The notion of intuitionistic fuzzy sets was introduced by Atanassov as a generalization of the notion of fuzzy sets. In this paper, we apply the concept of intuitionistic fuzzy sets to $H_v$-rings. We introduce the notion of an intuitionistic fuzzy $H_v$-ideal of an $H_v$-ring and then some related properties are investigated. We state some characterizations of intuitionistic fuzzy $H_v$-ideals. Also we investigate some natural equivalence relations on the set of all intuitionistic fuzzy $H_v$-ideals of an $H_v$-ring.

Copyright © 2006 Hindawi Publishing Corporation. All rights reserved.

1. Introduction and preliminaries

Hyperstructure theory was born in 1934 when Marty [11] defined hypergroups as a generalization of groups. This theory has been studied in the following decades and nowadays by many mathematicians. A recent book [3] contains a wealth of applications. There are applications to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence, and probabilities. Vougiouklis in the fourth Algebraic Hyperstructures and Applications Congress (1990) [15] introduced the notion of $H_v$-structures. The $H_v$-structures are hyperstructures where the equality is replaced by the nonempty intersection. The main tool in the study of $H_v$-structure is the fundamental structure which is the same as in the classical hyperstructures. In this paper, we deal with $H_v$-rings. $H_v$-rings are the largest class of algebraic systems that satisfy ring-like axioms. In [4], Darafsheh and Davvaz defined the $H_v$-ring of fractions of a commutative hyperring which is a generalization of the concept of ring of fractions. For the notion of an $H_v$-near-ring module, you can see [7]. In [13], Spartalis studied a wide class of $H_v$-rings resulting from an arbitrary ring by using the $P$-hyperoperations. In [18], Vougiouklis introduced the classes of $H_v$-rings useful in the theory of representations.

A hyperstructure is a nonempty set $H$ together with a map $*: H \times H \rightarrow \mathcal{P}(H)$ called hyperoperation, where $\mathcal{P}(H)$ denotes the set of all nonempty subsets of $H$. The image of the pair $(x, y)$ is denoted by $x * y$. If $x \in H$ and $A, B \subseteq H$, then by $A * B, A * x,$ and $x * B,$
we mean
\[ A \ast B = \bigcup_{a \in A, b \in B} a \ast b, \quad A \ast x = A \ast \{x\}, \quad x \ast B = \{x\} \ast B. \tag{1.1} \]

A hyperstructure \((H, \ast)\) is called an \(H\)-semigroup if
\[ (x \ast (y \ast z)) \cap ((x \ast y) \ast z) \neq \emptyset \quad \forall x, y, z \in H. \tag{1.2} \]

**Definition 1.1.** An \(H\)-ring is a system \((R, +, \cdot)\) with two hyperoperations satisfying the following ring-like axioms:
(i) \((R, +, \cdot)\) is an \(H\)-group, that is,
\[ ((x + y) + z) \cap (x + (y + z)) \neq \emptyset \quad \forall x, y \in R, \]
\[ a + R = R + a = R \quad \forall a \in R; \tag{1.3} \]
(ii) \((R, \cdot)\) is an \(H\)-semigroup;
(iii) \((\cdot)\) is weak distributive with respect to (+), that is, for all \(x, y, z \in R,\)
\[ (x \cdot (y + z)) \cap (x \cdot y + x \cdot z) \neq \emptyset, \]
\[ ((x + y) \cdot z) \cap (x \cdot z + y \cdot z) \neq \emptyset. \tag{1.4} \]

An \(H\)-ring \((R, +, \cdot)\) is called dual \(H\)-ring if \((R, \cdot, +)\) is an \(H\)-ring. If both operations (+) and (\cdot) are weak commutative, then \(R\) is called a weak commutative dual \(H\)-ring.

We see that \(H\)-rings are a nice generalization of rings. For more definitions, results, and applications on \(H\)-rings, see [4, 5, 7, 8, 13–15, 17, 18].

**Example 1.2** (cf. Vougiouklis [18]). Let \((H, \ast)\) be an \(H\)-group, then for every hyperoperation (\(\circ\)) such that \(\{x, y\} \subseteq x \circ y\) for all \(x, y \in H\), the hyperstructure \((H, \ast, \circ)\) is a dual \(H\)-ring.

**Example 1.3** (cf. Dramalidis [8]). On the set \(\mathbb{R}^n\), where \(\mathbb{R}\) is the set of real numbers, we define three hyperoperations:
\[
\begin{align*}
x \mathcal{V} y &= \{r(x + y) \mid r \in [0, 1] \}, \\
x \mathcal{O} y &= \{x + r(y - x) \mid r \in [0, 1] \}, \\
x \mathcal{D} y &= \{x + ry \mid r \in [0, 1] \}. \tag{1.5}
\end{align*}
\]

Then the hyperstructure \((\mathbb{R}^n, \ast, \circ)\), where \(\ast, \circ \in \{\mathcal{V}, \mathcal{O}, \mathcal{D}\}\), is a weak commutative dual \(H\)-ring.

**Definition 1.4.** Let \(R\) be an \(H\)-ring. A nonempty subset \(I\) of \(R\) is called a left (resp., right) \(H\)-ideal if the following axioms hold:
(i) \((I, +)\) is an \(H\)-subgroup of \((R, +)\),
(ii) \(R \cdot I \subseteq I\) (resp., \(I \cdot R \subseteq I\)).
2. Fuzzy sets and intuitionistic fuzzy sets

The concept of a fuzzy subset of a nonempty set was first introduced by Zadeh [19].

Let $X$ be a nonempty set, a mapping $\mu : X \rightarrow [0,1]$ is called a fuzzy subset of $X$. The complement of $\mu$, denoted by $\mu^c$, is the fuzzy set of $X$ given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$.

Note that using fuzzy subsets, we can introduce on any ring the structure of $H_v$-ring.

**Example 2.1** (cf. Davvaz [5]). Let $(R,+,\cdot)$ be an ordinary ring and let $\mu$ be a fuzzy subset of $R$. We define hyperoperations $\oplus, \otimes, \ast$ on $R$ as follows:

\[
x \oplus y = \{ t \mid \mu(t) = \mu(x + y) \},
\]
\[
x \otimes y = \{ t \mid \mu(t) = \mu(x \cdot y) \},
\]
\[
x \ast y = y \ast x = \{ t \mid \mu(t) \leq \mu(x \cdot y) \} \quad \text{(if } \mu(x) \leq \mu(y)\text{)}.
\]

Then $(R,\ast,\ast), (R,\ast,\otimes), (R,\otimes,\ast), (R,\otimes,\cdot), (R,\cdot,\otimes)$ are $H_v$-rings.

Rosenfeld [12] applied the concept of fuzzy sets to the theory of groups and defined the concept of fuzzy subgroups of a group. Since then, many papers concerning various fuzzy algebraic structures have appeared in the literature. In [5–7], Davvaz applied the concept of fuzzy set theory in the algebraic hyperstructures, in particular in [5] he defined the concept of fuzzy $H_v$-ideal of an $H_v$-ring which is a generalization of the concept of fuzzy ideal.

**Definition 2.2.** Let $(R,+,\cdot)$ be an $H_v$-ring and $\mu$ a fuzzy subset of $R$. Then $\mu$ is said to be a left (resp., right) fuzzy $H_v$-ideal of $R$ if the following axioms hold:

1. $\min \{ \mu(x), \mu(y) \} \leq \inf \{ \mu(z) \mid z \in x + y \}$ for all $x, y \in R$,
2. for all $x, a \in R$, there exists $y \in R$ such that $x \in a + y$ and
   \[
   \min \{ \mu(a), \mu(x) \} \leq \mu(y),
   \]
3. for all $x, a \in R$ there exists $z \in R$ such that $x \in z + a$ and
   \[
   \min \{ \mu(a), \mu(x) \} \leq \mu(z),
   \]
4. $\mu(y) \leq \inf \{ \mu(z) \mid z \in x \cdot y \}$ (resp., $\mu(x) \leq \inf \{ \mu(z) \mid z \in x \cdot y \}$) for all $x, y \in R$.

**Example 2.3** (cf. Davvaz [5]). Let $(R,+,\cdot)$ be an ordinary ring and let $\mu$ be a fuzzy ideal of $R$. We consider the $H_v$-ring $(R,\otimes,\otimes)$ defined in Example 2.1. Then $\mu$ is a left fuzzy $H_v$-ideal of $(R,\otimes,\otimes)$.

The concept of intuitionistic fuzzy set was introduced by Atanassov [1] as a generalization of the notion of fuzzy set. Some fundamental operations on intuitionistic fuzzy sets are defined by Atanassov in [2]. In [9], Kim et al. introduced the notion of an intuitionistic fuzzy subquasigroup of a quasigroup. Also in [10], Kim and Jun introduced the concept of intuitionistic fuzzy ideals of semirings.
4  Intuitionistic fuzzy $H_v$-ideals

Definition 2.4. An intuitionistic fuzzy set $A$ of a nonempty set $X$ is an object having the form

$$A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\},$$

where the functions $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ denote the degree of membership (namely, $\mu_A(x)$) and the degree of nonmembership (namely, $\lambda_A(x)$) of each element $x \in X$ to the set $A$, respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$.

Definition 2.5. For every two intuitionistic fuzzy sets $A$ and $B$, define the following operations:

1. $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in X$.
2. $A' = \{(x, \lambda_A(x), \mu_A(x)) \mid x \in X\}$.
3. $A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\}) \mid x \in X\}$.
4. $A \cup B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\lambda_A(x), \lambda_B(x)\}) \mid x \in X\}$.
5. $\square A = \{(x, \lambda_A(x), \mu_A(x)) \mid x \in X\}$.
6. $\triangle A = \{(x, \lambda_A(x), \mu_A(x)) \mid x \in X\}$.

For the sake of simplicity, we will use the symbol $A = (\mu_A, \lambda_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\}$.

Definition 2.6. Let $(R, +, \cdot)$ be an ordinary ring. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in $R$ is called a left (resp., right) intuitionistic fuzzy ideal of $R$ if

1. $\min\{\mu_A(x), \mu_A(y)\} \leq \mu_A(x - y)$ for all $x, y \in R$,
2. $\mu_A(y) \leq \mu_A(x \cdot y)$ (resp., $\mu_A(x) \leq \mu_A(x \cdot y)$) for all $x, y \in R$,
3. $\lambda_A(x - y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$ for all $x, y \in R$,
4. $\lambda_A(x \cdot y) \leq \lambda_A(y)$ (resp., $\lambda_A(x \cdot y) \leq \lambda_A(x)$) for all $x, y \in R$.

3. Intuitionistic fuzzy $H_v$-ideals

In what follows, let $R$ denote an $H_v$-ring, and we start by defining the notion of intuitionistic fuzzy $H_v$-ideals.

Definition 3.1. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in $R$ is called a left (resp., right) intuitionistic fuzzy $H_v$-ideal of $R$ if

1. $\min\{\mu_A(x), \mu_A(y)\} \leq \inf\{\mu_A(z) \mid z \in x + y\}$ for all $x, y \in R$,
2. for all $x, a \in R$, there exist $y, z \in R$ such that $x \in (a + y) \cap (z + a)$ and

$$\min\{\mu_A(a), \mu_A(x)\} \leq \min\{\mu_A(y), \mu_A(z)\},$$

(3.1)

3. $\mu_A(y) \leq \inf\{\mu_A(z) \mid z \in x \cdot y\}$ (resp., $\mu_A(x) \leq \inf\{\mu_A(z) \mid z \in x \cdot y\}$) for all $x, y \in R$,

4. $\sup\{\lambda_A(z) \mid z \in x + y\} \leq \max\{\lambda_A(x), \lambda_A(y)\}$ for all $x, y \in R$,
5. for all $x, a \in R$, there exist $y, z \in R$ such that $x \in (a + y) \cap (z + a)$ and

$$\max\{\lambda_A(y), \lambda_A(z)\} \leq \max\{\lambda_A(a), \lambda_A(x)\},$$

(3.2)

6. $\sup\{\lambda_A(z) \mid z \in x \cdot y\} \leq \lambda_A(y)$ (resp., $\sup\{\lambda_A(z) \mid z \in x \cdot y\} \leq \lambda_A(x)$) for all $x, y \in R$. 

Example 3.2. Let \( \mu \) be a left fuzzy \( H_v \)-ideal of \((R, \cup, \otimes)\) defined in Example 2.3. Then, as it is not difficult to see, \( A = (\mu_A, \mu_A^c) \) is a left intuitionistic fuzzy \( H_v \)-ideal of \((R, \cup, \otimes)\).

Here we present all the proofs for left \( H_v \)-ideals. For right \( H_v \)-ideals, similar results hold as well.

Lemma 3.3. If \( A = (\mu_A, \lambda_A) \) is a left intuitionistic fuzzy \( H_v \)-ideal of \( R \), then so is \( \square A = (\mu_A^c, \mu_A) \).

Proof. It is sufficient to show that \( \mu_A^c \) satisfies the conditions (4), (5), (6) of Definition 3.1.

For \( x, y \in R \), we have
\[
\min \{ \mu_A(x), \mu_A(y) \} \leq \inf \{ \mu_A(z) \mid z \in x \cdot y \},
\]
and so
\[
\min \{ 1 - \mu_A^c(x), 1 - \mu_A^c(y) \} \leq \inf \{ 1 - \mu_A^c(z) \mid z \in x \cdot y \}.
\]
Hence
\[
\min \{ 1 - \mu_A^c(x), 1 - \mu_A^c(y) \} \leq 1 - \sup \{ \mu_A^c(z) \mid z \in x \cdot y \},
\]
which implies that
\[
\sup \{ \mu_A^c(z) \mid z \in x \cdot y \} \leq 1 - \min \{ 1 - \mu_A^c(x), 1 - \mu_A^c(y) \}.
\]
Therefore
\[
\sup \{ \mu_A^c(z) \mid z \in x \cdot y \} \leq \max \{ \mu_A^c(x), \mu_A^c(y) \},
\]
in this way, Definition 3.1(4) is verified.

Now, let \( a, x \in R \), then there exist \( y, z \in R \) such that \( x \in (a + y) \cap (z + a) \) and
\[
\min \{ \mu_A(a), \mu_A(x) \} \leq \min \{ \mu_A(y), \mu_A(z) \}.
\]
So
\[
\min \{ 1 - \mu_A^c(a), 1 - \mu_A^c(x) \} \leq \min \{ 1 - \mu_A^c(y), 1 - \mu_A^c(z) \}.
\]
Hence
\[
\max \{ \mu_A^c(y), \mu_A^c(z) \} \leq \max \{ \mu_A^c(a), \mu_A^c(x) \},
\]
and Definition 3.1(5) is satisfied.

For the condition (6), let \( x, y \in R \), then since \( \mu_A \) is a left fuzzy \( H_v \)-ideal of \( R \), we have
\[
\mu_A(y) \leq \inf \{ \mu_A(z) \mid z \in x \cdot y \},
\]
and so
\[
1 - \mu_A^c(y) \leq \inf \{ 1 - \mu_A^c(z) \mid z \in x \cdot y \}.
\]
which implies that
\[ \sup \{ \mu_A'(z) \mid z \in x \cdot y \} \leq \mu_A'(y). \] (3.13)

Therefore Definition 3.1(6) is satisfied. \( \Box \)

**Lemma 3.4.** If \( A = (\mu_A, \lambda_A) \) is a left intuitionistic fuzzy \( H_v \)-ideal of \( R \), then so is \( \Diamond A = (\lambda_A, \lambda_A) \).

The proof is similar to the proof of Lemma 3.3.

Combining the above two lemmas, it is not difficult to see that the following theorem is valid.

**Theorem 3.5.** \( A = (\mu_A, \lambda_A) \) is a left intuitionistic fuzzy \( H_v \)-ideal of \( R \) if and only if \( \Box A \) and \( \Diamond A \) are left intuitionistic fuzzy \( H_v \)-ideals of \( R \).

**Corollary 3.6.** \( A = (\mu_A, \lambda_A) \) is a left intuitionistic fuzzy \( H_v \)-ideal of \( R \) if and only if \( \mu_A \) and \( \lambda_A \) are left fuzzy \( H_v \)-ideals of \( R \).

**Definition 3.7.** For any \( t \in [0,1] \) and fuzzy set \( \mu \) of \( R \), the set
\[ U(\mu; t) = \{ x \in R \mid \mu(x) \geq t \} \quad \text{(resp.,} \quad L(\mu; t) = \{ x \in R \mid \mu(x) \leq t \} \) (3.14)

is called an upper (resp., lower) \( t \)-level cut of \( \mu \).

**Theorem 3.8.** If \( A = (\mu_A, \lambda_A) \) is an intuitionistic fuzzy \( H_v \)-ideal of \( R \), then for every \( t \in \text{Im}(\mu_A) \cap \text{Im}(\lambda_A) \), the sets \( U(\mu_A; t) \) and \( L(\lambda_A; t) \) are \( H_v \)-ideals of \( R \).

**Proof.** Let \( t \in \text{Im}(\mu_A) \cap \text{Im}(\lambda_A) \subseteq [0,1] \) and let \( x, y \in U(\mu_A; t) \). Then \( \mu_A(x) \geq t \) and \( \mu_A(y) \geq t \), and so \( \min\{\mu_A(x), \mu_A(y)\} \geq t \). It follows from Definition 3.1(1) that \( \inf\{\mu_A(z) \mid z \in x \cdot y\} \geq t \). Therefore for all \( z \in x \cdot y \), we have \( z \in U(\mu_A; t) \). Hence for all \( a \in U(\mu_A; t) \), we have \( a + U(\mu_A; t) \subseteq U(\mu_A; t) \) and \( U(\mu_A; t) + a \subseteq U(\mu_A; t) \). Now, let \( x \in U(\mu_A; t) \), then there exist \( y, z \in R \) such that \( x \in (a + y) \cap (z + a) \) and \( \min\{\mu_A(x), \mu_A(a)\} \leq \min\{\mu_A(y), \mu_A(z)\} \). Since \( x, a \in U(\mu_A; t) \), we have \( t \leq \min\{\mu_A(x), \mu_A(a)\} \), and so \( t \leq \inf\{\mu_A(y), \lambda_A(z)\} \), which implies that \( y \in U(\mu_A; t) \) and \( z \in U(\mu_A; t) \). This proves that \( U(\mu_A; t) \subseteq a + U(\mu_A; t) \) and \( U(\mu_A; t) \subseteq U(\mu_A; t) + a \).

Now, for every \( x \in R \) and \( y \in U(\mu_A; t) \), we show that \( x \cdot y \subseteq U(\mu_A; t) \). Since \( A \) is a left intuitionistic fuzzy \( H_v \)-ideal of \( R \), we have
\[ t \leq \mu_A(y) \leq \inf\{\mu_A(z) \mid z \in x \cdot y\}. \] (3.15)

Therefore, for every \( z \in x \cdot y \), we get \( \mu_A(z) \geq t \), which implies that \( z \in U(\mu_A; t) \), so \( x \cdot y \subseteq U(\mu_A; t) \).

If \( x, y \in L(\lambda_A; t) \), then \( \max\{\lambda_A(x), \lambda_A(y)\} \leq t \). It follows from Definition 3.1(4) that
\[ \sup\{\lambda_A(z) \mid z \in x + y\} \leq t. \] Therefore for all \( z \in x + y \), we have \( z \in L(\lambda_A; t) \), so \( x + y \subseteq L(\lambda_A; t) \). Hence for all \( a \in L(\lambda_A; t) \), we have \( a + L(\lambda_A; t) \subseteq L(\lambda_A; t) \) and \( L(\lambda_A; t) + a \subseteq L(\lambda_A; t) \). Now, let \( x \in L(\lambda_A; t) \), then there exist \( y, z \in R \) such that \( x \in (a + y) \cap (z + a) \) and \( \max\{\lambda_A(y), \lambda_A(z)\} \leq \max\{\lambda_A(a), \lambda_A(x)\} \). Since \( x, a \in L(\lambda_A; t) \), we have \( \max\{\lambda_A(a), \lambda_A(x)\} \leq t \), and so \( \max\{\lambda_A(y), \lambda_A(z)\} \leq t \). Thus \( y \in L(\lambda_A; t) \) and \( z \in L(\lambda_A; t) \). Hence \( L(\lambda_A; t) \subseteq a + L(\lambda_A; t) \) and \( L(\lambda_A; t) \subseteq L(\lambda_A; t) + a \).
Now, we show that \( x \cdot y \subseteq L(\lambda_A; t) \) for every \( x \in R \) and \( y \in L(\lambda_A; t) \). Since \( A \) is a left intuitionistic fuzzy \( H_v \)-ideal of \( R \), we have
\[
\sup \{ \lambda_A(z) \mid z \in x \cdot y \} \leq \lambda_A(y) \leq t. \tag{3.16}
\]
Therefore, for every \( z \in x \cdot y \), we get \( \lambda_A(z) \leq t \), which implies that \( z \in L(\lambda_A; t) \), so \( x \cdot y \subseteq L(\lambda_A; t) \).

**Theorem 3.9.** If \( A = (\mu_A, \lambda_A) \) is an intuitionistic fuzzy set of \( R \) such that all nonempty levels \( U(\mu_A; t) \) and \( L(\lambda_A; t) \) are \( H_v \)-ideals of \( R \), then \( A = (\mu_A, \lambda_A) \) is an intuitionistic fuzzy \( H_v \)-ideal of \( R \).

**Proof.** Assume that all nonempty levels \( U(\mu_A; t) \) and \( L(\lambda_A; t) \) are \( H_v \)-ideals of \( R \). If \( t_0 = \min \{ \mu_A(x), \mu_A(y) \} \) and \( t_1 = \max \{ \lambda_A(x), \lambda_A(y) \} \), then \( x, y \in U(\mu_A; t_0) \) and \( x, y \in L(\lambda_A; t_1) \). Therefore for all \( z \in x + y \), we have \( \mu_A(z) \geq t_0 \) and \( \lambda_A(z) \leq t_1 \), that is,
\[
\inf \{ \mu_A(z) \mid z \in x + y \} \geq \min \{ \mu_A(x), \mu_A(y) \},
\]
\[
\sup \{ \lambda_A(z) \mid z \in x + y \} \leq \max \{ \lambda_A(x), \lambda_A(y) \}, \tag{3.17}
\]
which verifies the conditions (1) and (4) of Definition 3.1.

Now, if \( t_2 = \min \{ \mu_A(a), \mu_A(x) \} \) for \( a, x \in R \), then \( a, x \in U(\mu_A; t_2) \). So there exist \( y_1, z_1 \in U(\mu_A; t_2) \) such that \( x \in a + y_1 \) and \( x \in z_1 + a \). Also we have \( t_2 \leq \min \{ \mu_A(y_1), \mu_A(z_1) \} \). Therefore, Definition 3.1(2) is verified. If we put \( t_3 = \max \{ \lambda_A(a), \lambda_A(x) \} \), then \( a, x \in L(\lambda_A; t_3) \). So there exist \( y_2, z_2 \in L(\lambda_A; t_3) \) such that \( x \in a + y_2 \) and \( x \in z_2 + a \), and we have \( \lambda_A(y_2), \lambda_A(y_2) \leq t_3 \), and so Definition 3.1(3) is verified.

Now, we verify the conditions (3) and (6). Let \( t_4 = \mu_A(y) \) and \( t_5 = \lambda_A(y) \) for some \( x, y \in R \). Then \( y \in U(\mu_A; t_4) \) and \( y \in L(\lambda_A; t_5) \). Since \( U(\mu_A; t_4) \) and \( L(\lambda_A; t_5) \) are \( H_v \)-ideals of \( R \), then \( x \cdot y \subseteq U(\mu_A; t_4) \) and \( x \cdot y \subseteq L(\lambda_A; t_5) \). Therefore for every \( z \in x \cdot y \), we have \( z \in U(\mu_A; t_4) \) and \( z \in L(\lambda_A; t_5) \) which imply that \( \mu_A(z) \geq t_4 \) and \( \lambda_A(z) \leq t_5 \). Hence
\[
\inf \{ \mu_A(z) \mid z \in x \cdot y \} \geq t_4 = \mu_A(y),
\]
\[
\sup \{ \lambda_A(z) \mid z \in x \cdot y \} \leq t_5 = \lambda_A(y). \tag{3.18}
\]

This completes the proof. □

**Corollary 3.10.** Let \( I \) be a left \( H_v \)-ideal of an \( H_v \)-ring \( R \). If fuzzy sets \( \mu \) and \( \lambda \) are defined on \( R \) by
\[
\mu(x) = \begin{cases} 
\alpha_0 & \text{if } x \in I, \\
\alpha_1 & \text{if } x \in R \setminus I,
\end{cases} \quad \lambda(x) = \begin{cases} 
\beta_0 & \text{if } x \in I, \\
\beta_1 & \text{if } x \in R \setminus I,
\end{cases}
\tag{3.19}
\]
where \( 0 \leq \alpha_1 < \alpha_0 \leq \beta_0 < \beta_1 \), and \( \alpha_i + \beta_i \leq 1 \) for \( i = 0, 1 \), then \( A = (\mu, \lambda) \) is an intuitionistic fuzzy \( H_v \)-ideal of \( R \) and \( U(\mu; \alpha_0) = I = L(\lambda; \beta_0) \).

**Corollary 3.11.** Let \( \chi_I \) be the characteristic function of a left \( H_v \)-ideal \( I \) of \( R \). Then \( A = (\chi_I, \chi_I) \) is a left intuitionistic fuzzy \( H_v \)-ideal of \( R \).
Theorem 3.12. If \( A = (\mu_A, \lambda_A) \) is a left intuitionistic fuzzy Hv-ideal of \( R \), then for all \( x \in R \),
\[
\mu_A(x) = \sup \{ \alpha \in [0,1] \mid x \in U(\mu_A;\alpha) \}, \\
\lambda_A(x) = \inf \{ \alpha \in [0,1] \mid x \in L(\lambda_A;\alpha) \}.
\] (3.20)

Proof. Let \( \delta = \sup \{ \alpha \in [0,1] \mid x \in U(\mu_A;\alpha) \} \) and let \( \varepsilon > 0 \) be given. Then \( \delta - \varepsilon < \alpha \) for some \( \alpha \in [0,1] \) such that \( x \in U(\mu_A;\alpha) \). This means that \( \delta - \varepsilon < \mu_A(x) \) so that \( \delta \leq \mu_A(x) \) since \( \varepsilon \) is arbitrary.

We now show that \( \mu_A(x) \leq \delta \). If \( \mu_A(x) = \beta \), then \( x \in U(\mu_A;\beta) \), and so \( \beta \in \{ \alpha \in [0,1] \mid x \in U(\mu_A;\alpha) \} \). Hence
\[
\mu_A(x) = \beta \leq \sup \{ \alpha \in [0,1] \mid x \in U(\mu_A;\alpha) \} = \delta.
\] (3.21)

Therefore
\[
\mu_A(x) = \delta = \sup \{ \alpha \in [0,1] \mid x \in U(\mu_A;\alpha) \}.
\] (3.22)

Now let \( \eta = \inf \{ \alpha \in [0,1] \mid x \in L(\lambda_A;\alpha) \} \). Then
\[
\inf \{ \alpha \in [0,1] \mid x \in L(\lambda_A;\alpha) \} < \eta + \varepsilon
\] (3.24)
for any \( \varepsilon > 0 \), and so \( \alpha < \eta + \varepsilon \) for some \( \alpha \in [0,1] \) with \( x \in L(\lambda_A;\alpha) \). Since \( \lambda_A(x) \leq \alpha \) and \( \varepsilon \) is arbitrary, it follows that \( \lambda_A(x) \leq \eta \).

To prove that \( \lambda_A(x) \geq \eta \), let \( \lambda_A(x) = \zeta \). Then \( x \in L(\lambda_A;\zeta) \), and thus \( \zeta \in \{ \alpha \in [0,1] \mid x \in L(\lambda_A;\alpha) \} \). Hence
\[
\inf \{ \alpha \in [0,1] \mid x \in L(\lambda_A;\alpha) \} \leq \zeta,
\] (3.25)
that is, \( \eta \leq \zeta = \lambda_A(x) \). Consequently
\[
\lambda_A(x) = \eta = \inf \{ \alpha \in [0,1] \mid x \in L(\lambda_A;\alpha) \},
\] (3.26)
which completes the proof. \( \square \)

4. Relations
Let \( \alpha \in [0,1] \) be fixed and let \( \text{IF}(R) \) be the family of all intuitionistic fuzzy left Hv-ideals of an Hv-ring \( R \). For any \( A = (\mu_A, \lambda_A) \) and \( B = (\mu_B, \lambda_B) \) from \( \text{IF}(R) \), we define two binary relations \( \Lambda^\alpha \) and \( \Sigma^\alpha \) on \( \text{IF}(R) \) as follows:
\[
(A,B) \in \Lambda^\alpha \iff U(\mu_A;\alpha) = U(\mu_B;\alpha), \\
(A,B) \in \Sigma^\alpha \iff L(\lambda_A;\alpha) = L(\lambda_B;\alpha).
\] (4.1)

These two relations \( \Lambda^\alpha \) and \( \Sigma^\alpha \) are equivalence relations. Hence \( \text{IF}(R) \) can be divided into
the equivalence classes of $\mathcal{U}^\alpha$ and $\mathcal{L}^\alpha$, denoted by $[A]_{\mathcal{U}^\alpha}$ and $[A]_{\mathcal{L}^\alpha}$ for any $A = (\mu_A, \lambda_A) \in \text{IF}(R)$, respectively. The corresponding quotient sets will be denoted as $\text{IF}(R)/\mathcal{U}^\alpha$ and $\text{IF}(R)/\mathcal{L}^\alpha$, respectively.

For the family $L I(R)$ of all left $H_\alpha$-ideals of $R$, we define two maps $U_\alpha$ and $L_\alpha$ from $\text{IF}(R)$ to $L I(R) \cup \{\emptyset\}$ putting

$$U_\alpha(A) = U(\mu_A; \alpha), \quad L_\alpha(A) = L(\lambda_A; \alpha) \quad (4.2)$$

for each $A = (\mu_A, \lambda_A) \in \text{IF}(R)$.

It is not difficult to see that these maps are well defined.

**Lemma 4.1.** For any $\alpha \in (0, 1)$, the maps $U_\alpha$ and $L_\alpha$ are surjective.

**Proof.** Let $0$ and $1$ be fuzzy sets on $R$ defined by $0(x) = 0$ and $1(x) = 1$ for all $x \in R$. Then $0_\alpha = (0, 1) \in \text{IF}(R)$ and $U_\alpha(0_\alpha) = L_\alpha(0_\alpha) = \emptyset$ for any $\alpha \in (0, 1)$. Moreover, for any $K \in L I(R)$, we have $L_\alpha(\chi_K, \chi_K^c) \in \text{IF}(R)$, $U_\alpha(L_\alpha(\chi_K, \chi_K^c)) = K$, and $L_\alpha(L_\alpha(\chi_K, \chi_K^c)) = K$. Hence $U_\alpha$ and $L_\alpha$ are surjective. \hfill $\square$

**Theorem 4.2.** For any $\alpha \in (0, 1)$, the sets $\text{IF}(R)/\mathcal{U}^\alpha$ and $\text{IF}(R)/\mathcal{L}^\alpha$ are equipotent to $L I(R) \cup \{\emptyset\}$.

**Proof.** Let $\alpha \in (0, 1)$. Putting $U_\alpha^*([A]_{\mathcal{U}^\alpha}) = U_\alpha(A)$ and $L_\alpha^*([A]_{\mathcal{L}^\alpha}) = L_\alpha(A)$ for any $A = (\mu_A, \lambda_A) \in \text{IF}(R)$, we obtain two maps:

$$U_\alpha^*: \text{IF}(R)/\mathcal{U}^\alpha \longrightarrow L I(R) \cup \{\emptyset\}, \quad L_\alpha^*: \text{IF}(R)/\mathcal{L}^\alpha \longrightarrow L I(R) \cup \{\emptyset\}. \quad (4.3)$$

If $U(\mu_A; \alpha) = U(\mu_B; \alpha)$ and $L(\lambda_A; \alpha) = L(\lambda_B; \alpha)$ for some $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ from $\text{IF}(R)$, then $(A, B) \in \mathcal{U}^\alpha$ and $(A, B) \in \mathcal{L}^\alpha$, whence $[A]_{\mathcal{U}^\alpha} = [B]_{\mathcal{U}^\alpha}$ and $[A]_{\mathcal{L}^\alpha} = [B]_{\mathcal{L}^\alpha}$, which means that $U_\alpha^*$ and $L_\alpha^*$ are injective.

To show that the maps $U_\alpha^*$ and $L_\alpha^*$ are surjective, let $K \in L I(R)$. Then for $L_\alpha(\chi_K, \chi_K^c) \in \text{IF}(R)$, we have $U_\alpha^*(L_\alpha(\chi_K, \chi_K^c)) = U(\chi_K, \chi_K^c) = K$ and $L_\alpha^*(L_\alpha(\chi_K, \chi_K^c)) = L(\chi_K^c, \chi_K) = K$. Also $0_\alpha = (0, 1) \in \text{IF}(R)$. Moreover, $U_\alpha^*((0_\alpha)_{\mathcal{U}^\alpha}) = U(0; \alpha) = \emptyset$ and $L_\alpha^*((0_\alpha)_{\mathcal{L}^\alpha}) = L(1; \alpha) = \emptyset$. Hence $U_\alpha^*$ and $L_\alpha^*$ are surjective. \hfill $\square$

Now for any $\alpha \in [0, 1]$, we have the new relation $\mathcal{R}^\alpha$ on $\text{IF}(R)$ putting

$$(A, B) \in \mathcal{R}^\alpha \iff U(\mu_A; \alpha) \cap L(\lambda_A; \alpha) = U(\mu_B; \alpha) \cap L(\lambda_B; \alpha), \quad (4.4)$$

where $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$. Obviously, $\mathcal{R}^\alpha$ is an equivalence relation.

**Lemma 4.3.** The map $I_\alpha: \text{IF}(R) \rightarrow L I(R) \cup \{\emptyset\}$ defined by

$$I_\alpha(A) = U(\mu_A; \alpha) \cap L(\lambda_A; \alpha), \quad (4.5)$$

where $A = (\mu_A, \lambda_A)$, is surjective for any $\alpha \in (0, 1)$.

**Proof.** Indeed, if $\alpha \in (0, 1)$ is fixed, then for $0_\alpha = (0, 1) \in \text{IF}(R)$, we have

$$I_\alpha(0_\alpha) = U(0; \alpha) \cap L(1; \alpha) = \emptyset, \quad (4.6)$$
and for any $K \in LI(R)$, there exists $I_\gamma = (\gamma(x), \gamma(x)) \subseteq IF(R)$ such that $I_\gamma(I_\gamma) = U(\gamma(x); \alpha) \cap L(\gamma(x); \alpha) = K$.

**Theorem 4.4.** For any $\alpha \in (0,1)$, the quotient set $IF(R)/\gamma^*$ is equipotent to $LI(R) \cup \{\emptyset\}$.

**Proof.** Let $I^*_\alpha : IF(R)/\gamma \rightarrow LI(R) \cup \{\emptyset\}$, where $\alpha \in (0,1)$, be defined by the formula

$$I^*_\alpha ([A]_\gamma) = I_\alpha(A) \quad \text{for each } [A]_\gamma \in IF(R)/\gamma^*.$$  \hfill (4.7)

If $I^*_\alpha ([A]_\gamma) = I^*_\alpha ([B]_\gamma)$ for some $[A]_\gamma, [B]_\gamma \in IF(R)/\gamma^*$, then

$$U(\mu_A; \alpha) \cap L(\lambda_A; \alpha) = U(\mu_B; \alpha) \cap L(\lambda_B; \alpha),$$

which implies that $(A, B) \in \gamma^*$ and, in the consequence, $[A]_\gamma = [B]_\gamma$. Thus $I^*_\alpha$ is injective.

It is also onto because $I^*_\alpha([0\_]) = I_\alpha([0\_]) = \emptyset$ for $[0\_] = (0,1) \subseteq IF(R)$, and $I^*_\alpha(I_\gamma) = I_\alpha(K) = K$ for $K \in LI(R)$ and $I_\gamma = (\chi_{K}, \chi_{K}^c) \subseteq IF(R)$.

The relation $\gamma^*$ is the smallest equivalence relation on $R$ such that the quotient $R/\gamma^*$ is a ring. $\gamma^*$ is called the fundamental equivalence relation on $R$ and $R/\gamma^*$ is called the fundamental ring, see [16].

According to [16], if $\mathcal{U}$ denotes the set of all finite polynomials of elements of $R$ over $\mathbb{N}$, then a relation $\gamma$ can be defined on $R$ as follows:

$$xyy \iff \{x, y\} \subseteq u \text{ for some } u \in \mathcal{U}. \hfill (4.9)$$

According to [16], the transitive closure of $\gamma$ is the fundamental relation $\gamma^*$, that is, $a\gamma^*b$ if and only if there exist $x_1, \ldots, x_{m+1} \in R, u_1, \ldots, u_m \in \mathcal{U}$ with $x_1 = a, x_{m+1} = b$ such that

$$\{x_i, x_{i+1}\} \subseteq u_i, \quad i = 1, \ldots, m. \hfill (4.10)$$

Suppose that $\gamma^*(a)$ is the equivalence class containing $a \in R$. Then both the sum $\oplus$ and the product $\odot$ on $R/\gamma^*$ are defined as follows:

$$\gamma^*(a) \oplus \gamma^*(b) = \gamma^*(c) \quad \forall c \in \gamma^*(a) + \gamma^*(b),$$

$$\gamma^*(a) \odot \gamma^*(b) = \gamma^*(d) \quad \forall c \in \gamma^*(a) \cdot \gamma^*(b). \hfill (4.11)$$

We denote the unit of the group $(R/\gamma^*, \oplus)$ by $\omega_R$.

**Definition 4.5.** Let $R$ be an $H_\gamma$-ring and $\mu$ a fuzzy subset of $R$. The fuzzy subset $\mu_{\gamma^*}$ on $R/\gamma^*$ is defined as follows:

$$\mu_{\gamma^*} : R/\gamma^* \rightarrow [0,1], \quad \mu_{\gamma^*}(\gamma^*(x)) = \sup \{\mu(a) \mid a \in \gamma^*(x)\}. \hfill (4.12)$$

**Theorem 4.6.** Let $R$ be an $H_\gamma$-ring and $A = (\mu_A, \lambda_A)$ a left intuitionistic fuzzy $H_\gamma$-ideal of $R$. Then $A/\gamma^* = (\mu_{\gamma^*}, \lambda_{\gamma^*})$ is a left intuitionistic fuzzy ideal of the fundamental ring $R/\gamma^*$. 

References


Bijan Davvaz: Department of Mathematics, Yazd University, P.O. Box 89195-741, Yazd, Iran

*E-mail address*: davvaz@yazduni.ac.ir

Wieslaw A. Dudek: Institute of Mathematics and Computer Science, Wroclaw University of Technology, Ul. Wybrzeze Wyspianskiego 27, 50-370 Wroclaw, Poland

*E-mail address*: dudek@im.pwr.wroc.pl
Submit your manuscripts at http://www.hindawi.com