\ &\leq (xz)^2xzaty(ty)^2 = (xz)^3a(ty)^3 \leq (xz)^3xzaty(ty)^3 = (xz)^4a(ty)^4. \end{aligned} \quad (2.4)$$

Continuing this way, we have $a \leq (xz)^n a(ty)^n$ for each $n \in \mathbb{N}$. Since S is nil, there exists $n \in \mathbb{N}$ such that $(xz)^n = 0$. Then $a = 0$, and $b = 0$, so $a = b$. Thus S is $\mathcal{F}$ -trivial. $\square$

It might be noted that using the notation S^1 , we avoid to check each of the 16 (different) cases arising from $b \in I(a)$ and $a \in I(b)$, separately.

Definition 2.14 [9]. An ordered semigroup S having a zero element 0 is called *0-simple* if $S^2 \neq \{0\}$ and S and $\{0\}$ are the only ideals of S .

COROLLARY 2.15. *An ordered semigroup S cannot be 0-simple and nil (at the same time).*

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Proof. Suppose it is. If $S = \{0\}$, then $S^2 = \{0\}$ which is impossible. So there is at least one nonzero element a in S . Suppose b is another nonzero element of S . We have $I(a) \neq \{0\}$ and $I(b) \neq \{0\}$. Since S is 0-simple, the ideals S and $\{0\}$ are the only ideals of S . So we have $I(a) = S$ and $I(b) = S$. Since S is nil, by Theorem 2.13, it is $\mathcal{F}$ -trivial. Then, since $I(a) = I(b)$, we have $a = b$. Thus we have $S = \{0, a\}$ for some $a \in S$, $a \neq 0$. Then we have $a^2 = 0$. Indeed, if $a^2 = a$, then $a^n = a$ for each $n \in \mathbb{N}$. Since S is nil, there exists $n \in \mathbb{N}$ such that $a^n = 0$. Then we have $a = 0$ which is impossible. Since $S^2 = \{0, a^2\}$ and $a^2 = 0$, we have $S^2 = \{0\}$ which is impossible. $\square$

Notation 2.16. If S is an ordered semigroup and A an ideal of S , we denote by ρ_A the relation on S defined as follows:

$$\rho_A := \{(x, y) \in S \times S \mid x = y \text{ or } x, y \in A\}. \quad (2.5)$$

It is easy to check that the relation ρ_A is a congruence on S .

Definition 2.17. For an ordered semigroup S and an ideal A of S , the congruence ρ_A is called the *Rees congruence on S modulo A* . S is called the *ideal extension of A by S/ρ_A* .

THEOREM 2.18. *Let S be a nil ordered semigroup which is a chain with respect to the divisibility ordering. Then every complete congruence on S is a Rees congruence.*

Proof. Let ρ be a complete congruence on S . We define

$$A := \{a \in S \mid (a, 0) \in \rho\}. \quad (2.6)$$

The set A is an ideal of S . Indeed, let $a \in A$, $b \in S$. Since $a \in A$, we have $(a, 0) \in \rho$. Then $(ab, 0b) \in \rho$, $(ab, 0) \in \rho$, so $ab \in A$. Let $a \in S$, $b \in A$. Since $(b, 0) \in \rho$, we have $(ab, 0) \in \rho$, so $ab \in A$. Let $a \in A$ and $S \ni b \leq a$. Then $b \in A$. Indeed, since ρ is complete and $b \leq a$, we have $(b, ba) \in \rho$. Since $a \in A$, we have $(a, 0) \in \rho$, then $(ba, 0) \in \rho$. Since $(b, ba) \in \rho$ and $(ba, 0) \in \rho$, we have $(b, 0) \in \rho$, thus we have $b \in A$.

For the ideal A of S , we consider the Rees congruence

$$\rho_A := \{(x, y) \in S \times S \mid x = y \text{ or } x, y \in A\} \quad (2.7)$$

defined above. We have $\rho = \rho_A$. Indeed, let $(a, b) \in \rho$. Since $a, b \in S$ and S is a chain with respect to the divisibility ordering, we have $(a, b) \in \preceq$ or $(b, a) \in \preceq$, namely, $a \in I(b)$ or $b \in I(a)$. Suppose $a \in I(b)$. Then $a \leq xby$ for some $x, y \in S^1$. Then, since ρ is a complete congruence on S , we have $(a, axby) \in \rho$. Then $(a(xby), axby(xby)) \in \rho$, so $(a, a(xby)^2) \in \rho$. Then, since $(a(xby), a(xby)^3) \in \rho$, we have $(a, a(xby)^3) \in \rho$. Continuing this way, we have $(a, a(xby)^n) \in \rho$ for every $n \in \mathbb{N}$. Since S is nil, there exists $n \in \mathbb{N}$ such that $(xby)^n = 0$. Then we have $(a, 0) \in \rho$, and $a \in A$. Since $(b, a) \in \rho$ and $(a, 0) \in \rho$, we have $(b, 0) \in \rho$, and $b \in A$. Since $a, b \in A$, we have $(a, b) \in \rho_A$. If $b \in I(a)$, similarly, we get $a, b \in A$, so $(a, b) \in \rho_A$. Thus we have $\rho \subseteq \rho_A$.

Conversely, let $(a, b) \in \rho_A$. Then $a = b$ or $a, b \in A$. If $a = b$, then $(a, b) = (a, a) \in \rho$. If $a, b \in A$, then $(a, 0) \in \rho$ and $(b, 0) \in \rho$, from which $(a, b) \in \rho$. Thus $\rho_A \subseteq \rho$.

Hence, for the ideal A of S , we have $\rho = \rho_A$, and the proof is complete. $\square$

Definition 2.19. An ordered semigroup S is called a Δ -ordered semigroup if the complete congruences on S , with respect to the inclusion relation, form a chain.

PROPOSITION 2.20. Let S be a nil ordered semigroup. The ideals of S form a chain under inclusion if and only if S is a chain with respect to the divisibility ordering.

Proof. ($\Rightarrow$) By Proposition 2.10, the principal ideals of S form a chain under inclusion. Since S is nil, by Theorem 2.13, S is $\mathcal{J}$ -trivial. Then, by Proposition 2.9, S is a chain with respect to the divisibility ordering.

($\Leftarrow$) Let A, B be ideals of S such that $A \not\subseteq B$. Then $B \subseteq A$. Indeed, let $b \in B$, and $a \in A$, $a \notin B$. Since $a, b \in S$, by hypothesis, we have $(a, b) \in \preceq$ or $(b, a) \in \preceq$, that is, $a \in I(b)$ or $b \in I(a)$. Since $I(b) \subseteq B$ and $a \notin B$, we have $a \notin I(b)$. Thus we have $b \in I(a) \subseteq A$, and $b \in A$. $\square$

PROPOSITION 2.21. Let S be a nil ordered semigroup which is a chain with respect to the divisibility ordering. Then S is a Δ -ordered semigroup.

Proof. Let ρ, μ be two complete congruences on S . Since S is a nil ordered semigroup and it is a chain under the divisibility ordering, by Theorem 2.18, ρ and μ are Rees congruences on S . That is, there exist ideals A, B of S such that $\rho = \rho_A$ and $\mu = \rho_B$, where

$$\begin{aligned}\rho_A &:= \{(x, y) \in S \times S \mid x = y \text{ or } x, y \in A\}, \\ \rho_B &:= \{(x, y) \in S \times S \mid x = y \text{ or } x, y \in B\}.\end{aligned}\tag{2.8}$$

By Proposition 2.20, the ideals of S form a chain under inclusion, so we have $A \subseteq B$ or $B \subseteq A$. If $A \subseteq B$, then $\rho_A \subseteq \rho_B$. Indeed, if $(x, y) \in \rho_A$, then $x = y$ or $x, y \in A$, then $x = y$ or $x, y \in B$, which shows that $(x, y) \in \rho_B$. By symmetry, if $B \subseteq A$, then $\rho_B \subseteq \rho_A$. So we have $\rho \subseteq \mu$ or $\mu \subseteq \rho$ and the proof is complete. $\square$

THEOREM 2.22. Let $(S, \cdot, \leq)$ be a Δ -ordered semigroup, $(T, \cdot, \preceq)$ an ordered semigroup, and $f : S \rightarrow T$ a homomorphism and onto mapping. Then T is a Δ -ordered semigroup.

Proof. Let ρ, μ be two complete congruences on T . Let

$$\begin{aligned}\rho^* &:= \{(a, b) \in S \times S \mid (f(a), f(b)) \in \rho\}, \\ \mu^* &:= \{(a, b) \in S \times S \mid (f(a), f(b)) \in \mu\}.\end{aligned}\tag{2.9}$$

The relations ρ^* and μ^* are complete congruences on S .

In fact, it is easy to see that the relations ρ^* and μ^* are equivalence relations on S .

Let now $(a, b) \in \rho^*$ and $c \in S$. Then $(ac, bc) \in \rho^*$. Indeed, since $(a, b) \in \rho^*$, we have $a, b \in S$ and $(f(a), f(b)) \in \rho$. Since ρ is a right congruence on T , we have $(f(a)f(c), f(b)f(c)) \in \rho$. Then, since f is a homomorphism, we have $(f(ac), f(bc)) \in \rho$. Since $ac, bc \in S$ and $(f(ac), f(bc)) \in \rho$, we have $(ac, bc) \in \rho^*$. Similarly, ρ^* is a left congruence on S .

ρ^* is a complete congruence on S . Indeed, let $a, b \in S$, $a \leq b$. Since $a, b \in S$ and f is a homomorphism, we have $f(a) \preceq f(b)$. Then, since ρ is a complete congruence on T , we have $(f(a), f(a)f(b)) \in \rho$. Since f is a homomorphism, we have $(f(a), f(ab)) \in \rho$. Since $a, ab \in S$ and $(f(a), f(ab)) \in \rho$, we have $(a, ab) \in \rho^*$.

Since S is a $\triangle$ -ordered semigroup, we have $\rho^* \subseteq \mu^*$ or $\mu^* \subseteq \rho^*$. If $\rho^* \subseteq \mu^*$, then $\rho \subseteq \mu$. Indeed, let $(x, y) \in \rho \subseteq T \times T$. Since $x, y \in T$ and f is onto, there exist $a, b \in S$ such that $f(a) = x$, $f(b) = y$. Since $a, b \in S$ and $(f(a), f(b)) \in \rho$, we have $(a, b) \in \rho^* \subseteq \mu^*$. Then $(f(a), f(b)) \in \mu$, so $(x, y) \in \mu$. If $\mu^* \subseteq \rho^*$, by symmetry, we have $\mu \subseteq \rho$. So we have $\rho \subseteq \mu$ or $\mu \subseteq \rho$, and T is a $\triangle$ -ordered semigroup. $\square$

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