EXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR DELAYED PREDATOR-PREY PATCH SYSTEMS WITH STOCKING

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A sufficient condition is derived for the existence of positive periodic solutions for a delayed predator-prey patch system with stocking. Some known results are improved.

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1. Introduction

Predator-prey systems have been studied extensively. See, for instance, [1, 6, 8–10] and the references cited therein. Most of the previous papers focused on the predator-prey systems without stocking. Brauer and Soudack [2, 3] studied some predator-prey systems under constant rate stocking. To our knowledge, few papers have been published on the existence of positive periodic solutions for delayed predator-prey patch systems with periodic stocking.

In this paper, we investigate the following predator-prey system with stocking:

$$\begin{aligned}x_1'(t) &= x_1(t)(a_1(t) - b_1(t)x_1(t) - c(t)y(t)) + D_1(t)(x_2(t - \tau_1(t)) - x_1(t)) + S_1(t), \\x_2'(t) &= x_2(t)(a_2(t) - b_2(t)x_2(t)) + D_2(t)(x_1(t - \tau_2(t)) - x_2(t)) + S_2(t), \\y'(t) &= y(t)\left(-d(t) + p(t)x_1(t) - q(t)y(t) - \beta(t) \int_{-\tau}^0 k(s)y(t+s)ds\right) + S_3(t),\end{aligned}\tag{1.1}$$

with the initial conditions

$$\begin{aligned}x_1(s) &= \varphi_1(s) \geq 0, \quad s \in [-\sigma, 0], \quad \varphi_1(0) > 0, \\x_2(s) &= \varphi_2(s) \geq 0, \quad s \in [-\sigma, 0], \quad \varphi_2(0) > 0, \\y(s) &= \psi(s) \geq 0, \quad s \in [-\sigma, 0], \quad \psi(0) > 0,\end{aligned}\tag{1.2}$$

where x_1 and y are the population densities of prey species x and predator species y in patch 1, and x_2 is the density of species x in patch 2. Predator species y is confined to

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patch 1, while the prey species x can diffuse between two patches. $D_i(t)$ ($i = 1, 2$) are diffusion coefficients of species x . $S_i(t)$ ($i = 1, 2, 3$) denote the stocking rates. $\varphi_1(s)$, $\varphi_2(s)$, and $\psi(s)$ are continuous on $[-\sigma, 0]$, $\sigma = \max\{\tau, \sup_{t \in \mathbb{R}} \tau_1(t), \sup_{t \in \mathbb{R}} \tau_2(t)\}$. The delay τ_1 (τ_2) represents the time that species x migrates from patch 2 to patch 1 (patch 1 to patch 2).

When $S_i(t) \equiv 0$ ($i = 1, 2, 3$), $\tau_i \equiv 0$ ($i = 1, 2$), system (1.1) was considered by Zhang and Wang [15], Song and Chen [11], and Chen et al. [5].

The purpose of this paper is to derive a set of easily verifiable conditions for the existence of positive periodic solutions of system (1.1). The method in this paper is different from those of [4, 12–14].

2. Existence of positive periodic solutions

To show the existence of solutions to the considered problems, we will use an abstract theorem developed [7]. We first state this abstract theorem.

For a fixed $\sigma \geq 0$, let $\mathbb{C} := \mathbb{C}([-\sigma, 0]; \mathbb{R}^n)$. If $x \in \mathbb{C}([\gamma - \sigma, \gamma + \delta]; \mathbb{R}^n)$ for some $\delta > 0$ and $\gamma \in \mathbb{R}$, then $x_t \in \mathbb{C}$ for $t \in [\gamma, \gamma + \delta]$ is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-\sigma, 0]$. The supremum norm in $\mathbb{C}$ is denoted by $\|\cdot\|_c$, that is, $\|\phi\|_c = \max_{\theta \in [-\sigma, 0]} \|\phi(\theta)\|$ for $\phi \in \mathbb{C}$, where $\|\cdot\|$ denotes the norm in $\mathbb{R}^n$, and $\|u\| = \sum_{i=1}^n |u_i|$ for $u = (u_1, \dots, u_n) \in \mathbb{R}^n$.

We consider the following functional differential equation:

$$\frac{dx(t)}{dt} = f(t, x_t), \quad (2.1)$$

where $f : \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{R}^n$ is completely continuous, and there exists $T > 0$ such that for every $(t, \varphi) \in \mathbb{R} \times \mathbb{C}$, we have $f(t + T, \varphi) = f(t, \varphi)$.

The following lemma is a simple consequence of [7, Theorem 4.7.1].

LEMMA 2.1. *Suppose that there exists a constant $M > 0$ such that*

(i) *for any $\lambda \in (0, 1)$ and any T -periodic solution x of the system*

$$\frac{dx(t)}{dt} = \lambda f(t, x_t), \quad (2.2)$$

$\|x(t)\| < M$ for $t \in \mathbb{R}$;

(ii) *$g(u) := (1/T) \int_0^T f(s, \hat{u}) ds \neq 0$ for $u \in \partial B_M(\mathbb{R}^n)$, where $B_M(\mathbb{R}^n) = \{u \in \mathbb{R}^n : \|u\| < M\}$, and $\hat{u}$ denotes the constant mapping from $[-\sigma, 0]$ to $\mathbb{R}^n$ with the value $u \in \mathbb{R}^n$;*

(iii) *Brouwer degree $\deg(g, B_M(\mathbb{R}^n)) \neq 0$.*

Then there exists at least one T -periodic solution of the system

$$\frac{dx(t)}{dt} = f(t, x_t) \quad (2.3)$$

that satisfies $\sup_{t \in \mathbb{R}} \|x(t)\| < M$.

In the following, we set

$$\bar{g} = \frac{1}{T} \int_0^T g(t) dt, \quad g^l = \min_{t \in [0, T]} |g(t)|, \quad g^u = \max_{t \in [0, T]} |g(t)|, \quad (2.4)$$

where g is a continuous T -periodic function.

In system (1.1), we always assume the following.

- (H₁) $a_i(t)$, $b_i(t)$, $D_i(t)$ ($i = 1, 2$), $c(t)$, $d(t)$, $p(t)$, $q(t)$, and $\beta(t)$ are positive continuous T -periodic functions. $S_i(t)$ ($i = 1, 2, 3$), $\tau_i(t)$ ($i = 1, 2$) are nonnegative continuous T -periodic functions. $\tau'_i(t) < 1$ ($i = 1, 2$), $t \in \mathbb{R}$.
- (H₂) $k(s) \geq 0$ on $[-\tau, 0]$ ($0 \leq \tau < +\infty$); and $k(s)$ is a piecewise continuous and normalized function such that $\int_{-\tau}^0 k(s)ds = 1$.

Set

$$K = \frac{\bar{q} + \bar{\beta}}{\bar{p}},$$

$$K^* = \left(\frac{a_1 M_0 - D_1 M_0 + S_1}{b_1 M_0} \right)^l, \quad K_i^* = \left(\frac{a_i M_0 + S_i}{b_i M_0} \right)^l, \quad i = 1, 2,$$

$$M_0 = \max \left\{ \left(\frac{a_1 + \sqrt{a_1^2 + 4b_1 S_1}}{2b_1} \right)^u, \left(\frac{a_2 + \sqrt{a_2^2 + 4b_2 S_2}}{2b_2} \right)^u \right\},$$

$$m_0 = \min \left\{ \frac{(a_1/c)^l - \sqrt{(S_3/q)^u}}{b_1^u/c^l + (p/q)^u} \exp[-2T(\bar{D}_1 + \bar{b}_1 M_0 + \bar{c} \widetilde{M}_0)], \left(\frac{a_2 + \sqrt{a_2^2 + 4b_2 S_2}}{2b_2} \right)^l \right\},$$

$$\widetilde{M}_0 = \left(\frac{p M_0 + \sqrt{p^2 M_0^2 + 4q S_3}}{2q} \right)^u,$$

$$\widetilde{m}_0 = \min \left\{ \frac{K_1^* - \bar{d}/\bar{p}}{K + (c/b_1)^u}, \frac{K_2^* - \bar{d}/\bar{p}}{K}, \frac{K^* - \bar{d}/\bar{p}}{K + (c/b_1)^u} \right\} \exp[-2T(\bar{d} + \bar{q} \widetilde{M}_0 + \bar{\beta} \widetilde{M}_0)]. \quad (2.5)$$

THEOREM 2.2. *In addition to (H₁), (H₂), assume further that system (1.1) satisfies one of the following assumptions:*

$$(H_3) \quad (a_1/c)^l > \sqrt{(S_3/q)^u}, \quad K_i^* > \bar{d}/\bar{p} \quad (i = 1, 2);$$

$$(H_4) \quad (a_1/c)^l > \sqrt{(S_3/q)^u}, \quad K^* > \bar{d}/\bar{p}.$$

Then system (1.1) has at least one positive T -periodic solution, say $(x_1^(t), x_2^*(t), y^*(t))^T$ such that*

$$m_0 \leq x_i^*(t) \leq M_0 \quad (i = 1, 2), \quad \widetilde{m}_0 \leq y^*(t) \leq \widetilde{M}_0, \quad t \geq 0. \quad (2.6)$$

Proof. Consider the following system:

$$u'_1(t) = a_1(t) - D_1(t) - b_1(t)e^{u_1(t)} - c(t)e^{u_3(t)} + D_1(t)e^{u_2(t-\tau_1(t))-u_1(t)} + \frac{S_1(t)}{e^{u_1(t)}},$$

$$u'_2(t) = a_2(t) - D_2(t) - b_2(t)e^{u_2(t)} + D_2(t)e^{u_1(t-\tau_2(t))-u_2(t)} + \frac{S_2(t)}{e^{u_2(t)}}, \quad (2.7)$$

$$u'_3(t) = -d(t) + p(t)e^{u_1(t)} - q(t)e^{u_3(t)} - \beta(t) \int_{-\tau}^0 k(s)e^{u_3(t+s)} ds + \frac{S_3(t)}{e^{u_3(t)}},$$

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where $a_i(t)$, $b_i(t)$, $D_i(t)$ ($i = 1, 2$), $S_i(t)$ ($i = 1, 2, 3$), $c(t)$, $d(t)$, $p(t)$, $q(t)$, and $\beta(t)$ are the same as those in assumption (H₁), and τ , τ_i ($i = 1, 2$) and $k(s)$ are the same as those in assumption (H₂). We first show that system (2.7) has one T -periodic solution.

Let $\mathbb{C} := \mathbb{C}([- \sigma, 0]; \mathbb{R}^3)$. We define the following map:

$$\begin{aligned} f : \mathbb{R} \times \mathbb{C} &\longrightarrow \mathbb{R}^3, \quad f(t, \varphi) = (f_1(t, \varphi), f_2(t, \varphi), f_3(t, \varphi)), \quad \varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathbb{C}, \\ f_1(t, \varphi) &= a_1(t) - D_1(t) - b_1(t)e^{\varphi_1(0)} - c(t)e^{\varphi_3(0)} + D_1(t)e^{\varphi_2(-\tau_1(t)) - \varphi_1(0)} + \frac{S_1(t)}{e^{\varphi_1(0)}}, \\ f_2(t, \varphi) &= a_2(t) - D_2(t) - b_2(t)e^{\varphi_2(0)} + D_2(t)e^{\varphi_1(-\tau_2(t)) - \varphi_2(0)} + \frac{S_2(t)}{e^{\varphi_2(0)}}, \\ f_3(t, \varphi) &= -d(t) + p(t)e^{\varphi_1(0)} - q(t)e^{\varphi_3(0)} - \beta(t) \int_{-\tau}^0 k(s)e^{\varphi_3(s)} ds + \frac{S_3(t)}{e^{\varphi_3(0)}}. \end{aligned} \quad (2.8)$$

Clearly, $f : \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{R}^3$ is completely continuous. Now, the system (2.7) becomes

$$\frac{du(t)}{dt} = f(t, u_t). \quad (2.9)$$

Corresponding to

$$\frac{du(t)}{dt} = \lambda f(t, u_t), \quad \lambda \in (0, 1), \quad (2.10)$$

we have

$$\begin{aligned} u'_1(t) &= \lambda \left[a_1(t) - D_1(t) - b_1(t)e^{u_1(t)} - c(t)e^{u_3(t)} + D_1(t)e^{u_2(t-\tau_1(t)) - u_1(t)} + \frac{S_1(t)}{e^{u_1(t)}} \right], \\ u'_2(t) &= \lambda \left[a_2(t) - D_2(t) - b_2(t)e^{u_2(t)} + D_2(t)e^{u_1(t-\tau_2(t)) - u_2(t)} + \frac{S_2(t)}{e^{u_2(t)}} \right], \\ u'_3(t) &= \lambda \left[-d(t) + p(t)e^{u_1(t)} - q(t)e^{u_3(t)} - \beta(t) \int_{-\tau}^0 k(s)e^{u_3(t+s)} ds + \frac{S_3(t)}{e^{u_3(t)}} \right]. \end{aligned} \quad (2.11)$$

Suppose that $(u_1(t), u_2(t), u_3(t))^T$ is a T -periodic solution of system (2.11) for some $\lambda \in (0, 1)$. Choose $t_i^M, t_i^m \in [0, T]$, $i = 1, 2, 3$, such that

$$u_i(t_i^M) = \max_{t \in [0, T]} u_i(t), \quad u_i(t_i^m) = \min_{t \in [0, T]} u_i(t), \quad i = 1, 2, 3. \quad (2.12)$$

Then, it is clear that

$$u'_i(t_i^M) = 0, \quad u'_i(t_i^m) = 0, \quad i = 1, 2, 3. \quad (2.13)$$

From this and system (2.11), we obtain that

$$a_1(t_1^M) - D_1(t_1^M) - b_1(t_1^M)e^{u_1(t_1^M)} - c(t_1^M)e^{u_3(t_1^M)} + D_1(t_1^M)e^{u_2(t_1^M - \tau_1(t_1^M)) - u_1(t_1^M)} + \frac{S_1(t_1^M)}{e^{u_1(t_1^M)}} = 0, \quad (2.14)$$

$$a_2(t_2^M) - D_2(t_2^M) - b_2(t_2^M)e^{u_2(t_2^M)} + D_2(t_2^M)e^{u_1(t_2^M - \tau_2(t_2^M)) - u_2(t_2^M)} + \frac{S_2(t_2^M)}{e^{u_2(t_2^M)}} = 0, \quad (2.15)$$

$$-d(t_3^M) + p(t_3^M)e^{u_1(t_3^M)} - q(t_3^M)e^{u_3(t_3^M)} - \beta(t_3^M) \int_{-\tau}^0 k(s)e^{u_3(t_3^M+s)} ds + \frac{S_3(t_3^M)}{e^{u_3(t_3^M)}} = 0, \quad (2.16)$$

$$a_1(t_1^m) - D_1(t_1^m) - b_1(t_1^m)e^{u_1(t_1^m)} - c(t_1^m)e^{u_3(t_1^m)} + D_1(t_1^m)e^{u_2(t_1^m - \tau_1(t_1^m)) - u_1(t_1^m)} + \frac{S_1(t_1^m)}{e^{u_1(t_1^m)}} = 0, \quad (2.17)$$

$$a_2(t_2^m) - D_2(t_2^m) - b_2(t_2^m)e^{u_2(t_2^m)} + D_2(t_2^m)e^{u_1(t_2^m - \tau_2(t_2^m)) - u_2(t_2^m)} + \frac{S_2(t_2^m)}{e^{u_2(t_2^m)}} = 0. \quad (2.18)$$

Next we make the following claims.

Claim 1. For $u_i(t_i^M)$ ($i = 1, 2$), one of the following cases holds:

$$u_2(t_2^M) \leq u_1(t_1^M) \leq M_1^* \leq M_1, \quad (2.19)$$

$$u_1(t_1^M) < u_2(t_2^M) \leq M_2^* \leq M_1, \quad (2.20)$$

where $M_1 := \max\{M_1^*, M_2^*\}$, $M_j^* := \ln((a_j + \sqrt{a_j^2 + 4b_j S_j})/2b_j)^u$, $j = 1, 2$.

There are two cases to consider.

Case 1. Assume that $u_1(t_1^M) \geq u_2(t_2^M)$; then $u_1(t_1^M) \geq u_2(t_1^M - \tau_1(t_1^M))$.

From this and (2.14), we have

$$b_1(t_1^M)e^{u_1(t_1^M)} \leq a_1(t_1^M) + \frac{S_1(t_1^M)}{e^{u_1(t_1^M)}}. \quad (2.21)$$

That is,

$$b_1(t_1^M)e^{2u_1(t_1^M)} - a_1(t_1^M)e^{u_1(t_1^M)} - S_1(t_1^M) \leq 0. \quad (2.22)$$

Therefore,

$$e^{u_1(t_1^M)} \leq \frac{a_1(t_1^M) + \sqrt{a_1^2(t_1^M) + 4b_1(t_1^M)S_1(t_1^M)}}{2b_1(t_1^M)} \leq \left(\frac{a_1 + \sqrt{a_1^2 + 4b_1 S_1}}{2b_1} \right)^u. \quad (2.23)$$

Hence,

$$u_2(t_2^M) \leq u_1(t_1^M) \leq \ln \left(\frac{a_1 + \sqrt{a_1^2 + 4b_1 S_1}}{2b_1} \right)^u. \quad (2.24)$$

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Case 2. Assume that $u_1(t_1^M) < u_2(t_2^M)$; then $u_1(t_2^M - \tau_2(t_2^M)) < u_2(t_2^M)$.

From this and (2.15), we have

$$b_2(t_2^M)e^{u_2(t_2^M)} \leq a_2(t_2^M) + \frac{S_2(t_2^M)}{e^{u_2(t_2^M)}}. \quad (2.25)$$

By a similar argument to Case 1, we have

$$u_1(t_1^M) < u_2(t_2^M) \leq \ln \left(\frac{a_2 + \sqrt{a_2^2 + 4b_2S_2}}{2b_2} \right)^u. \quad (2.26)$$

It follows from (2.24) and (2.26) that Claim 1 holds.

Claim 2.

$$u_3(t_3^M) \leq \ln \left(\frac{pM_0 + \sqrt{p^2M_0^2 + 4qS_3}}{2q} \right)^u := M_2, \quad (2.27)$$

where $M_0 = e^{M_1}$.

By (2.16), we have

$$q(t_3^M)e^{u_3(t_3^M)} \leq p(t_3^M)e^{u_1(t_3^M)} + \frac{S_3(t_3^M)}{e^{u_3(t_3^M)}} \leq p(t_3^M)e^{u_1(t_1^M)} + \frac{S_3(t_3^M)}{e^{u_3(t_3^M)}}. \quad (2.28)$$

That is,

$$q(t_3^M)e^{2u_3(t_3^M)} - p(t_3^M)e^{u_1(t_1^M)}e^{u_3(t_3^M)} - S_3(t_3^M) \leq 0. \quad (2.29)$$

Therefore,

$$e^{u_3(t_3^M)} \leq \frac{p(t_3^M)e^{u_1(t_1^M)} + \sqrt{p^2(t_3^M)e^{2u_1(t_1^M)} + 4q(t_3^M)S_3(t_3^M)}}{2q(t_3^M)}, \quad (2.30)$$

which implies that Claim 2 holds.

Claim 3. For $u_i(t_i^m)$ ($i = 1, 2$), one of the following cases holds:

$$\begin{aligned} m_1 &\leq m_1^* - 2T(\bar{D}_1 + \bar{b}_1M_0 + \bar{c}\widetilde{M}_0) \leq u_1(t_1^m) \leq u_2(t_2^m), \\ m_1 &\leq m_2^* \leq u_2(t_2^m) < u_1(t_1^m), \end{aligned} \quad (2.31)$$

where

$$\begin{aligned} m_1 &:= \min \{m_1^* - 2T(\bar{D}_1 + \bar{b}_1M_0 + \bar{c}\widetilde{M}_0), m_2^*\}, \\ m_1^* &:= \ln \frac{(a_1/c)^l - \sqrt{(S_3/q)^u}}{b_1^u/c^l + (p/q)^u}, \\ m_2^* &:= \ln \left(\frac{a_2 + \sqrt{a_2^2 + 4b_2S_2}}{2b_2} \right)^l. \end{aligned} \quad (2.32)$$

There are two cases to consider.

Case 1. Assume that $u_1(t_1^m) \leq u_2(t_2^m)$; then $u_1(t_1^m) \leq u_2(t_1^m - \tau_1(t_1^m))$.

From this and (2.17), we have

$$a_1(t_1^m) \leq b_1(t_1^m)e^{u_1(t_1^m)} + c(t_1^m)e^{u_3(t_1^m)} \leq b_1(t_1^m)e^{u_1(t_1^M)} + c(t_1^m)e^{u_3(t_3^M)}. \quad (2.33)$$

From (2.30), by using the inequality

$$(a+b)^{1/2} < a^{1/2} + b^{1/2}, \quad a > 0, b > 0, \quad (2.34)$$

we have

$$e^{u_3(t_3^M)} < \frac{p(t_3^M)e^{u_1(t_1^M)} + \sqrt{q(t_3^M)S_3(t_3^M)}}{q(t_3^M)}. \quad (2.35)$$

From this and (2.33), we have

$$a_1(t_1^m) \leq \left[b_1(t_1^m) + \frac{c(t_1^m)p(t_3^M)}{q(t_3^M)} \right] e^{u_1(t_1^M)} + c(t_1^m) \sqrt{\frac{S_3(t_3^M)}{q(t_3^M)}}, \quad (2.36)$$

which implies

$$\left(\frac{a_1}{c} \right)^l \leq \left[\frac{b_1^u}{c^l} + \left(\frac{p}{q} \right)^u \right] e^{u_1(t_1^M)} + \sqrt{\left(\frac{S_3}{q} \right)^u}. \quad (2.37)$$

That is,

$$u_1(t_1^M) \geq \ln \frac{(a_1/c)^l - \sqrt{(S_3/q)^u}}{b_1^u/c^l + (p/q)^u} := m_1^*. \quad (2.38)$$

From the first equation of system (2.11), we obtain that

$$\begin{aligned} & \int_0^T a_1(t)dt + \int_0^T D_1(t)e^{u_2(t-\tau_1(t))-u_1(t)}dt + \int_0^T \frac{S_1(t)}{e^{u_1(t)}}dt \\ &= \int_0^T D_1(t)dt + \int_0^T b_1(t)e^{u_1(t)}dt + \int_0^T c(t)e^{u_3(t)}dt, \\ & \int_0^T |u_1'(t)|dt < \int_0^T a_1(t)dt + \int_0^T D_1(t)e^{u_2(t-\tau_1(t))-u_1(t)}dt + \int_0^T \frac{S_1(t)}{e^{u_1(t)}}dt \\ & \quad + \int_0^T D_1(t)dt + \int_0^T b_1(t)e^{u_1(t)}dt + \int_0^T c(t)e^{u_3(t)}dt. \end{aligned} \quad (2.39)$$

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It follows that

$$\begin{aligned}
 \int_0^T |u'_1(t)| dt &< 2 \left[\int_0^T D_1(t) dt + \int_0^T b_1(t) e^{u_1(t)} dt + \int_0^T c(t) e^{u_3(t)} dt \right] \\
 &\leq 2 \left[\int_0^T D_1(t) dt + e^{M_1} \int_0^T b_1(t) dt + e^{M_2} \int_0^T c(t) dt \right] \\
 &= 2T(\bar{D}_1 + \bar{b}_1 M_0 + \bar{c} \widetilde{M}_0).
 \end{aligned} \tag{2.40}$$

From (2.38) and (2.40), we have

$$u_1(t_1^m) \geq u_1(t_1^M) - \int_0^T |u'_1(t)| dt \geq m_1^* - 2T(\bar{D}_1 + \bar{b}_1 M_0 + \bar{c} \widetilde{M}_0). \tag{2.41}$$

Case 2. Assume that $u_1(t_1^m) > u_2(t_2^m)$; then $u_1(t_2^m - \tau_2(t_2^m)) > u_2(t_2^m)$.

From this and (2.18), we have

$$b_2(t_2^m) e^{u_2(t_2^m)} \geq a_2(t_2^m) + \frac{S_2(t_2^m)}{e^{u_2(t_2^m)}}, \tag{2.42}$$

which implies

$$e^{u_2(t_2^m)} \geq \frac{a_2(t_2^m) + \sqrt{a_2^2(t_2^m) + 4b_2(t_2^m)S_2(t_2^m)}}{2b_2(t_2^m)}. \tag{2.43}$$

That is,

$$u_2(t_2^m) \geq \ln \left(\frac{a_2 + \sqrt{a_2^2 + 4b_2 S_2}}{2b_2} \right)^l := m_2^*. \tag{2.44}$$

It follows from (2.41) and (2.44) that Claim 3 holds.

Claim 4.

$$u_3(t_3^m) \geq \min \{m_3^*, m_4^*, m_5^*\} - 2T(\bar{d} + \bar{q} \widetilde{M}_0 + \bar{\beta} \widetilde{M}_0) := m_2, \tag{2.45}$$

where

$$m_3^* = \ln \frac{K_1^* - \bar{d}/\bar{p}}{K + (c/b_1)^u}, \quad m_4^* = \ln \frac{K_2^* - \bar{d}/\bar{p}}{K}, \quad m_5^* = \ln \frac{K^* - \bar{d}/\bar{p}}{K + (c/b_1)^u}. \tag{2.46}$$

From the third equation of (2.11), we obtain

$$\begin{aligned}
 \int_0^T p(t) e^{u_1(t)} dt + \int_0^T \frac{S_3(t)}{e^{u_3(t)}} dt &= \int_0^T d(t) dt + \int_0^T q(t) e^{u_3(t)} dt + \int_0^T \beta(t) \int_{-\tau}^0 k(s) e^{u_3(t+s)} ds dt, \\
 \int_0^T |u'_3(t)| dt &< \int_0^T p(t) e^{u_1(t)} dt + \int_0^T \frac{S_3(t)}{e^{u_3(t)}} dt + \int_0^T d(t) dt \\
 &\quad + \int_0^T q(t) e^{u_3(t)} dt + \int_0^T \beta(t) \int_{-\tau}^0 k(s) e^{u_3(t+s)} ds dt.
 \end{aligned} \tag{2.47}$$

It follows that

$$\begin{aligned}
 \int_0^T |u'_3(t)| dt &< 2 \left[\int_0^T d(t) dt + \int_0^T q(t) e^{u_3(t)} dt + \int_0^T \beta(t) \int_{-\tau}^0 k(s) e^{u_3(t+s)} ds dt \right] \\
 &\leq 2 \left[\int_0^T d(t) dt + e^{M_2} \int_0^T q(t) dt + e^{M_2} \int_0^T \beta(t) dt \right] \\
 &= 2T(\bar{d} + \bar{q}\widetilde{M}_0 + \bar{\beta}\widetilde{M}_0),
 \end{aligned} \tag{2.48}$$

$$[\bar{q} + \bar{\beta}]e^{u_3(t_3^M)} \geq \bar{p}e^{u_1(t_1^m)} - \bar{d}. \tag{2.49}$$

There are two cases to consider.

Case 1. Assume that the assumption (H₃) holds.

If $u_1(t_1^m) \leq u_2(t_2^m)$, by (2.17), we have

$$\begin{aligned}
 e^{u_1(t_1^m)} &\geq \frac{a_1(t_1^m) - c(t_1^m)e^{u_3(t_1^m)}}{b_1(t_1^m)} + \frac{S_1(t_1^m)}{b_1(t_1^m)e^{u_1(t_1^m)}} \\
 &\geq \frac{a_1(t_1^m) - c(t_1^m)e^{u_3(t_3^M)}}{b_1(t_1^m)} + \frac{S_1(t_1^m)}{b_1(t_1^m)e^{M_1}}.
 \end{aligned} \tag{2.50}$$

Substituting this into (2.49) gives

$$[\bar{q} + \bar{\beta}]e^{u_3(t_3^M)} \geq \frac{\bar{p}a_1(t_1^m)}{b_1(t_1^m)} - \frac{\bar{p}c(t_1^m)e^{u_3(t_3^M)}}{b_1(t_1^m)} + \frac{\bar{p}S_1(t_1^m)}{b_1(t_1^m)e^{M_1}} - \bar{d}, \tag{2.51}$$

which implies

$$\left[\frac{\bar{q}}{\bar{p}} + \frac{\bar{\beta}}{\bar{p}} + \frac{c(t_1^m)}{b_1(t_1^m)} \right] e^{u_3(t_3^M)} \geq \frac{a_1(t_1^m)}{b_1(t_1^m)} + \frac{S_1(t_1^m)}{b_1(t_1^m)e^{M_1}} - \frac{\bar{d}}{\bar{p}}. \tag{2.52}$$

Therefore,

$$\left[K + \left(\frac{c}{b_1} \right)^u \right] e^{u_3(t_3^M)} \geq K_1^* - \frac{\bar{d}}{\bar{p}}. \tag{2.53}$$

That is,

$$u_3(t_3^M) \geq \ln \frac{K_1^* - \bar{d}/\bar{p}}{K + (c/b_1)^u} := m_3^*. \tag{2.54}$$

It follows from (2.48) and (2.54) that

$$u_3(t_3^m) \geq u_3(t_3^M) - \int_0^T |u'_3(t)| dt \geq m_3^* - 2T(\bar{d} + \bar{q}\widetilde{M}_0 + \bar{\beta}\widetilde{M}_0). \tag{2.55}$$

If $u_1(t_1^m) > u_2(t_2^m)$, by (2.42), (2.49), and (2.19), we have

$$[\bar{q} + \bar{\beta}]e^{u_3(t_3^M)} \geq \bar{p}e^{u_2(t_2^m)} - \bar{d} \geq \frac{\bar{p}[a_2(t_2^m) + S_2(t_2^m)e^{-M_1}]}{b_2(t_2^m)} - \bar{d}, \tag{2.56}$$

which implies

$$\left[\frac{\bar{q}}{\bar{p}} + \frac{\bar{\beta}}{\bar{p}} \right] e^{u_3(t_3^M)} \geq \frac{a_2(t_2^m) + S_2(t_2^m)e^{-M_1}}{b_2(t_2^m)} - \frac{\bar{d}}{\bar{p}}. \quad (2.57)$$

Therefore,

$$K e^{u_3(t_3^M)} \geq K_2^* - \frac{\bar{d}}{\bar{p}}. \quad (2.58)$$

That is,

$$u_3(t_3^M) \geq \ln \frac{K_2^* - \bar{d}/\bar{p}}{K} := m_4^*. \quad (2.59)$$

From (2.48) and (2.59), we have

$$u_3(t_3^m) \geq u_3(t_3^M) - \int_0^T |u_3'(t)| dt \geq m_4^* - 2T(\bar{d} + \bar{q}\widetilde{M}_0 + \bar{\beta}\widetilde{M}_0). \quad (2.60)$$

Case 2. Assume that the assumption (H₄) holds.

From (2.17), we have

$$\begin{aligned} b_1(t_1^m) e^{u_1(t_1^m)} &\geq a_1(t_1^m) - D_1(t_1^m) - c(t_1^m) e^{u_3(t_1^m)} + \frac{S_1(t_1^m)}{e^{u_1(t_1^m)}} \\ &\geq a_1(t_1^m) - D_1(t_1^m) - c(t_1^m) e^{u_3(t_3^M)} + \frac{S_1(t_1^m)}{e^{M_1}}. \end{aligned} \quad (2.61)$$

Therefore,

$$e^{u_1(t_1^m)} \geq \frac{a_1(t_1^m) - D_1(t_1^m) - c(t_1^m) e^{u_3(t_3^M)} + S_1(t_1^m) e^{-M_1}}{b_1(t_1^m)}. \quad (2.62)$$

Substituting this into (2.49) gives

$$[\bar{q} + \bar{\beta}] e^{u_3(t_3^M)} \geq \frac{\bar{p}[a_1(t_1^m) - D_1(t_1^m)]}{b_1(t_1^m)} - \frac{\bar{p}c(t_1^m) e^{u_3(t_3^M)}}{b_1(t_1^m)} + \frac{\bar{p}S_1(t_1^m)}{b_1(t_1^m) e^{M_1}} - \bar{d}, \quad (2.63)$$

which implies

$$\left[\frac{\bar{q}}{\bar{p}} + \frac{\bar{\beta}}{\bar{p}} + \frac{c(t_1^m)}{b_1(t_1^m)} \right] e^{u_3(t_3^M)} \geq \frac{a_1(t_1^m) - D_1(t_1^m) + S_1(t_1^m) e^{-M_1}}{b_1(t_1^m)} - \frac{\bar{d}}{\bar{p}}. \quad (2.64)$$

Therefore,

$$\left[K + \left(\frac{c}{b_1} \right)^u \right] e^{u_3(t_3^M)} \geq K^* - \frac{\bar{d}}{\bar{p}}. \quad (2.65)$$

That is,

$$u_3(t_3^M) \geq \ln \frac{K^* - \bar{d}/\bar{p}}{K + (c/b_1)^u} := m_5^*. \quad (2.66)$$

It follows from (2.48) and (2.66) that

$$u_3(t_3^m) \geq u_3(t_3^M) - \int_0^T |u_3'(t)| dt \geq m_5^* - 2T(\bar{d} + \bar{q}\widetilde{M}_0 + \bar{\beta}\widetilde{M}_0). \quad (2.67)$$

It follows from (2.55), (2.60), and (2.67) that Claim 4 holds.

Clearly, one of the following inequalities holds:

- (i) $M_1^* > m_2^*$,
- (ii) $M_1^* \leq m_2^*$.

Since $m_1^* < M_1^*$ and $m_2^* \leq M_2^*$, (ii) implies $M_2^* > m_1^*$. Thus, according to Claims 1–3, one of the following four cases must hold:

$$(P_1) \quad m_1 \leq m_1^* - 2T(\bar{D}_1 + \bar{b}_1 M_0 + \bar{c}\widetilde{M}_0) \leq u_1(t_1^m) \leq u_2(t_2^m), u_2(t_2^M) \leq u_1(t_1^M) \leq M_1^* \leq M_1;$$

$$(P_2) \quad m_1 \leq m_2^* \leq u_2(t_2^m) < u_1(t_1^m), u_2(t_2^M) \leq u_1(t_1^M) \leq M_1^* \leq M_1;$$

$$(P_3) \quad m_1 \leq m_1^* - 2T(\bar{D}_1 + \bar{b}_1 M_0 + \bar{c}\widetilde{M}_0) \leq u_1(t_1^m) \leq u_2(t_2^m), u_1(t_1^M) < u_2(t_2^M) \leq M_2^* \leq M_1;$$

$$(P_4) \quad m_1 \leq m_2^* \leq u_2(t_2^m) < u_1(t_1^m), u_1(t_1^M) < u_2(t_2^M) \leq M_2^* \leq M_1.$$

From this and Claims 3 and 4, we have

$$\max_{t \in [0, T]} |u_i(t)| \leq \max \{ |M_1|, |M_2|, |m_1|, |m_2| \} := M^*, \quad i = 1, 2, 3. \quad (2.68)$$

Obviously, M^* is independent of λ .

Set

$$B_i^* := \bar{a}_i + \sqrt{(\bar{a}_i)^2 + 4\bar{b}_i\bar{S}_i}, \quad i = 1, 2. \quad (2.69)$$

Take sufficiently large M such that

$$\begin{aligned} M &> 3 \max \{ M^*, |m_1^*|, |m_2^*|, |m_3^*|, |m_4^*|, |m_5^*| \}, \\ M &> |v_1^*| + |v_2^*| + |v_3^*|, \end{aligned} \quad (2.70)$$

where

$$\begin{aligned} v_1^* &= \ln \frac{B_1^*}{2\bar{b}_1}, \quad v_2^* = \ln \frac{B_2^*}{2\bar{b}_2}, \\ v_3^* &= \ln \frac{\bar{p}B_1^* + \sqrt{[\bar{p}B_1^*]^2 + 16(\bar{b}_1)^2[\bar{q} + \bar{\beta}]\bar{S}_3}}{4\bar{b}_1[\bar{q} + \bar{\beta}]}. \end{aligned} \quad (2.71)$$

Clearly, the condition (i) in Lemma 2.1 is satisfied by system (2.7).

Define $H(u_1, u_2, u_3, \mu) : \mathbb{R}^3 \times [0, 1] \rightarrow \mathbb{R}^3$ by

$$H(u_1, u_2, u_3, \mu) = \begin{pmatrix} \bar{a}_1 - \bar{b}_1 e^{u_1} + \frac{\bar{S}_1}{e^{u_1}} \\ \bar{a}_2 - \bar{b}_2 e^{u_2} + \frac{\bar{S}_2}{e^{u_2}} \\ \bar{p}e^{u_1} - [\bar{q} + \bar{\beta}]e^{u_3} + \frac{\bar{S}_3}{e^{u_3}} \end{pmatrix} + \mu \begin{pmatrix} \bar{D}_1 e^{u_2 - u_1} - \bar{c}e^{u_3} - \bar{D}_1 \\ \bar{D}_2 e^{u_1 - u_2} - \bar{D}_2 \\ -\bar{d} \end{pmatrix}. \quad (2.72)$$

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We show that

$$H(u_1, u_2, u_3, \mu) \neq 0 \quad \text{for any } u = (u_1, u_2, u_3) \in \partial B_M(\mathbb{R}^3), \mu \in [0, 1]. \quad (2.73)$$

Indeed, assume to the contrary, that

$$H(u_1^*, u_2^*, u_3^*, \mu^*) = 0 \quad \text{for some } u^* = (u_1^*, u_2^*, u_3^*) \in \partial B_M(\mathbb{R}^3), \mu^* \in [0, 1]. \quad (2.74)$$

Then, there exist $t_i \in [0, T]$, $i = 1, 2$, such that

$$\begin{aligned} a_1(t_1) - b_1(t_1)e^{u_1^*} + \frac{S_1(t_1)}{e^{u_1^*}} + \mu^* D_1(t_1)e^{u_2^* - u_1^*} - \mu^* c(t_1)e^{u_3^*} - \mu^* D_1(t_1) &= 0, \\ a_2(t_2) - b_2(t_2)e^{u_2^*} + \frac{S_2(t_2)}{e^{u_2^*}} + \mu^* D_2(t_2)e^{u_1^* - u_2^*} - \mu^* D_2(t_2) &= 0, \\ -\mu^* \bar{d} + \bar{p}e^{u_1^*} - [\bar{q} + \bar{\beta}]e^{u_3^*} + \frac{\bar{S}_3}{e^{u_3^*}} &= 0. \end{aligned} \quad (2.75)$$

By using the arguments of (2.19), (2.20), (2.27), (2.38), (2.44), (2.54), (2.59), (2.66), one can prove that

$$|u_i^*| \leq \max \{ |M_1|, |M_2|, |m_1^*|, |m_2^*|, |m_3^*|, |m_4^*|, |m_5^*|, \quad i = 1, 2, 3, \quad (2.76)$$

which implies that $\|u^*\| = |u_1^*| + |u_2^*| + |u_3^*| \leq 3 \max \{ |M^*|, |m_1^*|, |m_2^*|, |m_3^*|, |m_4^*|, |m_5^*| \} < M$. This contradicts the fact that $u^* \in \partial B_M(\mathbb{R}^3)$. Therefore, $H(u_1, u_2, u_3, \mu)$ is a homotopy.

Since

$$g(u) = \begin{pmatrix} \bar{a}_1 - \bar{D}_1 - \bar{b}_1 e^{u_1} - \bar{c} e^{u_3} + \bar{D}_1 e^{u_2 - u_1} + \frac{\bar{S}_1}{e^{u_1}} \\ \bar{a}_2 - \bar{D}_2 - \bar{b}_2 e^{u_2} + \bar{D}_2 e^{u_1 - u_2} + \frac{\bar{S}_2}{e^{u_2}} \\ -\bar{d} + \bar{p} e^{u_1} - [\bar{q} + \bar{\beta}] e^{u_3} + \frac{\bar{S}_3}{e^{u_3}} \end{pmatrix} = H(u_1, u_2, u_3, 1), \quad (2.77)$$

$g(u) \neq 0$ for any $(u_1, u_2, u_3) \in \partial B_M(\mathbb{R}^3)$. Thus, the condition (ii) in Lemma 2.1 is satisfied. Next we show that condition (iii) also holds. It is easy to see that $H(u_1, u_2, u_3, 0) = 0$ has a unique solution $v^* = (v_1^*, v_2^*, v_3^*)$, where v_1^*, v_2^*, v_3^* are the same as those in (2.71). Clearly, $\|v^*\| = |v_1^*| + |v_2^*| + |v_3^*| < M$, that is, $v^* \in B_M(\mathbb{R}^3)$. According to the invariance of homotopy, we obtain

$$\deg(g, B_M(\mathbb{R}^3)) = \deg(H(\cdot, 1), B_M(\mathbb{R}^3)) = \deg(H(\cdot, 0), B_M(\mathbb{R}^3)) = -1. \quad (2.78)$$

Therefore, all of the conditions required in Lemma 2.1 hold. According to Lemma 2.1, system (2.7) has one T -periodic solution $(u_1^*(t), u_2^*(t), u_3^*(t))^T$. It is easy to see that $(x_1^*(t), x_2^*(t), y^*(t))^T = (\exp[u_1^*(t)], \exp[u_2^*(t)], \exp[u_3^*(t)])^T$ is a positive T -periodic solution of system (1.1). By the arguments similar to Claims 1–4, one can show

$$m_1 \leq u_i^*(t) \leq M_1 \quad (i = 1, 2), \quad m_2 \leq u_3^*(t) \leq M_2, \quad t \geq 0, \quad (2.79)$$

which implies

$$m_0 \leq x_i^*(t) \leq M_0 \quad (i = 1, 2), \quad \tilde{m}_0 \leq y^*(t) \leq \tilde{M}_0, \quad t \geq 0. \quad (2.80)$$

The proof is complete. $\square$

Consider the special case of system (1.1) that $S_i(t) \equiv 0$, $i = 1, 2, 3$. In this case, by Theorem 2.2, we have the following.

COROLLARY 2.3. *In addition to (H_1) and (H_2) , assume further that system (1.1) satisfies one of the following conditions:*

$$(H_3)' \quad (a_i/b_i)^l > \bar{d}/\bar{p}, \quad i = 1, 2;$$

$$(H_4)' \quad ((a_1 - D_1)/b_1)^l > \bar{d}/\bar{p}.$$

Then system (1.1) has at least one positive T -periodic solution.

Remark 2.4. Corollary 2.3 greatly improves [15, Theorem 2.1] and [5, Theorem 1.1].

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