ON A CLASS OF SECOND-ORDER IMPULSIVE BOUNDARY VALUE PROBLEM AT RESONANCE

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Received 16 September 2004; Revised 4 December 2005; Accepted 18 December 2005

We consider the following impulsive boundary value problem, $x''(t) = f(t, x, x')$, $t \in J \setminus \{t_1, t_2, \dots, t_k\}$, $\Delta x(t_i) = I_i(x(t_i), x'(t_i))$, $\Delta x'(t_i) = J_i(x(t_i), x'(t_i))$, $i = 1, 2, \dots, k$, $x(0) = 0$, $x'(1) = \sum_{j=1}^{m-2} \alpha_j x'(\eta_j)$. By using the coincidence degree theory, a general theorem concerning the problem is given. Moreover, we get a concrete existence result which can be applied more conveniently than recent results. Our results extend some work concerning the usual m -point boundary value problem at resonance without impulses.

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1. Introduction

In the few past years, boundary value problems for impulsive differential equation have been studied (see [1, 5, 7]). They discussed the existence of solutions for first-order impulsive equations by the use of upper and lower solution methods. Dong [3] researched the periodic boundary value problem for second-order impulsive equations. Liu and Yu [6] considered the boundary value condition $x'(0) = 0$, $x(1) = \sum_{j=1}^{m-2} \alpha_j x(\eta_j)$ with $\sum_{j=1}^{m-2} \alpha_j = 1$ by making use of the coincidence degree which was developed by Gaines and Mawhin [4].

We are concerned with the m -point boundary value problem for the nonlinear impulsive differential equation:

$$\begin{aligned} x''(t) &= f(t, x(t), x'(t)), \quad t \in J', \\ \Delta x(t_i) &= I_i(x(t_i), x'(t_i)), \quad \Delta x'(t_i) = J_i(x(t_i), x'(t_i)), \quad i = 1, 2, \dots, k, \end{aligned} \tag{1.1}$$

associated with the boundary value condition

$$x(0) = 0, \quad x'(1) = \sum_{j=1}^{m-2} \alpha_j x'(\eta_j), \tag{1.2}$$

2 Second-order impulsive different equation BVP at resonance

where $J = [0, 1]$, $0 < t_1 < t_2 < \dots < t_k < 1$, $J' = J \setminus \{t_1, t_2, \dots, t_k\}$. $x \in R$, $f : J \times R \times R \rightarrow R$, $I_i : R \times R \rightarrow R$, $J_i : R \times R \rightarrow R$ are continuous, $0 < \eta_1 < \eta_2 < \dots < \eta_{m-2} < 1$, $\Delta x(t_i) = x(t_i + 0) - x(t_i)$, $\Delta x'(t_i) = x'(t_i + 0) - x'(t_i)$, $i = 1, 2, \dots, k$, $\sum_{j=1}^{m-2} \alpha_j = 1$, $\alpha_j > 0$, $j = 1, 2, \dots, m-2$. A map $x : J \rightarrow R$ is said to be solution of (1.1)–(1.2), if it satisfies

- (1) $x(t)$ is twice continuously differentiable for $t \in J'$, both $x(t+0)$ and $x(t-0)$ exist at $t = t_i$, and $x(t_i) = x(t_i - 0)$, $i = 1, 2, \dots, k$;
- (2) $x(t)$ satisfies the relations (1.1)–(1.2).

We will use the continuation theorem of coincidence degree [2] to show a general theorem for the existence of solutions to the problem (1.1)–(1.2) and then use it to get concrete existence conditions in Section 3. This paper is motivated by [2, 3, 6, 8, 9].

2. Preliminary lemmas

At first, we recall some notations and present a series of useful lemmas with respect to the problem (1.1)–(1.2) that is important in the proof of our results. Consider an operator equation

$$Lx = Nx, \quad (2.1)$$

where $L : \text{dom } L \cap X \rightarrow Z$ is a linear operator, $N : X \rightarrow Z$ is a nonlinear operator, X and Z are Banach spaces. If $\dim \text{Ker } L = \dim(Z/\text{Im } L) < \infty$, and $\text{Im } L$ is closed in Z , then L will be called a Fredholm mapping of index zero. And at the same time, there exist continuous projectors $P : X \rightarrow X$ and $Q : Z \rightarrow Z$ such that $\text{Im } P = \text{Ker } L$, $\text{Im } L = \text{Ker } Q$. It follows that $L|_{\text{dom } L \cap \text{Ker } P} : \text{dom } L \cap \text{Ker } P \rightarrow \text{Im } L$ is reversible. We denote the inverse of this map by K_p .

Let Ω be an open and bounded subset of X . The map N will be called L -compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q) : \overline{\Omega} \rightarrow X$ is compact. Since $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exists an isomorphism $\hat{f} : \text{Im } Q \rightarrow \text{Ker } L$.

LEMMA 2.1 (continuation theorem [4]). *Suppose that L is a Fredholm operator of index zero and N is L -compact on $\overline{\Omega}$, where Ω is an open bounded subset of X . If the following conditions are satisfied:*

- (i) *for each $\lambda \in (0, 1)$, every solution x of*

$$Lx = \lambda Nx \quad (2.2)$$

is such that $x \notin \partial\Omega$;

- (ii) *$QNx \neq 0$ for $x \in \partial\Omega \cap \text{Ker } L$, and $\deg(IQN, \Omega \cap \text{Ker } L, 0) \neq 0$, where $Q : Z \rightarrow Z$ is a continuous projector with $\text{Im } L = \text{Ker } Q$, $\hat{f} : Z/\text{Im } L \rightarrow \text{Ker } L$ is an isomorphism. Then the operator equation (2.1) has at least one solution in $\text{dom } L \cap \overline{\Omega}$.*

In the following, in order to obtain the existence theorem of (1.1)–(1.2) we first introduce the following:

- (i) $X = PC^1[J, R] = \{x : J \rightarrow R \mid x(t) \text{ is twice continuously differentiable for } t \in J', \text{ there exist } x'(t_i + 0), x'(t_i - 0) \text{ and } x(t_i) = x(t_i - 0), x'(t_i) = x'(t_i - 0), i = 1, 2, \dots, k \text{ and (1.2) is satisfied}\}$;

(ii) $Z = PC[J, R] \times R^{2k} = \{y : J \rightarrow R \mid y(t) \text{ is continuous for } t \in J', \text{ there exist } y(t_i - 0), y(t_i + 0), \text{ and } y(t_i - 0) = y(t_i), i = 1, 2, \dots, k\} \times R^{2k}$.

Let $\|x\|_\infty = \sup_{t \in J} |x(t)|$ for $x \in PC[J, R]$ and $x \in PC^1[J, R]$. $\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty\}$. And for every $z = (y, c) \in Z$, denote its norm by

$$\|z\| = \max\left\{\sup_{t \in J} |y(t)|, \|c\|\right\}. \quad (2.3)$$

We can prove that X and Z are Banach spaces. Let $\text{dom } L = \{x : J \rightarrow R \mid x(t) \text{ is twice differentiable for } t \in J'\} \cap X$,

$$\begin{aligned} L : \text{dom } L &\longrightarrow Z, x \longmapsto (x''(t), \Delta x(t_1), \dots, \Delta x(t_k), \Delta x'(t_1), \dots, \Delta x'(t_k)), \\ N : X &\longrightarrow Z, x \longmapsto (f(t, x(t), x'(t)), I_1(x(t_1), x'(t_1)), \dots, I_k(x(t_k), x'(t_k)), \\ &\quad J_1(x(t_1), x'(t_1)), \dots, J_k(x(t_k), x'(t_k))). \end{aligned} \quad (2.4)$$

Then problem (1.1)–(1.2) can be written as $Lx = Nx$, $x \in \text{dom } L$.

LEMMA 2.2. Assume that L is defined as above and $\sum_{j=1}^{m-2} \alpha_j = 1$. Then L is a Fredholm mapping of index zero. Furthermore, for the problem (1.1)–(1.2),

$$\text{Ker } L = \{x(t) \in X : x(t) = ct, c \in R\}, \quad (2.5)$$

$$\begin{aligned} \text{Im } L = \{ &(y, a_1, a_2, \dots, a_k, b_1, \dots, b_k) : x''(t) = y(t), \Delta x(t_i) = a_i, \Delta x'(t_i) = b_i, \\ &i = 1, 2, \dots, k, \text{ for some } x(t) \in \text{dom } L\} \end{aligned}$$

$$= \left\{ (y, a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k) \in PC[0, 1] \times R^{2k} : \sum_{j=1}^{m-2} \alpha_j \left[\int_{\eta_j}^1 y(s) ds + \sum_{t_i > \eta_j} b_i \right] = 0 \right\}. \quad (2.6)$$

Proof. Firstly, it is easily seen that (2.5) holds. Next we will show that (2.5) holds. Since problem

$$\begin{aligned} x''(t) &= y(t), \quad t \in J', \\ \Delta x(t_i) &= a_i, \quad \Delta x'(t_i) = b_i \end{aligned} \quad (2.7)$$

has solution $x(t)$ satisfying (1.2) if and only if

$$\sum_{j=1}^{m-2} \alpha_j \left[\int_{\eta_j}^1 y(s) ds + \sum_{t_i > \eta_j} b_i \right] = 0. \quad (2.8)$$

In fact, if (2.7) has solution $x(t)$ such that (1.2), then from (2.7) we have

$$x(t) = x(0) + x'(0)t + \int_0^t \int_0^s y(\tau) d\tau ds + \sum_{t_i < t} b_i(t - t_i) + \sum_{t > t_i} a_i. \quad (2.9)$$

Thus

$$x'(t) = x'(0) + \int_0^t y(s) ds + \sum_{t > t_i} b_i. \quad (2.10)$$

4 Second-order impulsive different equation BVP at resonance

In view of (1.2), we have

$$\begin{aligned} x(0) = 0, \quad x'(\eta_j) &= x'(0) + \int_0^{\eta_j} y(s)ds + \sum_{\eta_j > t_i} b_i, \\ 0 = x'(1) - \sum_{j=1}^{m-2} \alpha_j x'(\eta_j) &= \sum_{j=1}^{m-2} \alpha_j \left[\int_{\eta_j}^1 y(s)ds + \sum_{t_i > \eta_j} b_i \right]. \end{aligned} \quad (2.11)$$

Hence, (2.8) holds.

On the other hand, if (2.8) holds setting

$$x(t) = ct + \int_0^t \int_0^s y(\tau) d\tau ds + \sum_{t_i < t} b_i(t - t_i) + \sum_{t_i > t} a_i, \quad (2.12)$$

where $c \in R$ is an arbitrary constant, then it is clear that $x(t)$ is a solution of (2.7) and satisfies (1.2). Hence, (2.5) holds.

Take the projector $Q : Z \rightarrow Z$ as follows:

$$Q(y, a_1, \dots, a_k, b_1, \dots, b_k) = \left(\frac{2}{\sum_{j=1}^{m-2} \alpha_j (1 - \eta_j^2)} \sum_{j=1}^{m-2} \alpha_j \left[\int_{\eta_j}^1 y(\tau) d\tau + \sum_{t_i > \eta_j} b_i \right] \cdot t, 0, \dots, 0 \right) \quad (2.13)$$

and for $(y, a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k) \in Z$. Let

$$\begin{aligned} z &= (y_1, a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k) \\ &= (y, a_1, \dots, a_k, b_1, b_2, \dots, b_k) - Q(y, a_1, a_2, \dots, a_k, b_1, \dots, b_k), \end{aligned} \quad (2.14)$$

then $z \in \text{Im } L$. Thus, we have

$$\dim(Z/\text{Im } L) = \dim \text{Im } Q = 1 = \dim \text{Ker } L, \quad (2.15)$$

moreover by the Ascoli-Arzela theorem, L is a Fredholm mapping of index zero. $\square$

3. Main results

By applying Lemma 2.1, a general theorem for the existence of solutions to the problem (1.1)–(1.2) is obtained. And concrete existence conditions for the same problem are also obtained.

For any subset $G \subset R^2$, let

$$\begin{aligned} \Omega &= \{x \in X \mid (x(t), x'(t)) \in G, (x(t_i^+), x'(t_i^+)) \in G, \forall t \in J, i = 1, 2, \dots, k\}, \\ \Omega \cap \text{Ker } L &= \{x = ct \mid (ct, c) \in G, \forall t \in J\} = G_1, \quad \hat{G}_1 = \{c \in R : ct \in G_1\}. \end{aligned} \quad (3.1)$$

THEOREM 3.1. Assume that the following conditions are satisfied:

- (1) let $G \subset \mathbb{R}^2$ be an open bounded subset such that for every $\lambda \in (0, 1)$, each possible solution $x(t)$ of the auxiliary system

$$\begin{aligned} x''(t) &= \lambda f(t, x, x'), \quad t \in J', \\ \Delta x(t_i) &= \lambda I_i(x(t_i), x'(t_i)), \\ \Delta x'(t_i) &= \lambda J_i(x(t_i), x'(t_i)), \quad i = 1, 2, \dots, k, \\ x(0) &= 0, \quad x'(1) = \sum_{j=1}^{m-2} \alpha_j x'(\eta_j) \end{aligned} \quad (3.2)$$

satisfies $x \notin \partial\Omega$;

- (2) $h(c) \neq 0$, for $c \in \partial\hat{G}_1$, $\deg(h, \hat{G}_1, 0) \neq 0$, where h is defined by

$$h(c) = \frac{2}{\sum_{j=1}^{m-2} \alpha_j (1 - \eta_j^2)} \cdot \sum_{j=1}^{m-2} \alpha_j \left[\int_{\eta_j}^1 f(\tau, c\tau, c) d\tau + \sum_{t_i > \eta_j} J_i(ct_i, c) \right]. \quad (3.3)$$

Then the BVP (1.1)–(1.2) has at least one solution $x(t)$ satisfying $(x(t), x'(t)) \in G$, for $t \in J$.

Proof. By Lemma 2.2, we know that L is a Fredholm operator of index zero, and the problem (3.2) can be written as $Lx = \lambda Nx$. Set

$$\Omega = \{x \in X : (x(t), x'(t)) \in G, (x(t_i + 0), x'(t_i + 0)) \in G, \forall t \in J, i = 1, \dots, k\}. \quad (3.4)$$

Then Ω is open and bounded. To use Lemma 2.1, we show at first N is L -compact on $\overline{\Omega}$. Defining a projector

$$P : X \longrightarrow \text{Ker } L, \quad P(x(t)) = x'(0)t, \quad (3.5)$$

then $K_P : \text{Im } L \rightarrow \text{Ker } P \cap \text{dom } L$ can be written in

$$K_P z = K_P(y, a_1, \dots, a_k, b_1, \dots, b_k) = \int_0^t \int_0^s y(\tau) d\tau ds + \sum_{t_i < t} b_i(t - t_i) + \sum_{t_i < t} a_i. \quad (3.6)$$

6 Second-order impulsive different equation BVP at resonance

In fact, we have $K_p L = I - P$, thus for any $x \in \text{dom} L$, $K_p Lx = x - x'(0)t$, so (3.6) holds. Again from (2.13) and (3.6), we have

$$\begin{aligned} QNx &= \left(\frac{2}{\sum_{j=1}^{m-2} \alpha_j (1 - \eta_j^2)} \cdot \sum_{j=1}^{m-2} \alpha_j \left[\int_{\eta_j}^1 f(\tau, x(\tau), x'(\tau)) d\tau \right. \right. \\ &\quad \left. \left. + \sum_{t_i > \eta_j} J_i(x(t_i), x'(t_i)) \right] \cdot t, 0, \dots, 0 \right), \\ K_p(I - Q)Nx &= \int_0^t \int_0^s f(\tau, x(\tau), x'(\tau)) d\tau ds + \int_0^t \sum_{t_i < s} J_i(x(t_i), x'(t_i)) ds \\ &\quad + \sum_{t_i < t} I_i(x(t_i), x'(t_i)) - \int_0^t \int_0^s \frac{2}{\sum_{j=1}^{m-2} \alpha_j (1 - \eta_j^2)} \\ &\quad \cdot \sum_{j=1}^{m-2} \alpha_j \left[\int_{\eta_j}^1 f(\tau, x(\tau), x'(\tau)) d\tau + \sum_{t_i > \eta_j} J_i(x(t_i), x'(t_i)) \right] \cdot s ds dt. \end{aligned} \quad (3.7)$$

By using the Ascoli-Arzelà theorem, we can prove that $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \rightarrow X$ is compact, thus N is L -compact on $\overline{\Omega}$.

At last, we will prove that (i), (ii) of Lemma 2.1 are satisfied. Note that $x \in \partial\Omega$, if and only if $(x(t), x'(t)) \in \overline{G}$, for $t \in J$, and either $(x(s), x'(s)) \in \partial G$, for some $s \in J$, or $(x(t_{i_0} + 0), x'(t_{i_0} + 0)) \in \partial G$, for some $i_0 = \{1, 2, \dots, k\}$, then the assumption (i) follows from condition (1).

Let $\hat{f} : \text{Im } Q \rightarrow \text{Ker } L : (ct, 0, \dots, 0) \rightarrow ct$ be the isomorphism. Then

$$\hat{f}QNx = \frac{2}{\sum_{j=1}^{m-2} \alpha_j (1 - \eta_j^2)} \cdot \sum_{j=1}^{m-2} \alpha_j \left[\int_{\eta_j}^1 f(\tau, x(\tau), x'(\tau)) d\tau + \sum_{t_i > \eta_j} J_i(x(t_i), x'(t_i)) \right] \cdot t. \quad (3.8)$$

Take an isomorphism $g : G_1 \rightarrow R$, $g(ct) = c$,

$$\begin{aligned} h(c) &= g(\hat{f}QNg^{-1}c) = \frac{2}{\sum_{j=1}^{m-2} \alpha_j (1 - \eta_j^2)} \cdot \sum_{j=1}^{m-2} \alpha_j \left[\int_{\eta_j}^1 f(\tau, c\tau, c) d\tau + \sum_{t_i > \eta_j} J_i(ct_i, c) \right], \\ g(\Omega \cap \text{ker } L) &= \hat{G}_1, \end{aligned} \quad (3.9)$$

then

$$\deg \{\hat{f}QN, \Omega \cap \text{ker } L, 0\} = \deg \{g\hat{f}QNg^{-1}, g(\Omega \cap \text{ker } L), g(0)\} = \deg \{h, \hat{G}_1, 0\} \neq 0 \quad (3.10)$$

in view of (2), $h(c) \neq 0$, for $c \in \partial\hat{G}_1$, then $\hat{f}QNx \neq 0$, $x \in \partial\Omega \cap \text{Ker } L$, that is, condition (2) yields (ii) of Lemma 2.1, and the proof is finished. $\square$

Remark 3.2. A similar result is given in [3] for the periodic boundary value problems (see Theorem 3.1 therein). However, there is something wrong in its proof. For a subset $G \subset R^{2n}$, define

$$\Omega = \{x \in X : (x(t), x'(t)) \in G \text{ for } t \in [0, T]; (x(t_i + 0), x'(t_i + 0)) \in G, \text{ for } i = 1, 2, \dots, k\}. \quad (3.11)$$

Then $\Omega \cap \text{Ker } L = \{x \in R^n : (x, 0) \in G\}$, which is not the set $G_1 = \{x \in R^n : \text{there exists } y \in R \text{ such that } (x, y) \in G\}$ defined by the author.

For example, take $G = \{(x, y) \in R^2 : x^2 + (y - 1)^2 < 4\}$. We have

$$\begin{aligned} \Omega \cap \text{Ker } L &= \{x \in R : x^2 + 1 < 4\} = (-\sqrt{3}, \sqrt{3}), \\ G_1 &= \{x \in R : x^2 < 4\} = (-2, 2) \end{aligned} \quad (3.12)$$

since $\text{Ker } L = R$. So the proof of Theorem 1 in [3] is not correct. The same problem appears in the proof of Theorem 1 in [6].

Let

$$\begin{aligned} a_1 &= \max_{|y| \leq M, |x| \leq Mt_1} |I_1(x, y)|, \\ a_2 &= \max_{|y| \leq M, |x| \leq Mt_2 + a_1} |I_2(x, y)|, \\ a_3 &= \max_{|y| \leq M, |x| \leq Mt_3 + a_1 + a_2} |I_3(x, y)| \cdots, \\ a_k &= \max_{|y| \leq M, |x| \leq Mt_k + \sum_{l=1}^{k-1} a_l} |I_k(x, y)|. \end{aligned} \quad (3.13)$$

Since $x(t) = \int_0^t x'(s)ds + \sum_{t_i < t} I_i(x(t_i), x'(t_i))$, if $|x'(t)| \leq M$, $t \in J$, then

$$|x(t)| \leq M + \sum_{i=1}^k a_i := \widehat{M}. \quad (3.14)$$

THEOREM 3.3. *Let $f : J \times R \times R \rightarrow R$ be a continuous function and suppose that there exists a constant $M > 0$ such that*

$$yf(t, x, y) > 0, \quad yI_i(x, y) > 0, \quad (3.15)$$

for $|y| \geq M$, $(t, x) \in J \times [-\widehat{M}, \widehat{M}]$, $i = 1, 2, \dots, k$.

Then BVP(1.1)–(1.2) with $\sum_{j=1}^{m-2} \alpha_j = 1$ has at least one solution $x(t) \in PC^1[0, 1]$.

Proof. Let

$$\begin{aligned} f^*(t, x, y) &= \begin{cases} f(t, x, y), & |x| \leq \widehat{M}, \\ f(t, \widehat{M}, y), & x > \widehat{M}, \\ f(t, -\widehat{M}, y), & x < -\widehat{M}, \end{cases} \\ J_i^*(x, y) &= \begin{cases} J_i(x, y), & |x| \leq \widehat{M}, \\ J_i(\widehat{M}, y), & x > \widehat{M}, \\ J_i(-\widehat{M}, y), & x < -\widehat{M}. \end{cases} \end{aligned} \quad (3.16)$$

Suppose $x(t)$ is a solution to BVP (3.2). We show at first that $\|x'\| < M$, when $\lambda \in (0, 1)$. Otherwise, there is $t_0 \in [0, 1]$ such that $\|x'\| = |x'(t_0)| = \sup_{t \in J} |x'(t)| \geq M$.

Without loss of generality we suppose that $x'(t_0) \geq M$.

If $t_0 \notin \{t_i, i = 1, 2, \dots, k\} \cup \{0, 1\}$, then one has

$$x'(t_0) = \sup_{t \in J} x'(t) \geq M, \quad x''(t_0) \leq 0. \quad (3.17)$$

However, by condition (3.15), $x''(t_0) = \lambda f^*(t_0, x(t_0), x'(t_0)) > 0$, a contradiction.

If $t_0 \in \{t_i, i = 1, 2, \dots, k\}$, say $t_0 = t_i$, then $J_i^*(x(t_i), x'(t_i)) > 0$ and hence

$$x'(t_i^+) = x'(t_i) + \lambda J_i^*(x(t_i), x'(t_i)) > x'(t_i) \quad (3.18)$$

which contradicts the assumption $x'(t_i) = \sup_{t \in J} x'(t)$.

If $t_0 = t_i^+ := t_i + \epsilon$, then there is $\sigma \in (0, t_{i+1} - t_i)$, (if $i = k, t_{i+1}$ is replaced by 1), such that $x'(t) > M$, $t \in (t_i, t_i + \sigma)$. Since $x''(t) = \lambda f^*(t, x(t), x'(t))$, $t \in (t_i, t_i + \sigma)$, $x''(t_i^+) = \lambda f^*(t_i, x(t_i^+), x'(t_i^+)) > 0$, then

$$x'(t_i + \sigma) = x'(t_i^+) + \int_{t_i}^{t_i + \sigma} x''(s) ds > x'(t_i^+) \quad (3.19)$$

which contradicts $x'(t_i^+) = \sup_{t \in J} x'(t)$.

If $t_0 = 1$, since $\sum_{j=1}^{m-2} \alpha_j = 1$, $\alpha_j \in (0, 1)$, and $x'(1) = \sum_{j=1}^{m-2} \alpha_j x'(\eta_j)$ yield that

$$x'(1) = x'(\eta_1) = \dots = x'(\eta_{m-2}) = \sup_{t \in J} x'(t) \quad (3.20)$$

which is the case we have discussed in case 1.

If $t_0 = 0$, $x'(0) = \sup_{t \in J} |x(t)| \geq M$, then $x''(0) = \lambda f(0, x(0), x'(0)) > 0$. So there is a $\sigma > 0$ small enough, such that $x''(t) > 0$, $t \in (0, \sigma)$, which yields

$$x'(\sigma) = x'(0) + \int_0^\sigma x''(s) ds > x'(0), \quad (3.21)$$

a contraction.

So $\|x'\| < M$ holds for all cases.

Since $x(t) = \int_0^t x'(s)ds + \sum_{t_i < t} I_i(x(t_i), x'(t_i))$, and $|x'(t)| \leq M$, $t \in J$, then

$$|x(t)| \leq Mt + \sum_{i=1}^k a_i \leq M + \sum_{i=1}^k a_i = \widehat{M} \quad (3.22)$$

and in turn the prior bound is obtained. Let $\Omega = \{x \in X \mid \|x\|_X < \widehat{M} + 1\}$. We have $x \notin \partial\Omega$.

By the proof of Theorem 3.1, we know that $h(c) = g\widehat{f}QNg^{-1}c$,

$$h(c) = 0 \iff g\widehat{f}QNg^{-1}c = 0 \iff \widehat{f}QNct = 0 \iff QNct = 0 \iff Nct \in \text{Im}L, \quad (3.23)$$

one has $x \in \{x = ct : |ct| < \widehat{M} + 1, t \in J, |c| < \widehat{M} + 1\} = \{ct : |c| < \widehat{M} + 1\}$ and $\widehat{G}_1 = (-\widehat{M} - 1, \widehat{M} + 1)$. When $c = \widehat{M} + 1$ or $c = -(\widehat{M} + 1)$ by condition (3.15), it holds that

$$\text{sgnc} \cdot \sum_{j=1}^{m-2} \alpha_j \left[\int_{\eta_j}^1 f(\tau, c\tau, c) d\tau + \sum_{t_i > \eta_j} J_i(ct_i, c) \right] > 0, \quad (3.24)$$

$c \in \partial\widehat{G}_1 = \{-\widehat{M} - 1, \widehat{M} + 1\}$.

Obviously,

$$\begin{aligned} \text{sgnc} \cdot h(c) &= \text{sgnc} \cdot g\widehat{f}QNg^{-1}(ct) \\ &= \text{sgnc} \cdot \frac{2}{\sum_{j=1}^{m-2} \alpha_j (1 - \eta_j^2)} \cdot \sum_{j=1}^{m-2} \alpha_j \left[\int_{\eta_j}^1 f(\tau, c\tau, c) d\tau + \sum_{t_i > \eta_j} J_i(ct_i, c) \right] > 0 \end{aligned} \quad (3.25)$$

for $c \in \partial\widehat{G}_1 = \{-\widehat{M} - 1, \widehat{M} + 1\}$. Then

$$\deg \{g\widehat{f}QNg^{-1}, \widehat{G}_1, 0\} = \deg \{h, (-\widehat{M} - 1, \widehat{M} + 1), 0\} = 1. \quad (3.26)$$

Hence, the conditions of Theorem 3.1 are satisfied and the proof of Theorem 3.3 is completed. $\square$

Finally, we present an example to check our result.

Example 3.4. Consider the boundary value problem

$$\begin{aligned} x''(t) &= x'(t)[t^2 + \ln(2 + x^2(t)) + x'^2(t)] + 3t \sin(x(t) + 1), \quad t \in [0, 1], t \neq \frac{1}{2}, \\ \Delta x\left(\frac{1}{2}\right) &= \sin x\left(\frac{1}{2}\right), \quad \Delta x'\left(\frac{1}{2}\right) = x'\left(\frac{1}{2}\right) \left[1 + x'^2\left(\frac{1}{2}\right)\right] - \left[2 + \cos x\left(\frac{1}{2}\right)\right], \quad t = \frac{1}{2}, \\ x(0) &= 0, \quad x'(1) = x'\left(\frac{1}{3}\right), \end{aligned} \quad (3.27)$$

where $f(t, x, x') = x'(t)[t^2 + \ln(2 + x^2(t)) + x'^2(t)] + 3t \sin(x(t) + 1)$, $I(x, x') = \sin x(t)$, $J = x'(t)[1 + x'^2(t)] - [2 + \cos x(t)]$.

10 Second-order impulsive different equation BVP at resonance

In this example, we note that $t_k = 1/2$, $k = 1$, $\alpha = 1$, $\eta_j = 1/3$, $j = 1$.

We choose a constant $M > 0$ large enough, let

$$a = \max_{|x'| < M, |x| \leq Mt_1} |I(x, x')| = \max_{|x'| < (1/2)M, |x| \leq M} \left| \sin x \left(\frac{1}{2} \right) \right| = 1. \quad (3.28)$$

Since $x(t) = \int_0^t x'(s)ds + I(x, x') = \int_0^t x'(s)ds + \sin x(1/2)$,

$$|x(t)| \leq M + 1 := \widehat{M}. \quad (3.29)$$

When $|x'| \geq M$, $(t, x) \in J \times [-\widehat{M}, \widehat{M}]$, obviously

$$\begin{aligned} x' \cdot f(t, x, x') &= x'^2(t)[t^2 + \ln(2 + x^2(t)) + x'^2(t)] + x'(t)[3t \sin(x(t) + 1)] > 0, \\ x' \cdot J &= x'^2(t)[1 + x'^2(t)] - x'(t)[2 + \cos x(t)] > 0, \end{aligned} \quad (3.30)$$

that is to say, the condition of Theorem 3.3 is satisfied. The BVP (3.27) has at least one solution.

Acknowledgment

This work was supported by National Natural Science Foundation of China no. (10371006) and Science Project of Central University for Nationalities for Youth Teachers.

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