EXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR DELAYED PREDATOR-PREY PATCH SYSTEMS WITH STOCKING

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A sufficient condition is derived for the existence of positive periodic solutions for a delayed predator-prey patch system with stocking. Some known results are improved.

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1. Introduction

Predator-prey systems have been studied extensively. See, for instance, [1, 6, 8–10] and the references cited therein. Most of the previous papers focused on the predator-prey systems without stocking. Brauer and Soudack [2, 3] studied some predator-prey systems under constant rate stocking. To our knowledge, few papers have been published on the existence of positive periodic solutions for delayed predator-prey patch systems with periodic stocking.

In this paper, we investigate the following predator-prey system with stocking:

\[
\begin{align*}
    x'_1(t) &= x_1(t)\left(a_1(t) - b_1(t)x_1(t) - c(t)y(t)\right) + D_1(t)\left(x_2(t - \tau_1(t)) - x_1(t)\right) + S_1(t), \\
    x'_2(t) &= x_2(t)\left(a_2(t) - b_2(t)x_2(t)\right) + D_2(t)\left(x_1(t - \tau_2(t)) - x_2(t)\right) + S_2(t), \\
    y'(t) &= y(t)\left(-d(t) + p(t)x_1(t) - q(t)y(t) - \beta(t) \int_{-\tau}^{0} k(s)y(t+s)ds\right) + S_3(t),
\end{align*}
\]

with the initial conditions

\[
\begin{align*}
    x_1(s) &= \varphi_1(s) \geq 0, \quad s \in [-\sigma, 0], \quad \varphi_1(0) > 0, \\
    x_2(s) &= \varphi_2(s) \geq 0, \quad s \in [-\sigma, 0], \quad \varphi_2(0) > 0, \\
    y(s) &= \psi(s) \geq 0, \quad s \in [-\sigma, 0], \quad \psi(0) > 0,
\end{align*}
\]

where \(x_1\) and \(y\) are the population densities of prey species \(x\) and predator species \(y\) in patch 1, and \(x_2\) is the density of species \(x\) in patch 2. Predator species \(y\) is confined to
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patch 1, while the prey species $x$ can diffuse between two patches. $D_i(t)$ $(i = 1, 2)$ are diffusion coefficients of species $x$. $S_i(t)$ $(i = 1, 2, 3)$ denote the stocking rates. $\varphi_1(s)$, $\varphi_2(s)$, and $\psi(s)$ are continuous on $[-\sigma, 0]$, $\sigma = \max\{\tau_1(t), \sup_{t \in \mathbb{R}} \tau_2(t)\}$. The delay $\tau_1(\tau_2)$ represents the time that species $x$ migrates from patch 2 to patch 1 (patch 1 to patch 2).

When $S_i(t) \equiv 0$ $(i = 1, 2, 3)$, $\tau_i \equiv 0$ $(i = 1, 2)$, system $(1.1)$ was considered by Zhang and Wang [15], Song and Chen [11], and Chen et al. [5].

The purpose of this paper is to derive a set of easily verifiable conditions for the existence of positive periodic solutions of system $(1.1)$. The method in this paper is different from those of [4, 12–14].

2. Existence of positive periodic solutions

To show the existence of solutions to the considered problems, we will use an abstract theorem developed [7]. We first state this abstract theorem.

For a fixed $\sigma \geq 0$, let $C := C([-\sigma, 0]; \mathbb{R}^n)$. If $x \in C([\gamma - \sigma, \gamma + \delta]; \mathbb{R}^n)$ for some $\delta > 0$ and $\gamma \in \mathbb{R}$, then $x_t \in C$ for $t \in [\gamma, \gamma + \delta]$ is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-\sigma, 0]$. The supremum norm in $C$ is denoted by $\| \cdot \|$, that is, $\| \phi \|_{c} = \max_{\theta \in [-\sigma, 0]} \| \phi(\theta) \|$ for $\phi \in C$, where $\| \cdot \|$ denotes the norm in $\mathbb{R}^n$, and $\| u \| = \sum_{i=1}^{n} | u_i |$ for $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$.

We consider the following functional differential equation:

$$
\frac{dx(t)}{dt} = f(t, x_t),
$$

where $f : \mathbb{R} \times C \to \mathbb{R}^n$ is completely continuous, and there exists $T > 0$ such that for every $(t, \varphi) \in \mathbb{R} \times C$, we have $f(t + T, \varphi) = f(t, \varphi)$.

The following lemma is a simple consequence of [7, Theorem 4.7.1].

**Lemma 2.1.** Suppose that there exists a constant $M > 0$ such that

(i) for any $\lambda \in (0, 1)$ and any $T$-periodic solution $x$ of the system

$$
\frac{dx(t)}{dt} = \lambda f(t, x_t),
$$

$\| x(t) \| < M$ for $t \in \mathbb{R}$;

(ii) $g(u) := (1/T) \int_{0}^{T} f(s, \hat{u})ds \neq 0$ for $u \in \partial B_M(\mathbb{R}^n)$, where $B_M(\mathbb{R}^n) = \{ u \in \mathbb{R}^n : \| u \| < M \}$, and $\hat{u}$ denotes the constant mapping from $[-\sigma, 0]$ to $\mathbb{R}^n$ with the value $u \in \mathbb{R}^n$;

(iii) Brouwer degree $\deg(g, B_M(\mathbb{R}^n)) \neq 0$.

Then there exists at least one $T$-periodic solution of the system

$$
\frac{dx(t)}{dt} = f(t, x_t)
$$

that satisfies $\sup_{t \in \mathbb{R}} \| x(t) \| < M$.

In the following, we set

$$
\hat{g} = \frac{1}{T} \int_{0}^{T} g(t)dt, \quad g^l = \min_{t \in [0, T]} | g(t) |, \quad g^u = \max_{t \in [0, T]} | g(t) |,
$$

where $g$ is a continuous $T$-periodic function.
In system (1.1), we always assume the following.

(H1) \( a_i(t), b_i(t), D_i(t) \) \( i = 1, 2 \), \( c(t), d(t), p(t), q(t), \) and \( \beta(t) \) are positive continuous \( T \)-periodic functions. \( S_i(t) \) \( (i = 1, 2, 3) \), \( \tau_i(t) \) \( (i = 1, 2) \) are nonnegative continuous \( T \)-periodic functions. \( \tau_i'(t) < 1 \) \( (i = 1, 2) \), \( t \in \mathbb{R} \).

(H2) \( k(s) \geq 0 \) on \([-\tau, 0] \) \( (0 \leq \tau < +\infty) \); and \( k(s) \) is a piecewise continuous and normalized function such that \( \int_{-\tau}^{0} k(s)ds = 1 \).

Set

\[
K = \frac{q + \beta}{p},
\]

\[
K^* = \left( \frac{a_1M_0 - D_1M_0 + S_1}{b_1M_0} \right)^l, \quad K_i^* = \left( \frac{a_iM_0 - S_i}{b_iM_0} \right)^l, \quad i = 1, 2,
\]

\[
M_0 = \max \left\{ \left( \frac{a + \sqrt{a^2 + 4bS_1}}{2b} \right)^u, \left( \frac{a + \sqrt{a^2 + 4bS_2}}{2b} \right)^u \right\},
\]

\[
m_0 = \min \left\{ \frac{(a/c)^l \sqrt{(S_3/q)}u}{b_1/c^l + (p/q)^u} \exp \left[ -2T(D_1 + b_1M_0 + c\tilde{M}_0) \right], \left( \frac{a + \sqrt{a^2 + 4bS_2}}{2b} \right)^l \right\},
\]

\[
\tilde{M}_0 = \left( \frac{pM_0 + \sqrt{p^2M_0^2 + 4aqS_3}}{2q} \right)^u,
\]

\[
\tilde{m}_0 = \min \left\{ \frac{K^* - \bar{d}/\bar{p}}{K + (c/b_1)^u}, \frac{K^* - \bar{d}/\bar{p}}{K + (c/b_1)^u} \right\} \exp \left[ -2T(\bar{d} + \bar{q}\tilde{M}_0 + \bar{\beta}\tilde{M}_0) \right].
\]

Theorem 2.2. In addition to (H1), (H2), assume further that system (1.1) satisfies one of the following assumptions:

(H3) \( (a_1/c)^l > \sqrt{(S_3/q)^u}, K^* > \bar{d}/\bar{p} \) \( (i = 1, 2) \);

(H4) \( (a_1/c)^l > \sqrt{(S_3/q)^u}, K^* > \bar{d}/\bar{p} \).

Then system (1.1) has at least one positive \( T \)-periodic solution, say \( (x_1^*(t), x_2^*(t), y^*(t))^T \) such that

\[
m_0 \leq x_i^*(t) \leq M_0 \quad (i = 1, 2), \quad \tilde{m}_0 \leq y^*(t) \leq \tilde{M}_0, \quad t \geq 0.
\]

Proof. Consider the following system:

\[
u_1'(t) = a_1(t) - D_1(t) - b_1(t)e^{\mu_1(t)} - c(t)e^{\mu_2(t)} + D_1(t)e^{\mu_2(t - \tau_1(t)) - u_1(t)} + \frac{S_1(t)}{e^{\mu_1(t)}},
\]

\[
u_2'(t) = a_2(t) - D_2(t) - b_2(t)e^{\mu_2(t)} + D_2(t)e^{\mu_1(t - \tau_2(t)) - u_2(t)} + \frac{S_2(t)}{e^{\mu_1(t)}},
\]

\[
u_3'(t) = -d(t) + p(t)e^{\mu_1(t)} - q(t)e^{\mu_3(t)} - \beta(t) \int_{-\tau}^{0} k(s)e^{\mu_3(t + s)}ds + \frac{S_3(t)}{e^{\mu_3(t)}},
\]

(2.7)
where \( a_i(t), b_i(t), D_i(t) \) for \( i = 1, 2, 3 \), \( S_i(t) \) for \( i = 1, 2, 3 \), \( c(t), d(t), p(t), q(t) \), and \( \beta(t) \) are the same as those in assumption (H1), and \( \tau, \tau_i \) for \( i = 1, 2 \) and \( k(s) \) are the same as those in assumption (H2). We first show that system (2.7) has one \( T \)-periodic solution.

Let \( \mathbb{C} : = \mathbb{C}([-\sigma, 0]; \mathbb{R}^3) \). We define the following map:

\[
\begin{align*}
\varphi(t, \varphi) &= (f_1(t, \varphi), f_2(t, \varphi), f_3(t, \varphi)), \\
\varphi &= (\varphi_1, \varphi_2, \varphi_3) \in \mathbb{C}, \\
f_1(t, \varphi) &= a_1(t) - D_1(t) - b_1(t)e^{\varphi_1(0)} - c(t)e^{\varphi_2(0)} + D_1(t) e^{\varphi_3(-\tau_1(t)) - \varphi_1(t)} + \frac{S_1(t)}{e^{\varphi_1(0)}}, \\
f_2(t, \varphi) &= a_2(t) - D_2(t) - b_2(t)e^{\varphi_2(0)} + D_2(t) e^{\varphi_3(-\tau_2(t)) - \varphi_2(t)} + \frac{S_2(t)}{e^{\varphi_2(0)}}, \\
f_3(t, \varphi) &= -d(t) + p(t)e^{\varphi_1(0)} - q(t)e^{\varphi_3(0)} - \beta(t) \int_{-\tau}^{0} k(s)e^{\varphi_3(s)}ds + \frac{S_3(t)}{e^{\varphi_3(0)}},
\end{align*}
\]

Clearly, \( f : \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{R}^3 \) is completely continuous. Now, the system (2.7) becomes

\[
\frac{du(t)}{dt} = f(t, u_t).
\]

Corresponding to

\[
\frac{du(t)}{dt} = \lambda f(t, u_t), \quad \lambda \in (0, 1),
\]

we have

\[
\begin{align*}
u_1'(t) &= \lambda \left[ a_1(t) - D_1(t) - b_1(t)e^{u_1(t)} - c(t)e^{u_2(t)} + D_1(t) e^{u_3(t - \tau_1(t)) - u_1(t)} + \frac{S_1(t)}{e^{u_1(t)}} \right], \\
u_2'(t) &= \lambda \left[ a_2(t) - D_2(t) - b_2(t)e^{u_2(t)} + D_2(t) e^{u_3(t - \tau_2(t)) - u_2(t)} + \frac{S_2(t)}{e^{u_2(t)}} \right], \\
u_3'(t) &= \lambda \left[ -d(t) + p(t)e^{u_1(t)} - q(t)e^{u_3(t)} - \beta(t) \int_{-\tau}^{0} k(s)e^{u_3(s)}ds + \frac{S_3(t)}{e^{u_3(t)}} \right].
\end{align*}
\]

Suppose that \((u_1(t), u_2(t), u_3(t))^T\) is a \( T \)-periodic solution of system (2.11) for some \( \lambda \in (0, 1) \). Choose \( t_i^M, t_i^m \in [0, T], \) \( i = 1, 2, 3 \), such that

\[
u_i(t_i^M) = \max_{t \in [0, T]} u_i(t), \quad u_i(t_i^m) = \min_{t \in [0, T]} u_i(t), \quad i = 1, 2, 3.\]

Then, it is clear that

\[
u_i'(t_i^M) = 0, \quad u_i'(t_i^m) = 0, \quad i = 1, 2, 3.\]
From this and system (2.11), we obtain that
\[
\begin{align*}
    a_1(t_1^M)& - D_1(t_1^M) - b_1(t_1^M) e^{u_1(t_1^M)} - c(t_1^M) e^{u_1(t_1^M)} + D_1(t_1^M) e^{\mu_1(t_1^M - \tau_1(t_1^M))} - u_1(t_1^M) + \frac{S_1(t_1^M)}{e^{u_1(t_1^M)}} = 0, \\
    a_2(t_2^M) & - D_2(t_2^M) - b_2(t_2^M) e^{\mu_2(t_2^M)} + D_2(t_2^M) e^{\mu_1(t_2^M - \tau_1(t_2^M))} - u_2(t_2^M) + \frac{S_2(t_2^M)}{e^{u_2(t_2^M)}} = 0, \\
    -d(t_3^M) + p(t_3^M) e^{\mu_3(t_3^M)} - q(t_3^M) e^{\mu_3(t_3^M)} - \beta(t_3^M) \int_{-\tau}^{0} k(s) e^{\mu_3(t_3^M + s)} ds + \frac{S_3(t_3^M)}{e^{u_3(t_3^M)}} = 0,
\end{align*}
\]
(2.14)
(2.15)
(2.16)
(2.17)
(2.18)

Next we make the following claims.

**Claim 1.** For \( u_i(t_i^M) \) (\( i = 1, 2 \)), one of the following cases holds:
\[
\begin{align*}
    u_2(t_2^M) & \leq u_1(t_1^M) \leq M_1^* \leq M_1, \\
    u_1(t_1^M) & < u_2(t_2^M) \leq M_2^* \leq M_1,
\end{align*}
\]
(2.19)
(2.20)

where \( M_1 := \max\{M_1^*, M_2^*\} \), \( M_j^* := \ln((a_j + \sqrt{a_j^2 + 4b_jS_j})/2b_j)^j \), \( j = 1, 2 \).

There are two cases to consider.

**Case 1.** Assume that \( u_1(t_1^M) \geq u_2(t_2^M) \); then \( u_1(t_1^M) \geq u_2(t_2^M - \tau_1(t_1^M)) \).

From this and (2.14), we have
\[
b_1(t_1^M) e^{u_1(t_1^M)} \leq a_1(t_1^M) + \frac{S_1(t_1^M)}{e^{u_1(t_1^M)}},
\]
(2.21)

That is,
\[
b_1(t_1^M) e^{2u_1(t_1^M)} - a_1(t_1^M) e^{u_1(t_1^M)} - S_1(t_1^M) \leq 0.
\]
(2.22)

Therefore,
\[
e^{u_1(t_1^M)} \leq \frac{a_1(t_1^M) + \sqrt{a_1^2(t_1^M) + 4b_1(t_1^M)}S_1(t_1^M)}{2b_1(t_1^M)} \leq \left( a_1 + \sqrt{a_1^2 + 4b_1S_1} \right)^u/2b_1.
\]
(2.23)

Hence,
\[
u_2(t_2^M) \leq u_1(t_1^M) \leq \ln \left( \frac{a_1 + \sqrt{a_1^2 + 4b_1S_1}}{2b_1} \right)^u.
\]
(2.24)
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Case 2. Assume that \( u_1(t^M_1) < u_2(t^M_2) \); then \( u_1(t^M_2 - t^M_2) < u_2(t^M_2) \).

From this and (2.15), we have

\[
b_2(t^M_2) e^{u_2(t^M_2)} \leq a_2(t^M_2) + \frac{S_2(t^M_2)}{e^{u_2(t^M_2)}}.
\]

By a similar argument to Case 1, we have

\[
u_1(t^M_1) < u_2(t^M_2) \leq \ln \left( \frac{a_2 + \sqrt{a^2_2 + 4b_2S_2}}{2b_2} \right).
\]

It follows from (2.24) and (2.26) that Claim 1 holds.

Claim 2.

\[
u_3(t^M_3) \leq \ln \left( \frac{pM_0 + \sqrt{p^2 M_0^2 + 4qS_3}}{2q} \right) := M_2,
\]

where \( M_0 = e^{M_1} \).

By (2.16), we have

\[
q(t^M_3) e^{u_3(t^M_3)} \leq p(t^M_3) e^{u_1(t^M_3)} + \frac{S_3(t^M_3)}{e^{u_3(t^M_3)}} \leq p(t^M_3) e^{u_1(t^M_3)} + \frac{S_3(t^M_3)}{e^{u_3(t^M_3)}}.
\]

That is,

\[
q(t^M_3) e^{2u_3(t^M_3)} - p(t^M_3) e^{u_1(t^M_3)} e^{u_3(t^M_3)} - S_3(t^M_3) \leq 0.
\]

Therefore,

\[
e^{u_3(t^M_3)} \leq \frac{p(t^M_3) e^{u_1(t^M_3)} + \sqrt{p^2(t^M_3) e^{2u_1(t^M_3)} + 4q(t^M_3) S_3(t^M_3)}}{2q(t^M_3)},
\]

which implies that Claim 2 holds.

Claim 3. For \( u_i(t^m_i)(i = 1, 2) \), one of the following cases holds:

\[
m_1 \leq m^*_1 = 2T(\bar{D}_1 + \bar{b}_1 M_0 + \bar{c}\hat{M}_0) \leq u_1(t^m_1) \leq u_2(t^m_2),
\]

\[
m_1 \leq m^*_2 \leq u_2(t^m_2) < u_1(t^m_1),
\]

where

\[
m_1 := \min \{ m^*_1 - 2T(\bar{D}_1 + \bar{b}_1 M_0 + \bar{c}\hat{M}_0), m^*_2 \},
\]

\[
m^*_1 := \ln \left( \frac{(a_1/c)^l - \sqrt{(S_3/q)^u}}{b_1^l/c + (p/q)^u} \right),
\]

\[
m^*_2 := \ln \left( \frac{a_2 + \sqrt{a^2_2 + 4b_2S_2}}{2b_2} \right).
\]

There are two cases to consider.
**Case 1.** Assume that \( u_1(t_1^m) \leq u_2(t_2^m) \); then \( u_1(t_1^m) \leq u_2(t_1^m - \tau_1(t_1^m)) \).

From this and (2.17), we have

\[
a_1(t_1^m) \leq b_1(t_1^m)e^{u_1(t_1^m)} + c(t_1^m) \leq b_1(t_1^m)e^{u_3(t_1^m)} + c(t_1^m)e^{u_3(t_1^m)}.
\] (2.33)

From (2.30), by using the inequality

\[
(a + b)^{1/2} < a^{1/2} + b^{1/2}, \quad a > 0, \quad b > 0,
\] (2.34)

we have

\[
e^{u_3(t_1^M)} < \frac{p(t_3^M)e^{u_1(t_1^M)} + \sqrt{q(t_3^M)}S_3(t_3^M)}{q(t_3^M)}.
\] (2.35)

From this and (2.33), we have

\[
a_1(t_1^m) \leq b_1(t_1^m) + \frac{c(t_1^m)p(t_3^M)}{q(t_3^M)}e^{u_3(t_1^M)} + c(t_1^m)\sqrt{\frac{S_3(t_3^M)}{q(t_3^M)}},
\] (2.36)

which implies

\[
\left( \frac{a_1}{c} \right)^l \leq \left[ \frac{b_1^u}{c^l} + \left( \frac{p}{q} \right)^u \right]e^{u_1(t_1^M)} + \sqrt{\frac{S_3^u}{q^u}}.
\] (2.37)

That is,

\[
u_1(t_1^M) \geq \ln \left( c_1 \right) - \sqrt{\frac{S_3}{q}} u := m^*_1.
\] (2.38)

From the first equation of system (2.11), we obtain that

\[
\int_0^T a_1(t)dt + \int_0^T D_1(t)e^{u_1(t-\tau(t)) - u_1(t)}dt + \int_0^T \frac{S_1(t)}{e^{u_1(t)}}dt
\]

\[
= \int_0^T D_1(t)dt + \int_0^T b_1(t)e^{u_1(t)}dt + \int_0^T c(t)e^{u_3(t)}dt,
\]

\[
\int_0^T \left| u_1'(t) \right| dt < \int_0^T a_1(t)dt + \int_0^T D_1(t)e^{u_3(t-\tau(t)) - u_1(t)}dt + \int_0^T \frac{S_1(t)}{e^{u_3(t)}}dt
\]

\[
+ \int_0^T D_1(t)dt + \int_0^T b_1(t)e^{u_3(t)}dt + \int_0^T c(t)e^{u_3(t)}dt.
\] (2.39)
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It follows that

\[
\int_0^T |u_1'(t)| \, dt < 2 \left[ \int_0^T D_1(t) \, dt + \int_0^T b_1(t) e^{u_1(t)} \, dt + \int_0^T c(t) e^{u_1(t)} \, dt \right]
\leq 2 \left[ \int_0^T D_1(t) \, dt + e^{M_1} \int_0^T b_1(t) \, dt + e^{M_2} \int_0^T c(t) \, dt \right]
= 2T(\tilde{D}_1 + \tilde{b}_1 M_0 + \tilde{c}M_0).
\]

From (2.38) and (2.40), we have

\[
u_1(t^m_1) \geq u_1(t^M_1) - \int_0^T |u_1'(t)| \, dt \geq m_1^* - 2T(\tilde{D}_1 + \tilde{b}_1 M_0 + \tilde{c}M_0).
\]  

Case 2. Assume that \(u_1(t^m_1) > u_2(t^m_2)\); then \(u_1(t^m_2 - \tau_2(t^m_2)) > u_2(t^m_2)\).

From this and (2.18), we have

\[
b_2(t^m_2) e^{u_2(t^m_2)} \geq a_2(t^m_2) + \frac{S_2(t^m_2)}{e^{a_2(t^m_2)}},
\]

which implies

\[
e^{u_2(t^m_2)} \geq \frac{a_2(t^m_2) + \sqrt{a_2^2(t^m_2) + 4b_2(t^m_2)S_2(t^m_2)}}{2b_2(t^m_2)}.
\]

That is,

\[
u_2(t^m_2) \geq \ln \left( \frac{a_2 + \sqrt{a_2^2 + 4b_2S_2}}{2b_2} \right) := m_2^*.
\]

It follows from (2.41) and (2.44) that Claim 3 holds.

Claim 4.

\[
u_3(t^m_3) \geq \min \{m_3^*, m_4^*, m_5^*\} - 2T(\tilde{d} + \tilde{q}M_0 + \tilde{b}M_0) := m_2,
\]

where

\[
m_3^* = \ln \frac{K^*_1 - \tilde{d}/\tilde{p}}{K + (c/b_1)}\pi, \quad m_4^* = \ln \frac{K^*_2 - \tilde{d}/\tilde{p}}{K}, \quad m_5^* = \ln \frac{K^*_3 - \tilde{d}/\tilde{p}}{K + (c/b_1)}\pi.
\]

From the third equation of (2.11), we obtain

\[
\int_0^T p(t) e^{u_1(t)} \, dt + \int_0^T \frac{S_3(t)}{e^{u_3(t)}} \, dt = \int_0^T \beta(t) \int_{-\tau}^0 k(s) e^{u_3(t+s)} ds \, dt,
\]

\[
\int_0^T |u_3'(t)| \, dt < \int_0^T p(t) e^{u_1(t)} \, dt + \int_0^T \frac{S_3(t)}{e^{u_3(t)}} \, dt + \int_0^T \beta(t) \int_{-\tau}^0 k(s) e^{u_3(t+s)} ds \, dt.
\]
It follows that
\[
\int_0^T |u'(t)| dt < 2 \left[ \int_0^T d(t) dt + \int_0^T q(t)e^{u_3(t)} dt + \int_0^T \beta(t) \int_{-\infty}^0 k(s)e^{u_3(t+s)} ds dt \right]
\]
\[
\leq 2 \left[ \int_0^T d(t) dt + e^{M_2} \int_0^T q(t) dt + e^{M_2} \int_0^T \beta(t) dt \right]
\]
\[
= 2T(\tilde{d} + \tilde{q}\tilde{M}_0 + \tilde{\beta}\tilde{M}_0),
\]
\[
[q + \bar{\beta}]e^{u_3(t^m)} \geq \tilde{p}e^{u_1(t^m)} - \tilde{d}.
\] (2.49)

There are two cases to consider.

Case 1. Assume that the assumption (H3) holds.

If \(u_1(t_1^m) \leq u_2(t_2^m)\), by (2.17), we have
\[
e^{u_1(t^m)} \geq \frac{a_1(t_1^m) - c(t_1^m)e^{u_3(t^m)}}{b_1(t_1^m)} + \frac{S_1(t_1^m)}{b_1(t_1^m)e^{u_3(t^m)}}
\]
\[
\geq \frac{a_1(t_1^m) - c(t_1^m)e^{u_3(t^m)}}{b_1(t_1^m)} + \frac{S_1(t_1^m)}{b_1(t_1^m)e^{M_1}}.
\] (2.50)

Substituting this into (2.49) gives
\[
[q + \bar{\beta}]e^{u_3(t^m)} \geq \frac{\tilde{p}a_1(t_1^m)}{b_1(t_1^m)} - \frac{\tilde{p}c(t_1^m)e^{u_3(t^m)}}{b_1(t_1^m)} + \frac{\tilde{p}S_1(t_1^m)}{b_1(t_1^m)e^{M_1}} - \tilde{d},
\] (2.51)

which implies
\[
\left[ \frac{\tilde{q}}{\tilde{p}} + \frac{\bar{\beta}}{\tilde{p}} + \frac{c(t_1^m)}{b_1(t_1^m)} \right] e^{u_3(t^m)} \geq \frac{a_1(t_1^m)}{b_1(t_1^m)} + \frac{S_1(t_1^m)}{b_1(t_1^m)e^{M_1}} - \frac{\tilde{d}}{\tilde{p}}.
\] (2.52)

Therefore,
\[
\left[ K + \left( \frac{c}{b_1} \right)^u \right] e^{u_3(t^m)} \geq K_1^* - \frac{\tilde{d}}{\tilde{p}}.
\] (2.53)

That is,
\[
u_3(t_3^M) \geq \ln \frac{K_1^* - \tilde{d}/\tilde{p}}{K + (c/b_1)^u} := m_3^*.
\] (2.54)

It follows from (2.48) and (2.54) that
\[
u_3(t_3^m) \geq u_3(t_3^M) - \int_0^T |u'_3(t)| dt \geq m_3^* - 2T(\tilde{d} + \tilde{q}\tilde{M}_0 + \tilde{\beta}\tilde{M}_0).
\] (2.55)

If \(u_1(t_1^m) > u_2(t_2^m)\), by (2.42), (2.49), and (2.19), we have
\[
[q + \bar{\beta}]e^{u_3(t^m)} \geq \tilde{p}e^{u_2(t^m)} - \tilde{d} \geq \frac{\tilde{p}[a_2(t_2^m) + S_2(t_2^m)e^{-M_1}]}{b_2(t_2^m)} - \tilde{d},
\] (2.56)
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which implies

\[ \left( \frac{\bar{q}}{\bar{p}} + \frac{\bar{\beta}}{\bar{p}} \right) e^{\mu_3(t_M^M)} \geq \frac{a_2(t_2^M) + S_2(t_2^M) e^{-M_1}}{b_2(t_2^M)} - \frac{\bar{d}}{\bar{p}}. \]  

(2.57)

Therefore,

\[ K e^{\mu_3(t_M^M)} \geq K^* - \frac{\bar{d}}{\bar{p}}. \]  

(2.58)

That is,

\[ u_3(t_M^M) \geq \ln \frac{K^* - \bar{d}/\bar{p}}{K} := m^*_5. \]  

(2.59)

From (2.48) and (2.59), we have

\[ u_3(t_M^m) \geq u_3(t_M^3) - \int_0^T |u_3(t)| dt \geq m^*_4 - 2T(\bar{d} + \bar{q}\bar{M}_0 + \bar{\beta}\bar{M}_0). \]  

(2.60)

Case 2. Assume that the assumption (H_4) holds.

From (2.17), we have

\[ b_1(t_1^m) e^{\mu_1(t_1^m)} \geq a_1(t_1^m) - D_1(t_1^m) c(t_1^m) e^{\mu_1(t_1^m)} + \frac{S_1(t_1^m)}{e^{\mu_1(t_1^m)}} \]  

(2.61)

\[ \geq a_1(t_1^m) - D_1(t_1^m) c(t_1^m) e^{\mu_1(t_1^m)} + \frac{S_1(t_1^m)}{e^{M_1}}. \]

Therefore,

\[ e^{\mu_1(t_1^m)} \geq \frac{a_1(t_1^m) - D_1(t_1^m) c(t_1^m) e^{\mu_1(t_1^m)} + S_1(t_1^m) e^{-M_1}}{b_1(t_1^m)}. \]  

(2.62)

Substituting this into (2.49) gives

\[ [\bar{q} + \bar{\beta}] e^{\mu_3(t_M^M)} \geq \bar{p} \left[ a_1(t_1^m) - D_1(t_1^m) \right] - \frac{\bar{p} c(t_1^m) e^{\mu_1(t_1^m)}}{b_1(t_1^m)} + \frac{\bar{p} S_1(t_1^m)}{b_1(t_1^m) e^{M_1}} - \bar{d}, \]  

(2.63)

which implies

\[ \left[ \frac{\bar{q}}{\bar{p}} + \frac{\bar{\beta}}{\bar{p}} + \frac{c(t_1^m)}{b_1(t_1^m)} \right] e^{\mu_3(t_1^m)} \geq \frac{a_1(t_1^m) - D_1(t_1^m) + S_1(t_1^m) e^{-M_1}}{b_1(t_1^m)} - \frac{\bar{d}}{\bar{p}}. \]  

(2.64)

Therefore,

\[ \left[ K + \left( \frac{c}{b_1} \right)^u \right] e^{\mu_3(t_M^M)} \geq K^* - \frac{\bar{d}}{\bar{p}}. \]  

(2.65)

That is,

\[ u_3(t_M^M) \geq \ln \frac{K^* - \bar{d}/\bar{p}}{K + (c/b_1)^u} := m^*_5. \]  

(2.66)
It follows from (2.48) and (2.66) that
\[ u_3(t_3^m) \geq u_3(t_3^M) - \int_0^T |u_3'(t)| \, dt \geq m_2^* - 2T(\bar{d} + \bar{q} \tilde{M}_0 + \bar{\beta} \tilde{M}_0). \] (2.67)

It follows from (2.55), (2.60), and (2.67) that Claim 4 holds. Clearly, one of the following inequalities holds:
(i) \( M_1^* > m_2^* \),
(ii) \( M_1^* \leq m_2^* \).

Since \( m_1^* < M_1^* \) and \( m_2^* \leq M_2^* \), (ii) implies \( M_2^* > m_1^* \). Thus, according to Claims 1–3, one of the following four cases must hold:

1. \( m_1^* - 2T(\bar{D}_1 + \bar{b}_1 M_0 + \bar{c} \tilde{M}_0) \leq u_1(t_1^m) \leq u_2(t_1^m), u_2(t_1^M) \leq u_1(t_1^M) \leq M_1^* \leq M_1 \);  
2. \( m_2^* \leq u_2(t_2^m) < u_1(t_2^m), u_2(t_2^M) \leq u_1(t_2^M) \leq M_1^* \leq M_1 \);  
3. \( m_1^* - 2T(\bar{D}_1 + \bar{b}_1 M_0 + \bar{c} \tilde{M}_0) \leq u_1(t_1^m) \leq u_2(t_1^m), u_1(t_1^M) \leq u_2(t_1^M) \leq M_1^* \leq M_1 \);  
4. \( m_2^* \leq u_2(t_2^m) < u_1(t_2^m), u_1(t_2^M) < u_2(t_2^M) \leq M_2^* \leq M_1 \).

From this and Claims 3 and 4, we have
\[ \max_{t \in [0,T]} |u_i(t)| \leq \max \{ |M_1|, |M_2|, |m_1|, |m_2| \} := M^*, \quad i = 1, 2, 3. \] (2.68)

Obviously, \( M^* \) is independent of \( \lambda \).

Set
\[ B_i^* := a_i + \sqrt{(a_i)^2 + 4b_i S_i}, \quad i = 1, 2. \] (2.69)

Take sufficiently large \( M \) such that
\[ M > 3 \max \{ M^*, |m_1^*|, |m_2^*|, |m_3^*|, |m_4^*|, |m_5^*| \}, \]
\[ M > |v_1^*| + |v_2^*| + |v_3^*|, \] (2.70)

where
\[ v_1^* = \ln \frac{B_1^*}{2b_1}, \quad v_2^* = \ln \frac{B_2^*}{2b_2}, \]
\[ v_3^* = \ln \frac{\bar{\rho} B_1^* + \sqrt{[\bar{\rho} B_1^*]^2 + 16(\bar{b}_1)^2 [\bar{q} + \bar{\beta} S_3]}}{4b_1(\bar{q} + \bar{\beta})}. \] (2.71)

Clearly, the condition (i) in Lemma 2.1 is satisfied by system (2.7).

Define \( H(u_1, u_2, u_3, \mu) : \mathbb{R}^3 \times [0, 1] \rightarrow \mathbb{R}^3 \) by
\[ H(u_1, u_2, u_3, \mu) = \begin{pmatrix} \bar{a}_1 - \bar{b}_1 e^{u_1} + \frac{\bar{S}_1}{e^{u_1}} \\ \bar{a}_2 - \bar{b}_2 e^{u_2} + \frac{\bar{S}_2}{e^{u_2}} \\ \bar{\rho} e^{u_1} - [\bar{q} + \bar{\beta}] e^{u_3} + \frac{\bar{S}_3}{e^{u_3}} \end{pmatrix} + \mu \begin{pmatrix} \bar{D}_1 e^{u_1 - u_1} - \bar{c} e^{u_3} - \bar{D}_1 \\ \bar{D}_2 e^{u_1 - u_2} - \bar{D}_2 \\ -\bar{d} \end{pmatrix}. \] (2.72)
We show that

\[ H(u_1, u_2, u_3, \mu) \neq 0 \quad \text{for any } u = (u_1, u_2, u_3) \in \partial B_M(\mathbb{R}^3), \; \mu \in [0, 1]. \] (2.73)

Indeed, assume to the contrary, that

\[ H(u_1^*, u_2^*, u_3^*, \mu^*) = 0 \quad \text{for some } u^* = (u_1^*, u_2^*, u_3^*) \in \partial B_M(\mathbb{R}^3), \; \mu^* \in [0, 1]. \] (2.74)

Then, there exist \( t_i \in [0, T], \; i = 1, 2 \), such that

\[
\begin{align*}
    a_1(t_1) - b_1(t_1) e^{u_1^*} + \frac{S_1(t_1)}{e^{u_1^*}} + \mu^* D_1(t_1) e^{-u_1^*} - \mu^* c(t_1) e^{u_2^*} - \mu^* D_1(t_1) &= 0, \\
    a_2(t_2) - b_2(t_2) e^{u_2^*} + \frac{S_2(t_2)}{e^{u_2^*}} + \mu^* D_2(t_2) e^{-u_2^*} - \mu^* D_2(t_2) &= 0, \\
    -\mu^* \bar{d} + \bar{p} e^{u_3^*} - [\bar{q} + \bar{\beta}] e^{u_3^*} + \frac{S_3}{e^{u_3^*}} &= 0.
\end{align*}
\] (2.75)

By using the arguments of (2.19), (2.20), (2.27), (2.38), (2.44), (2.54), (2.59), (2.66), one can prove that

\[|u_i^*| \leq \max |M_1|, |M_2|, |m_1^*|, |m_2^*|, |m_3^*|, |m_4^*|, |m_5^*|, \; i = 1, 2, 3, \] (2.76)

which implies that \( \|u^*\| = |u_1^*| + |u_2^*| + |u_3^*| \leq 3 \max \{M^*, |m_1^*|, |m_2^*|, |m_3^*|, |m_4^*|, |m_5^*|\} < M \). This contradicts the fact that \( u^* \in \partial B_M(\mathbb{R}^3) \). Therefore, \( H(u_1, u_2, u_3, \mu) \) is a homotopy.

Since

\[
g(u) = \left( \begin{array}{c}
\tilde{a}_1 - \tilde{b}_1 e^{u_1} - \tilde{c} e^{u_3} + \tilde{D}_1 e^{-u_1} + \frac{\tilde{S}_1}{e^{u_1}} \\
\tilde{a}_2 - \tilde{b}_2 e^{u_2} + \tilde{D}_2 e^{-u_2} + \frac{\tilde{S}_2}{e^{u_2}} \\
-\tilde{d} + \tilde{p} e^{u_3} - [\tilde{q} + \tilde{\beta}] e^{u_3} + \frac{\tilde{S}_3}{e^{u_3}}
\end{array} \right) = H(u_1, u_2, u_3, 1),
\] (2.77)

\( g(u) \neq 0 \) for any \( (u_1, u_2, u_3) \in \partial B_M(\mathbb{R}^3) \). Thus, the condition (ii) in Lemma 2.1 is satisfied. Next we show that condition (iii) also holds. It is easy to see that \( H(u_1, u_2, u_3, 0) = 0 \) has a unique solution \( v^* = (v_1^*, v_2^*, v_3^*) \), where \( v_1^*, v_2^*, v_3^* \) are the same as those in (2.71). Clearly, \( \|v^*\| = |v_1^*| + |v_2^*| + |v_3^*| < M \), that is, \( v^* \in B_M(\mathbb{R}^3) \). According to the invariance of homotopy, we obtain

\[
\deg (g, B_M(\mathbb{R}^3)) = \deg (H(\cdot, 1), B_M(\mathbb{R}^3)) = \deg (H(\cdot, 0), B_M(\mathbb{R}^3)) = -1.
\] (2.78)

Therefore, all of the conditions required in Lemma 2.1 hold. According to Lemma 2.1, system (2.7) has one \( T \)-periodic solution \( (u_1^*(t), u_2^*(t), u_3^*(t))^T \). It is easy to see that \( (x_1^*(t), x_2^*(t), y^*(t))^T = (\exp([u_1^*(t)], \exp[u_2^*(t)], \exp[u_3^*(t)])^T \) is a positive \( T \)-periodic solution of system (1.1). By the arguments similar to Claims 1–4, one can show

\[m_1 \leq u_i^*(t) \leq M_1 \quad (i = 1, 2), \quad m_2 \leq u_3^*(t) \leq M_2, \quad t \geq 0, \] (2.79)
which implies

\[ m_0 \leq x_i^*(t) \leq M_0 \quad (i = 1, 2), \quad \tilde{m}_0 \leq y^*(t) \leq \tilde{M}_0, \quad t \geq 0. \quad (2.80) \]

The proof is complete. \( \square \)

Consider the special case of system (1.1) that \( S_i(t) \equiv 0, \ i = 1, 2, 3. \) In this case, by

Theorem 2.2, we have the following.

**Corollary 2.3.** In addition to \((H_1)\) and \((H_2)\), assume further that system (1.1) satisfies

one of the following conditions:

\( \text{(H3)} \) \( (a_i/b_i)^l > \bar{d}/\bar{p}, \ i = 1, 2; \)

\( \text{(H4)} \) \( ((a_1 - D_1)/b_1)^l > \bar{d}/\bar{p}. \)

Then system (1.1) has at least one positive \( T \)-periodic solution.

**Remark 2.4.** Corollary 2.3 greatly improves [15, Theorem 2.1] and [5, Theorem 1.1].

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**References**


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