STRUCTURE OF RINGS WITH CERTAIN CONDITIONS ON ZERO DIVISORS

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Let $R$ be a ring such that every zero divisor $x$ is expressible as a sum of a nilpotent element and a potent element of $R$: $x = a + b$, where $a$ is nilpotent, $b$ is potent, and $ab = ba$. We call such a ring a $D^*$-ring. We give the structure of periodic $D^*$-ring, weakly periodic $D^*$-ring, Artinian $D^*$-ring, semiperfect $D^*$-ring, and other classes of $D^*$-ring.

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1. Introduction

Throughout this paper, $R$ is an associative ring; and $N$, $C$, $C(R)$, and $J$ denote, respectively, the set of nilpotent elements, the center, the commutator ideal, and the Jacobson radical. An element $x$ of $R$ is called potent if $x^n = x$ for some positive integer $n = n(x) > 1$. A ring $R$ is called periodic if for every $x$ in $R$, $x^m = x^n$ for some distinct positive integers $m = m(x)$, $n = n(x)$. A ring $R$ is called weakly periodic if every element of $R$ is expressible as a sum of a nilpotent element and a potent element of $R$: $R = N + P$, where $P$ is the set of potent elements of $R$. A ring $R$ such that every zero divisor is nilpotent is called a $D$-ring. The structure of certain classes of $D$-rings was studied in [1]. Following [7], $R$ is called normal if all of its idempotents are in $C$. A ring $R$ is called a $D^*$-ring, if every zero divisor $x$ in $R$ can be written as $x = a + b$, where $a \in N$, $b \in P$, and $ab = ba$. Clearly every $D$-ring is a $D^*$-ring. In particular every nil ring is a $D^*$-ring, and every domain is a $D^*$-ring. A Boolean ring is a $D^*$-ring but not a $D$-ring. Our objective is to study the structure of certain classes of $D^*$-ring.

2. Main results

We start by stating the following known lemmas: Lemmas 2.1 and 2.2 were proved in [5], Lemmas 2.3 and 2.4 were proved in [4].

**Lemma 2.1.** Let $R$ be a weakly periodic ring. Then the Jacobson radical $J$ of $R$ is nil. If, furthermore, $xR \subseteq N$ for all $x \in N$, then $N = J$ and $R$ is periodic.

**Lemma 2.2.** If $R$ is a weakly periodic division ring, then $R$ is a field.
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Lemma 2.3. Let \( R \) be a periodic ring and \( x \) any element of \( R \). Then

(a) some power of \( x \) is idempotent;
(b) there exists an integer \( n > 1 \) such that \( x - x^n \in N \).

Lemma 2.4. Let \( R \) be a periodic ring and let \( \sigma : R \to S \) be a homomorphism of \( R \) onto a ring \( S \). Then the nilpotents of \( S \) coincide with \( \sigma(N) \), where \( N \) is the set of nilpotents of \( R \).

Definition 2.5. A ring is said to be a \( D \)-ring if every zero divisor is nilpotent. A ring \( R \) is called a \( D^\ast \)-ring if every zero divisor \( x \) in \( R \) can be written as \( x = a + b \), where \( a \in N \), \( b \in P \), and \( ab = ba \).

Theorem 2.6. A ring \( R \) is a \( D^\ast \)-ring if and only if every zero divisor of \( R \) is periodic.

Proof. Assume \( R \) is a \( D^\ast \)-ring and let \( x \) be any zero divisor. Then

\[
x = a + b, \quad a \in N, \ b \in P, \ ab = ba.
\] (2.1)

So, \( (x - a) = b = b^n = (x - a)^n \). This implies, since \( x \) commutes with \( a \), that \( (x - a) = (x - a)^n = x^n + \text{sum of pairwise commuting nilpotent elements} \).

Hence

\[
x - x^n \in N \quad \text{for every zero divisor} \; x. \tag{2.2}
\]

Since each such \( x \) is included in a subring of zero divisors, which is periodic by Chacron’s theorem, \( x \) is periodic.

Suppose, conversely, that each zero divisor is periodic. Then by the proof of [4, Lemma 1], \( R \) is a \( D^\ast \)-ring.

Theorem 2.7. If \( R \) is any normal \( D^\ast \)-ring, then either \( R \) is periodic or \( R \) is a \( D \)-ring. Moreover, \( aR \subseteq N \) for each \( a \in N \).

Proof. If \( R \) is a normal \( D^\ast \)-ring which is not a \( D \)-ring, then \( R \) has a central idempotent \( e \). Then \( R = eR \oplus A(e) \), where \( eR \) and \( A(e) \) both consist of zero divisors of \( R \), hence (in view of Theorem 2.6) are periodic. Therefore \( R \) is periodic.

Now consider \( a \in N \) and \( x \in R \). Since \( ax \) is a zero divisor, hence a periodic element, \( (ax)^j = e \) is a central idempotent for some \( j \). Thus \( (ax)^{j+1} = (ax)^j ax = a^2 y \) for some \( y \in R \). Repeating this argument, one can show that for each positive integer \( k \), there exists \( m \) such that \( (ax)^m = a^2 w \) for some \( w \in R \). Therefore \( aR \subseteq N \).

Corollary 2.8. Let \( R \) be a \( D^\ast \)-ring which is not a \( D \)-ring. If \( N \subseteq C \), then \( R \) is commutative.

Proof. Since \( N \subseteq C \), \( R \) is normal. Therefore commutativity follows from Theorem 2.7 and a theorem of Herstein.

Now, we prove the following result for \( D^\ast \)-rings.

Theorem 2.9. Let \( R \) be a normal \( D^\ast \)-ring.

(i) If \( R \) is weakly periodic, then \( N \) is an ideal of \( R \), \( R \) is periodic, and \( R \) is a subdirect sum of nil rings and/or local rings \( R_i \). Furthermore, if \( N_i \) is the set of nilpotents of the local ring \( R_i \), then \( R_i/N_i \) is a periodic field.
(ii) If $R$ is Artinian, then $N$ is an ideal and $R/N$ is a finite direct product of division rings.

**Proof.** (i) Using Theorem 2.7, we have

$$aR \subseteq N \quad \text{for every } a \in N. \quad (2.3)$$

This implies, using Lemma 2.1, that $N = J$ is an ideal of $R$, and $R$ is periodic. As is well-known, we have

$$R \cong \text{a subdirect sum of subdirectly irreducible rings } R_i. \quad (2.4)$$

Let $\sigma : R \to R_i$ be the natural homomorphism of $R$ onto $R_i$. Since $R$ is periodic, $R_i$ is periodic and by Lemma 2.4,

$$N_i = \text{the set of nilpotents of } R_i = \sigma(N) \text{ is an ideal of } R_i. \quad (2.5)$$

We now distinguish two cases.

**Case 1** $1 \notin R_i$. Let $x_i \in R_i$, and let $\sigma : x \to x_i$. Then by Lemma 2.3, $x^k$ is a central idempotent of $R$, and hence $x_i^k$ is a central idempotent in the subdirectly irreducible ring $R_i$, for some positive integer $k$. Hence $x_i^k = 0$ ($1 \notin R_i$). Thus $R_i = N_i$ is a nil ring.

**Case 2** $1 \in R_i$. The above argument in Case 1 shows that $x_i^k$ is a central idempotent in the subdirectly irreducible ring $R_i$. Hence $x_i^k = 0$ or $x_i^k = 1$ for all $x_i \in R_i$. So, $R_i$ is a local ring and for every $x_i + N_i \in R_i/N_i$,

$$x_i + N_i = N_i \quad \text{or} \quad (x_i + N_i)^k = 1 + N_i. \quad (2.6)$$

So $R_i/N_i$ is a periodic division ring, and hence by Lemma 2.2, $R_i/N_i$ is a periodic field.

(ii) Suppose $R$ is Artinian. Using (2.3), $aR$ is a nil right ideal for every $a \in N$. So, $N \subseteq J$. But $J \subseteq N$ since $R$ is Artinian. Therefore $N = J$ is an ideal of $R$ and $R/N = R/J$ is semisimple Artinian. This implies that $R/N$ is isomorphic to a finite direct product $R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ is a complete $t_i \times t_i$ matrix ring over a division ring $D_i$. Since $R$ is Artinian, the idempotents of $R/J$ lift to idempotents in $R$ [2], and hence the idempotents of $R/J$ are central. If $t_j > 1$, then $E_{11} \in R_j$, and $(0, \ldots, 0, E_{11}, 0, \ldots, 0)$ is an idempotent element of $R/J$ which is not central in $R/J$. This is a contradiction. So $t_i = 1$ for every $i$. Therefore each $R_i$ is a division ring and $R/N$ is isomorphic to a finite direct product of division rings. □

The next result deals with a special kind of $D^*$-rings.

**Theorem 2.10.** Let $R$ be a ring such that every zero divisor $x$ can be written uniquely as $x = a + e$, where $a \in N$ and $e$ is idempotent.

(i) If $R$ is weakly periodic, then $N$ is an ideal of $R$, and $R/N$ is isomorphic to a subdirect sum of fields.

(ii) If $R$ is Artinian, then $N$ is an ideal and $R/N$ is a finite direct product of division rings.
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Proof. Let \( e^2 = e \in R, x \in R, \) and let \( f = e + ex - exe. \) Then \( f^2 = f \) and hence \((ef - e)f = 0.\) So if \( f \) is not a zero divisor, then \( ef - e = 0.\) So \( ef = e, \) and thus \( f = e, \) which implies that \( ex = exe. \) The net result is \( ex - exe = 0 \) if \( f \) is not a zero divisor. Next, suppose \( f \) is a zero divisor. Then since

\[
\begin{align*}
f & = e + ex - exe, \\
f^2 & = f \\
(ef - e)f & = 0.
\end{align*}
\]

So if \( f \) is not a zero divisor, then \( ef - e = 0. \) So \( ef = e, \) and thus

\[
f = e, \quad \text{which implies that } \quad ex = exe.
\]

The net result is \( ex - exe = 0 \) if \( f \) is not a zero divisor. Next, suppose \( f \) is a zero divisor. Then since \( f = en + f \),

\[
(2.7)
\]

it follows from uniqueness that \( ex - exe = 0, \) and hence \( ex = exe \) in all cases. Similarly \( xe = exe, \) and thus

all idempotents of \( R \) are central, and hence \( R \) is a normal \( D^* \)-ring. \((2.8)\)

(i) Using \((2.8), \) \( R \) satisfies all the hypotheses of Theorem 2.9(i), and hence \( N \) is an ideal, and \( R \) is periodic. Using Lemma 2.2, for each \( x \in R, \) there exists an integer \( k > 1, \) such that

\[
(x + N)^k = (x + N), \quad k = k(x) > 1.
\]

\((2.9)\)

By a well-known theorem of Jacobson [6], \((2.9)\) implies that \( R/N \) is a subdirect sum of fields.

(ii) If \( R \) is Artinian, then using \((2.8), \) \( R \) satisfies the hypotheses of Theorem 2.9(ii). Therefore \( N \) is an ideal and \( R/N \) is a finite direct product of division rings. \( \square \)

Theorem 2.11. Let \( R \) be a semiprime \( D^* \)-ring with \( N \) commutative. Then \( R \) is either a domain or a \( J \)-ring.

Proof. As in the proof of [3, Theorem 1] we can show that if \( a^k = 0, \) then \( (ar)^k = 0 \) for all \( r \in R. \) Therefore, by Levitzkii’s theorem, \( N = \{0\}. \) Assume \( R \) is not a domain, and let \( a \) be any nonzero divisor of zero. Then \( a \) is potent and \( aR \) consists of zero divisors, hence is a \( J \)-ring containing \( a. \) Therefore \([ax, a] = 0 \) for all \( x \in R, \) hence \((ax)^n = a^n x^n \) for all \( x \in R, \) and all \( n \geq 2. \) For \( x \) not a zero divisor, choose \( n > 1, \) such that \( a^n = a \) and \((ax)^n = ax. \) Then

\[
a^n x^n = ax, \quad \text{so } a(x^n - x) = 0 \quad \text{and } x^n - x \quad \text{is a zero divisor, hence is periodic. It follows by Chacron’s theorem that } \quad R \quad \text{is a periodic ring; and since } \quad N = \{0\}, \quad R \quad \text{is a } J \text{-ring.} \quad \square\)

Example 2.12. Let

\[
R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \quad 0, 1 \in GF(2).
\]

\((2.10)\)

Then \( R \) is a normal weakly periodic \( D^* \)-ring with commuting nilpotents. \( R \) is not semiprime since the set of nilpotent elements \( N \) is a nonzero nilpotent ideal. This example shows that we cannot drop the hypothesis “\( R \) is semiprime” in Theorem 2.11.

In Theorem 2.14 below, we study the structure of a special kind of \( D^* \)-rings, the class of rings in which every zero divisor is potent. Recall that a ring is semiperfect [2] if and
only if \( R/J \) is semisimple (Artinian) and idempotents lift modulo \( J \). We need the following lemma.

**Lemma 2.13.** Let \( R \) be a ring in which every zero divisor is potent. Then \( N = \{0\} \) and \( R \) is normal. Moreover, if \( R \) is not a domain, then \( J = \{0\} \).

**Proof.** If \( a \in N \), then \( a \) is a zero divisor and hence potent by hypothesis. So \( a^n = a \) for some positive integer \( n \), and since \( a \in N \), there exists a positive integer \( k \) such that \( 0 = a^{nk} = a \). So \( N = \{0\} \). Let \( e \) be any idempotent element of \( R \) and \( x \) is any element of \( R \). Then \( ex - exe \in N \), and hence \( ex - exe = 0 \). Similarly, \( xe = exe \). So \( ex = xe \) and \( R \) is normal.

Let \( x \) be a nonzero divisor of zero. Then \( xJ \) consists of zero divisors, which are potent. Therefore \( xJ = \{0\} \). But then \( J \) consists of zero divisors, hence potent elements, and therefore \( J = \{0\} \). \( \square \)

**Theorem 2.14.** Let \( R \) be a ring such that every zero divisor is potent.

(i) If \( R \) is weakly periodic, then every element of \( R \) is potent and \( R \) is a subdirect sum of fields.

(ii) If \( R \) is prime, then \( R \) is a domain.

(iii) If \( R \) is Artinian, then \( R \) is a finite direct product of division rings.

(iv) If \( R \) is semiperfect, then \( R/J \) is a finite direct product of division rings.

**Proof.** (i) Since \( R \) is weakly periodic, every element \( x \in R \) can be written as

\[ x = a + b, \quad \text{where } a \in N, \ b \text{ is potent}. \tag{2.11} \]

But \( N = \{0\} \) (Lemma 2.13), so every \( x \in R \) is potent and hence \( R \) is isomorphic to a subdirect sum of fields by a well-known theorem of Jacobson.

(ii) Suppose \( R \) is a prime, then \( R \) is a prime ring with \( N = \{0\} \), and hence \( R \) is a domain.

(iii) Let \( R \) be an Artinian ring such that every zero divisor is potent. Since \( N = \{0\} \) (Lemma 2.13) and \( R \) is Artinian, \( J = N = \{0\} \). So \( R \) is semisimple Artinian and hence it is isomorphic to a finite direct product \( R_1 \times R_2 \times \cdots \times R_n \), where each \( R_i \) is a complete \( t_i \times t_i \) matrix ring over a division ring \( D_i \). If \( t_i > 1 \), then \( E_{11} \in R_j \), and \( (0, \ldots, 0, E_{11}, 0, \ldots, 0) \) is an idempotent element of \( R \) which is not central in \( R \) contradicting Lemma 2.13. So \( t_i = 1 \) for every \( i \). Therefore each \( R_i \) is a division ring and \( R \) is isomorphic to a finite direct product of division rings.

(iv) Let \( R \) be a semiperfect ring such that every zero divisor is potent. Then \( R/J \) is semisimple Artinian and hence it is isomorphic to a finite direct product \( R_1 \times R_2 \times \cdots \times R_n \), where each \( R_i \) is a complete \( t_i \times t_i \) matrix ring over a division ring \( D_i \). Since \( R \) is semiperfect, the idempotents of \( R/J \) lift to idempotents in \( R \), and hence the argument of part (iii) above implies that each \( R_i \) is a division ring and \( R/J \) is isomorphic to a finite direct product of division rings. \( \square \)

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