A RADICAL FOR RIGHT NEAR-RINGS: THE RIGHT JACOBSON RADICAL OF TYPE-0

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The notions of a right quasiregular element and right modular right ideal in a near-ring are initiated. Based on these $J_0^r(R)$, the right Jacobson radical of type-0 of a near-ring R is introduced. It is obtained that J_0^r is a radical map and $N(R) \subseteq J_0^r(R)$, where $N(R)$ is the nil radical of a near-ring R . Some characterizations of $J_0^r(R)$ are given and its relation with some of the radicals is also discussed.

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1. Introduction

Throughout this paper, R stands for a right near-ring. The structure of matrix near-rings was studied by Rao in [3, 4]. It is clear from these papers that right Jacobson-type radicals have an important role to play in the study of Meldrum-van der Walt matrix near-rings. This motivated the authors to develop the right Jacobson-type radicals of near-rings. The aim of this paper is to give a good beginning in this direction of investigation. Left quasiregularity was introduced and studied in near-rings. In this paper, right quasiregularity is developed, and right modules of near-rings and the right Jacobson radical of type-0 are studied.

In Section 2, the notions of a right quasiregular element and a right modular right ideal are introduced. Using these right 0-primitive ideal, right 0-primitive near-ring, and $J_0^r(R)$, the right Jacobson radical of R of type-0 are introduced. A nil subset of R is right quasiregular and the constant part of R is also right quasiregular. It is shown that J_0^r is a radical map and $J_0^r(R)$ is the largest right quasiregular ideal of R . Moreover, $N(R)$, the nil radical of R , is contained in $J_0^r(R)$ and that $P(R) \subseteq N(R) \subseteq J_0^r(R)$. If R is a zero-symmetric near-ring with DCC on the left R -subgroups of R , then $J_0(R) \subseteq J_0^r(R)$.

In Section 3, a right R -group of type-0 is introduced. It is shown that G is a right R -group of type-0 if and only if G is right R -isomorphic to R/K , for some maximal right modular right ideal K of R . A right 0-primitive ideal of R is a prime ideal of R . An ideal P of a d.g. near-ring R is a right 0-primitive ideal of R if and only if $P = (0 : G)$, for some right R -group G of type-0.

2 The right Jacobson radical of type-0

In Section 4, the right Jacobson radical of type-0 of a biregular near-ring is studied. For such a near-ring R , it is shown that $J'_0(R) = J_0(R) = \{0\}$. Moreover, if R is biregular, then an ideal P of R is right 0-primitive if and only if P is left 0-primitive if and only if P is a maximal right and left modular ideal. It is shown by an example that in general $J'_0(R)$ differs from the existing Jacobson-type radicals of R and the J'_0 -radical class contains almost all the classes of near-rings with trivial multiplication.

2. Right quasiregularity and the right J_0 -radical

Throughout this paper, R is a right near-ring.

Definition 2.1. An element $a \in R$ is called *right quasiregular* if and only if the right ideal of R generated by the set $\{x - ax \mid x \in R\}$ is R .

Remark 2.2. Note that unlike in the left quasiregularity case, if R is a near-ring and $a \in R$, then the right ideal of R generated by the set $\{x - ax \mid x \in R\}$ is R if and only if a is in the right ideal of R generated by the set $\{x - ax \mid x \in R\}$.

Definition 2.3. A right ideal (left ideal, ideal, subset) K of R is called a *right quasiregular right ideal (left ideal, ideal, subset)* of R , if each element of K is right quasiregular.

LEMMA 2.4. *A nilpotent element of R is right quasiregular.*

Proof. Let a be a nilpotent element in R , say $a^n = 0$ for some positive integer n . Let K be the right ideal of R generated by $x - ax$, $x \in R$. Observe that $x = x - a^n x = (x - ax) + (ax - a(ax)) + (a^2 x - a(a^2 x)) + \cdots + (a^{n-1} x - a(a^{n-1} x)) \in K$. Therefore, $K = R$ and hence a is right quasiregular. $\square$

One of the major differences between right and left quasiregular elements of R and hence between its right and left Jacobson-type radicals is the following.

LEMMA 2.5. *The constant part of R is right quasiregular.*

Proof. Let R_c be the constant part of R and $a \in R_c$. For $x \in R$, $(x + a) - a(x + a) = (x + a) - a = x$. Therefore, a is right quasiregular. So, R_c is right quasiregular. $\square$

LEMMA 2.6. *Let e be a nonzero distributive idempotent in R . Then e is not a right quasiregular element of R .*

Proof. Let e be a nonzero distributive idempotent in R . Now $(0 : e) = \{a \in R \mid ea = 0\}$ is a right ideal of R . Also $x - ex \in (0 : e)$ for all $x \in R$. But e is not in $(0 : e)$. So $(0 : e) \neq R$. Therefore, e is not right quasiregular. $\square$

Definition 2.7. A right ideal K of R is called *right modular* if there is an element $e \in R$ such that $x - ex \in K$ for all $x \in R$. In this case K is said to be *right modular by e* .

PROPOSITION 2.8. *Let $e \in R$. Then e is right quasiregular if and only if no proper right ideal of R is right modular by e .*

The proof follows from the definitions.

Remark 2.9. Let K be a right ideal of R . If K is right modular by e , then $e \in K$ if and only if $K = R$.

PROPOSITION 2.10. *Let K be a proper right ideal of R . If K is, right modular, then K is contained in a maximal right ideal of R which is also right modular.*

The proof of this result is easy and hence omitted.

PROPOSITION 2.11. *Let K and L be right modular right ideals of R and $R = K + L$. Then $K \cap L$ is also a right modular right ideal of R .*

Proof. Suppose that K is right modular by e_1 and L is right modular by e_2 . Let $e_1 = b_{11} + b_{12}$ and let $e_2 = b_{21} + b_{22}$, where $b_{11}, b_{21} \in K$ and $b_{12}, b_{22} \in L$. Let $e = b_{21} + b_{12}$. Let $r \in R$; $r - er = r - (b_{21} + b_{12})r = r - b_{12}r - b_{21}r = (r - e_1r) + (b_{11}r - b_{21}r) \in K$; $r - er = r - (b_{21} + b_{12})r = r - b_{12}r - b_{21}r = (r - b_{12}r + b_{22}r - r) + (r - e_2r) \in L$. Therefore, $r - er \in K \cap L$, and hence $K \cap L$ is a right modular right ideal. $\square$

PROPOSITION 2.12. *If $K_1, K_2, \dots, K_n$ are maximal right modular right ideals of R such that $\bigcap_{i=1}^n K_i = \{0\}$, then R has a left identity.*

The proof follows from Proposition 2.11.

Definition 2.13. $J'_{1/2}(R)$ is the intersection of all maximal right modular right ideals of R and if R has no maximal right modular right ideals, then $J'_{1/2}(R) = R$.

THEOREM 2.14. $J'_{1/2}(R)$ is the largest right quasiregular right ideal of R .

Proof. Let $q \in J'_{1/2}(R)$. Suppose that q is not right quasiregular. Let K be the right ideal of R generated by the set $\{x - qx \mid x \in R\}$. Now $q \notin K$. By Zorn's lemma, we get a right ideal M , which is maximal for the property that $K \subseteq M$, $q \notin M$. Now M is a maximal right ideal of R . M is right modular right ideal of R as $x - qx \in M$, for all $x \in R$. As $q \in J'_{1/2}(R)$, $q \in M$, a contradiction. Therefore each element of $J'_{1/2}(R)$ is right quasiregular. We see now that each right quasiregular right ideal of R is contained in $J'_{1/2}(R)$. Let K be a right quasiregular right ideal of R . We claim that $K \subseteq J'_{1/2}(R)$. Suppose that K is not contained in $J'_{1/2}(R)$. We get a maximal right modular right ideal M such that $R = M + K$. Suppose that M is right modular by e ; $e = m + k$, $m \in M$, $k \in K$; $x - kx = x - (-m + e)x = x - (-mx + ex) = x - ex + mx \in M$. Since $M \neq R$, k is not right quasiregular, which is a contradiction. Therefore, $K \subseteq J'_{1/2}(R)$. Hence, $J'_{1/2}(R)$ is the largest right quasiregular right ideal of R . $\square$

Remark 2.15. As expected, $J'_{1/2}(R)$ is not an ideal of R . For this, consider the nonabelian group G of order 6. Then $M_0(G)$ is a simple near-ring. Since G has only one nonzero maximal normal subgroup, $M_0(G)$ has only one nonzero maximal right ideal which is right modular by the identity element of $M_0(G)$. So, $J'_{1/2}(M_0(G))$ is not an ideal of $M_0(G)$.

Remark 2.16. Let K be a right ideal of R . We show that there is a unique largest ideal of R contained in K . Now $\{0\}$ is an ideal of R and $\{0\} \subseteq K$. Let I and J be ideals of R and $I \subseteq K$, $J \subseteq K$. Now $I + J$ is an ideal of R . Since K is a subgroup of $(R, +)$, $I + J \subseteq K$. Similarly, if $I_1, I_2, \dots, I_n$ are ideals of R and $I_j \subseteq K$, for all $1 \leq j \leq n$, then we get that $I_1 + I_2 + \dots + I_n \subseteq K$. Let $\{J_i \mid i \in \Delta\}$ be the collection of all ideals T of R such that $T \subseteq K$. Let $J = \sum_{i \in \Delta} J_i$. Now J is an ideal of R . It is clear that $J \subseteq K$ as any element $a \in J$ can be written as $a = a_{i_1} + a_{i_2} + \dots + a_{i_k}$, $a_{i_j} \in J_{i_j}$, $i_j \in \Delta$. Obviously, J is the largest ideal of R contained in K .

4 The right Jacobson radical of type-0

Definition 2.17. The largest ideal of R contained in $J_{1/2}^r(R)$ is denoted by $J_0^r(R)$ and is called the *right Jacobson radical of R of type-0*.

THEOREM 2.18. $R \rightarrow J_0^r(R)$ is a radical map.

Proof. (1) First suppose that R has no maximal right modular right ideal. Now $R = J_0^r(R)$ and that $R/J_0^r(R) = \{0\}$. So $J_0^r(R/J_0^r(R)) = \{0\}$. Suppose now that R has a maximal right modular right ideal. Let $\{M_\alpha \mid \alpha \in \Delta\}$ be the collection of all maximal right modular right ideals of R . Since M_α is a maximal right modular right ideal of R and $J_0^r(R) \subseteq M_\alpha$, $M_\alpha/J_0^r(R)$ is a maximal right modular right ideal of $R/J_0^r(R)$ for all $\alpha \in \Delta$. So $J_0^r(R/J_0^r(R)) \subseteq \bigcap_{\alpha \in \Delta} (M_\alpha/J_0^r(R)) = (\bigcap_{\alpha \in \Delta} M_\alpha)/J_0^r(R)$. Since $J_0^r(R)$ is the largest ideal of R contained in $\bigcap_{\alpha \in \Delta} M_\alpha$, we get that the largest ideal of $R/J_0^r(R)$ contained in $\bigcap_{\alpha \in \Delta} (M_\alpha/J_0^r(R))$ is the zero ideal. Therefore, $J_0^r(R/J_0^r(R)) = \{0\}$.

(2) Let h be a homomorphism of the near-ring R onto a near-ring S . If S has no maximal right modular right ideal, then $J_0^r(S) = S$. Then clearly $h(J_0^r(R)) \subseteq S = J_0^r(S)$. Suppose that S has a maximal right modular right ideal. Let $\{N_\alpha \mid \alpha \in \Delta\}$ be the collection of all maximal right modular right ideals of S . Now $h^{-1}(N_\alpha)$ is a maximal right modular right ideal of R for each $\alpha \in \Delta$. Let $M_\alpha = h^{-1}(N_\alpha)$, $\alpha \in \Delta$. We have that $h(h^{-1}(N_\alpha)) = N_\alpha$, for all $\alpha \in \Delta$, and also $J_0^r(R) \subseteq \bigcap_{\alpha \in \Delta} M_\alpha$. So $h(J_0^r(R)) \subseteq h(\bigcap_{\alpha \in \Delta} M_\alpha) \subseteq \bigcap_{\alpha \in \Delta} h(M_\alpha) = \bigcap_{\alpha \in \Delta} N_\alpha$. Since $h(J_0^r(R))$ is an ideal of S and $J_0^r(S)$ is the largest ideal of S contained in $\bigcap_{\alpha \in \Delta} N_\alpha$, $h(J_0^r(R)) \subseteq J_0^r(S)$. Therefore, $R \rightarrow J_0^r(R)$ is a radical map. $\square$

We denote the ideal of R generated by an element a of R by (a) . The following result is obvious in view of Theorem 2.14.

THEOREM 2.19. $J_0^r(R) = \{a \in R \mid (a) \text{ is a right quasiregular ideal}\}$.

THEOREM 2.20. $J_0^r(R)$ is the largest right quasiregular ideal of R .

The proof follows from Theorem 2.14.

THEOREM 2.21. The nil radical $N(R)$ of R is contained in $J_0^r(R)$.

The proof follows from Lemma 2.4 and Theorem 2.20.

COROLLARY 2.22. $P(R) \subseteq N(R) \subseteq J_0^r(R)$, where $P(R)$ is the prime radical of R .

Proof. We know that $P(R) \subseteq N(R)$. Therefore from Theorem 2.21, $P(R) \subseteq N(R) \subseteq J_0^r(R)$. $\square$

THEOREM 2.23. Let R be a zero symmetric right near-ring with DCC on left R -subgroups of R . Then $J_0(R) \subseteq J_0^r(R)$.

Proof. By Pilz [2, Theorem 5.40], $J_0(R)$ is nilpotent. Therefore, $J_0(R) \subseteq N(R)$. By Theorem 2.21, $N(R) \subseteq J_0^r(R)$. Hence, $J_0(R) \subseteq J_0^r(R)$. $\square$

Definition 2.24. The largest ideal contained in a maximal right modular right ideal of R is called a *right 0-primitive ideal* of R .

Remark 2.25. If R is a ring, then $J_0^r(R)$ is the (right) Jacobson radical of R and a right 0-primitive ideal of the near-ring R is a (right) primitive ideal of the ring R .

THEOREM 2.26. *If $J_0^r(R) \neq R$, then $J_0^r(R)$ is the intersection of all right 0-primitive ideals of R .*

Proof. $J_0^r(R)$ is an ideal contained in each maximal right modular right ideal of R . So $J_0^r(R)$ is contained in each right 0-primitive ideal of R . Hence it is contained in the intersection of all right 0-primitive ideals of R . On the other hand, the intersection of all right 0-primitive ideals of R is an ideal contained in each maximal right modular right ideal of R and that it is contained in $J_0^r(R)$. $\square$

THEOREM 2.27. *A maximal right modular ideal of a near-ring R is a right 0-primitive ideal of R .*

Proof. Let R be a near-ring and let K be a maximal right modular ideal of R . Since K is a proper right modular right ideal of R , K is contained in a maximal right modular right ideal M of R . Since K is a maximal ideal of R , K is the largest ideal contained in M . Hence, K is a right 0-primitive ideal of R . $\square$

Definition 2.28. R is called a *right 0-primitive near-ring*, if $\{0\}$ is a right 0-primitive ideal of R .

Definition 2.29. A 0-primitive ideal of R defined in Pilz [2] is called a *left 0-primitive ideal* of R and similarly a *left 0-primitive near-ring*.

THEOREM 2.30. *Let P be an ideal of R . P is a right 0-primitive ideal of R if and only if R/P is a right 0-primitive near-ring.*

Proof. Let P be a right 0-primitive ideal of R . So we get a maximal right modular right ideal M of R such that P is the largest ideal of R contained in M . Now M/P is a maximal right modular right ideal of R/P . Since P is the largest ideal of R contained in M , the zero ideal of R/P is the largest ideal of R/P contained in M/P . Therefore R/P is a right 0-primitive near-ring. Suppose now that R/P is a right 0-primitive near-ring. So we get a maximal right modular right ideal M/P of R/P such that the zero ideal of R/P is the largest ideal of R/P contained in M/P . Clearly M is a maximal right modular right ideal of R . Since the zero ideal of R/P is the largest ideal of R/P contained in M/P , P is the largest ideal of R contained in M . Therefore, P is a right 0-primitive ideal of R . $\square$

THEOREM 2.31. *A commutative right 0-primitive near-ring is a field.*

Proof. Let R be a commutative right 0-primitive near-ring. We get a modular maximal right ideal M of R such that $\{0\}$ is the largest ideal of R contained in M . Suppose that M is right modular by e . $x - ex \in M$, for all $x \in R$. Since R is commutative, M is an ideal of R . Therefore $M = \{0\}$. Since $x - ex \in M = \{0\}$, $x = ex = xe$. So e is the identity element of R . Now R is a commutative ring with identity. Since $M = \{0\}$ is a maximal ideal of R , R is a field. $\square$

3. Right R -groups of type-0

Definition 3.1. A group $(G, +)$ is called a *right R -group* if there is a mapping $(g, r) \rightarrow gr$ of $G \times R$ into G such that (1) $(g + h)r = gr + hr$, (2) $g(rs) = (gr)s$, for all $g, h \in G$ and $r, s \in R$.

6 The right Jacobson radical of type-0

A subgroup (normal subgroup) H of a right R -group of G is called an R -subgroup (ideal) of G , if $hr \in H$ for all $h \in H$ and $r \in R$.

R is a right R -group. If K is a subgroup of $(R, +)$ and $kr \in K$ for all $k \in K$ and $r \in R$, then K is a right R -subgroup of R . Every right ideal of R is an ideal of the right R -group R . Also, if K is a right ideal of R , then R/K is a right R -group, where $(x + K)r = xr + K$, for all $x + K \in R/K$ and $r \in R$.

Definition 3.2. Let G be a right R -group. An element $g \in G$ is called a *generator* of G if $gR = G$ and $g(r + s) = gr + gs$ for all $r, s \in R$. G is said to be *monogenic* if G has a generator.

Definition 3.3. Let G and H be right R -groups. A mapping $f : G \rightarrow H$ is called an R -homomorphism if $f(x + y) = f(x) + f(y)$ and $f(xr) = f(x)r$ for all $x, y \in G$ and for all $r \in R$. G is said to be R -isomorphic to H if there is a one-one R -homomorphism of G onto H .

PROPOSITION 3.4. Let G be a right R -group. Then G is monogenic if and only if there is a right modular right ideal K of R such that G is R -isomorphic to R/K .

Proof. Let G be a right R -group. Suppose that G is monogenic. Let g be a generator of G . Define $h : R \rightarrow G$ by $h(r) = gr$, for all $r \in R$. h is an R -homomorphism of R onto G . Let K be the kernel of h . $K = \{r \in R \mid h(r) = 0\}$ is a right ideal of R . Therefore R/K is R -isomorphic to G . We get $b \in R$ such that $g = gb$. For each $x \in R$, $gx = gbx$. Now $g(x - bx) = 0$, that is, $x - bx \in K$. So K is modular by b . Conversely, suppose that K is a right ideal of R modular by e and R/K is R -isomorphic to G . Let f be an R -isomorphism of R/K onto G . Let $f(e + K) = g$. We see that $e + K$ is a generator of the right R -group R/K . Let $r, s \in R$. Now $r - er \in K$. So $r + K = er + K = (e + K)r \in (e + K)R$ and hence $(e + K)R = R/K$. Also $(r + s) - e(r + s), r - er, s - es \in K$. Let $k = r - er$ and let $t = s - es$. So $r = k + er, s = t + es$. Since $(r + s) - e(r + s) \in K$, we get that $(k + er) + (t + es) - e(r + s) = k + (er + t - er) + er + es - e(r + s) \in K$ and that $er + es - e(r + s) \in K$. Therefore $e(r + s) + K = (er + es) + K$. So $(e + K)(r + s) = (er + K) + (es + K) = (e + K)r + (e + K)s$. This shows that $e + K$ is a generator of R/K . So g is a generator of G and hence G is monogenic. $\square$

PROPOSITION 3.5. Let K be a right ideal of R . Then K is right modular if and only if there is a right R -group G with a generator g such that $K = (0 : g)$.

Proof. Suppose that K is right modular by e . As seen in the above proposition $e + K$ is a generator of the right R -group R/K . Now $r \in (K : e + K) \Leftrightarrow er + K = K \Leftrightarrow er \in K \Leftrightarrow r \in K$. Therefore $K = (K : e + K)$. Conversely suppose that g is a generator of the right R -group G and $(0 : g) = K$. Since $gR = G$, we get $e \in R$ such that $ge = g$. Let $r \in R$. Now $g(r - er) = gr - gr = 0$. Therefore $r - er \in (0 : g) = K$. Hence, K is right modular by e . $\square$

Definition 3.6. Let G be a right R -group. G is said to be *simple* if $G \neq \{0\}$ and $\{0\}$ and G are the only ideals of G .

Definition 3.7. A monogenic right R -group G is said to be a *right R -group of type-0* if G is simple.

PROPOSITION 3.8. *Let G be a right R -group. G is a right R -group of type-0 if and only if there is a maximal right modular right ideal K of R such that G is R -isomorphic to R/K .*

Proof. G is a right R -group. Suppose that G is of type-0. Let $g \in G$ be a generator. Therefore from the proof of Proposition 3.4, G is R -isomorphic to R/K for some right modular right ideal K of R . Since G is simple, we get that R/K is also simple. Hence, K is a maximal right ideal of R . Conversely, suppose that G is R -isomorphic to R/K , where K is a maximal right modular right ideal of R . Since $R/K \neq \{K\}$ has exactly two ideals, we get that $\{0\}$ and G are the only ideals of G , where $\{0\} \neq G$. Let K be right modular by e . So $e + K$ is a generator of R/K . Therefore, G is also monogenic. Hence, G is a right R -group of type-0. $\square$

Definition 3.9. Let G be a right R -group. The annihilator of G denoted by $(0 : G)$ is defined as $(0 : G) = \{a \in R \mid Ga = \{0\}\}$.

If A and B are nonempty subsets of R , then $(A : B)$ denotes the set $\{r \in R \mid Br \subseteq A\}$.

COROLLARY 3.10. *Let K be a right modular right ideal of R . Then $(K : R) \subseteq K$.*

Proof. Since K is a right modular right ideal of R , by Proposition 3.5, there is a right R -group G with a generator g such that $K = (0 : g)$. Therefore, $K = (0 : g) \supseteq (0 : G) = (0 : R/K) = (K : R)$. $\square$

PROPOSITION 3.11. *Let R be a zero-symmetric near-ring and let K be a right ideal of R right modular by e . Then $(K : R) = (K : eR)$ and the largest ideal of R contained in K is the largest ideal of R contained in $(K : R)$.*

Proof. Since $eR \subseteq R$, $(K : R) \subseteq (K : eR)$. Let $x \in (K : eR)$. Now $eyx \in K$, for all $y \in R$. But $yx - eyx \in K$, for all $y \in R$. Therefore, $yx \in K$, for all $y \in R$, that is, $x \in (K : R)$. So $(K : eR) \subseteq (K : R)$. Therefore, $(K : R) = (K : eR)$. Let J be the largest ideal of R contained in K . For $x \in J$, $Rx \subseteq J \subseteq K$. Therefore, $J \subseteq (K : R)$. Let I be an ideal of R contained in $(K : R)$. By Corollary 3.10, $(K : R) \subseteq K$. So, $I \subseteq K$. Therefore, $I \subseteq J$. Hence, J is the largest ideal of R contained in $(K : R)$. $\square$

PROPOSITION 3.12. *Let P be an ideal of a zero-symmetric near-ring R . P is right 0-primitive if and only if P is the largest ideal of R contained in $(0 : G)$ for some right R -group G of type-0.*

Proof. Let P be an ideal of a zero-symmetric near-ring R . Suppose that P is a right 0-primitive ideal of R . So we get a maximal right modular right ideal K of R such that P is the largest ideal of R contained in K . Now by Proposition 3.8, R/K is a right R -group of type-0. By Proposition 3.11, P is the largest ideal of R contained in $(K : R) = (0 : R/K)$. Conversely, suppose that P is the largest ideal of R contained in $(0 : G)$, where G is a right R -group of type-0. Now G is R -isomorphic to R/K for some maximal right modular right ideal K of R . So $(0 : G) = (0 : R/K) = (K : R)$. Since P is the largest ideal of R contained in $(0 : G) = (K : R)$, by Proposition 3.11, P is the largest ideal of R contained in K . Hence, P is a right 0-primitive ideal of R . $\square$

8 The right Jacobson radical of type-0

PROPOSITION 3.13. *Let G be a monogenic right R -group. If R is a distributively generated (d.g.) near-ring then there is a subset T of G such that $h(a+b) = ha + hb$, for all $h \in T$ and $a, b \in R$, and T generates $(G, +)$.*

Proof. Let G be a monogenic right R -group. Suppose that R is a d.g. near-ring. Since G is a monogenic right R -group, by Proposition 3.4, we get a right modular right ideal K of R such that G is R -isomorphic to R/K . Let f be a R -isomorphism of G onto R/K . Let S be the set of distributive elements of R , where S generates $(R, +)$. It is clear that $\bar{S} = \{s + K \mid s \in S\}$ generates $(R/K, +)$. Let $s \in S$ and let $a, b \in R$. Since $s(a+b) = sa + sb$, we have that $s(a+b) + K = (sa + sb) + K$, that is, $(s + K)[a + b] = (s + K)a + (s + K)b$. Therefore, $T = \{f^{-1}(s + K) \mid s \in S\}$ is the required subset of G . $\square$

PROPOSITION 3.14. *Let G be a monogenic right R -group. If R is a d.g. near-ring, then $(0 : G)$ is an ideal of R .*

Proof. G is a monogenic right R -group and R is d.g. Let S be the set of distributive elements of R . By Proposition 3.13, we get a subset T of G such that $h(x+y) = hx + hy$, for all $h \in T$ and $x, y \in R$, and T generates $(G, +)$. Let $a, b \in (0 : G)$, $p, q \in R$, and let $g \in G$. $h(a+b) = ha + hb = 0 + 0 = 0$, $h(-b) = -hb = 0$ for all $h \in T$. Now $p = \delta_1 s_1 + \delta_2 s_2 + \dots + \delta_n s_n$, where $\delta_i = \pm 1$ and $s_i \in S$, for $1 \leq i \leq n$:

$$\begin{aligned} p(q+a) - pq &= (\delta_1 s_1 + \delta_2 s_2 + \dots + \delta_n s_n)(q+a) - (\delta_1 s_1 + \delta_2 s_2 + \dots + \delta_n s_n)q \\ &= \delta_1 s_1(q+a) + \delta_2 s_2(q+a) + \dots + \delta_n s_n(q+a) \\ &\quad - (\delta_1 s_1 q + \delta_2 s_2 q + \dots + \delta_n s_n q) \\ &= \delta_1 (s_1 q + s_1 a) + \delta_2 (s_2 q + s_2 a) + \dots + \delta_n (s_n q + s_n a) \\ &\quad - \delta_n s_n q - \delta_{n-1} s_{n-1} q - \dots - \delta_1 s_1 q. \end{aligned} \tag{3.1}$$

Now

$$\begin{aligned} h[p(q+a) - pq] &= h(\delta_1 (s_1 q + s_1 a)) + h(\delta_2 (s_2 q + s_2 a)) + \dots + h(\delta_n (s_n q + s_n a)) \\ &\quad - h(\delta_n s_n q) - h(\delta_{n-1} s_{n-1} q) - \dots - h(\delta_1 s_1 q) \\ &= \delta_1 (h(s_1 q) + (hs_1)a) + \delta_2 (h(s_2 q) + (hs_2)a) + \dots + \delta_n (h(s_n q) \\ &\quad + (hs_n)a) - \delta_n h(s_n q) - \delta_{n-1} h(s_{n-1} q) - \dots - \delta_1 h(s_1 q) \\ &= \delta_1 h(s_1 q) + \delta_2 h(s_2 q) + \dots + \delta_n h(s_n q) - \delta_n h(s_n q) \\ &\quad - \delta_{n-1} h(s_{n-1} q) - \dots - \delta_1 h(s_1 q) = 0. \end{aligned} \tag{3.2}$$

Also $h(ap) = (ha)p = 0p = 0$. Since every element of G is a finite sum of elements h , where $h \in T$ or $-h \in T$, we get that

- (1) $g(a+b) = 0$;
- (2) $g(-a) = 0$;

- (3) $g(p + a - p) = 0$;
- (4) $g(p(q + a) - pq) = 0$;
- (5) $g(ap) = 0$.

Therefore, $(0 : G)$ is an ideal of R . □

COROLLARY 3.15. *Let R be a d.g. near-ring and let P be an ideal of R . Then P is a right 0-primitive ideal of R if and only if $P = (0 : G)$ for some right R -group G of type-0.*

Proof. Since a d.g. near-ring is zero-symmetric, the proof follows from Propositions 3.14 and 3.12. □

THEOREM 3.16. *A right 0-primitive ideal of R is a prime ideal of R .*

Proof. Let P be a right 0-primitive ideal of R . We get a right R -module G of type-0 with a generator g such that P is the largest ideal of R contained in $M = (0 : g) = \{r \in R \mid gr = 0\}$, M is a maximal right modular right ideal of R . Let A and B be ideals of R and $AB \subseteq P$. Suppose that $A \not\subseteq P$ and $B \not\subseteq P$. Since $A \not\subseteq P$, $gA \neq \{0\}$. Clearly gA is a subgroup of G . Let $h \in G$ and let $a \in A$. $h = gr$, for some $r \in R$. $h + ga - h = gr + ga - gr = g(r + a - r) \in gA$, as $r + a - r \in A$. So gA is a normal subgroup of G . Also $(gA)R = g(AR) \subseteq gA$. This shows that gA is an ideal of G . Since $gA \neq \{0\}$ and G is a right R -group of type-0, $gA = G$. Similarly, for B , also $gB = G$. Now $G \supseteq gAB = (gA)B = GB \supseteq gB = G$. Therefore, $gAB = G$, a contradiction to the fact that $gAB = \{0\}$. So either $A \subseteq P$ or $B \subseteq P$. Hence, P is a prime ideal of R . □

4. The right J_0 -radical of a biregular near-ring

In this section, it will be shown that the right and left Jacobson radicals of type-0 of a biregular near-ring R are equal, and an ideal P of R is right 0-primitive if and only if P is left 0-primitive.

We recall the following definition of Betsch (Pilz [2, Remark 3.49]).

Definition 4.1. A near-ring R is called *biregular* if there exists a set E of central idempotents of R such that

- (1) Re is an ideal of R for all $e \in E$;
- (2) for each $r \in R$, there exists an $e \in E$ such that $Re = (r)$;
- (3) $e + f = f + e$, for all $e, f \in E$;
- (4) $ef, e + f - ef \in E$, if $e, f \in E$.

A biregular near-ring is zero-symmetric. Let R be a biregular near-ring. Now $\{0\} = (0) = Re$, for some central idempotent e of R . Also $e = ee \in Re = \{0\}$. So $e = 0$ and hence $R0 = \{0\}$. Therefore R is zero-symmetric.

THEOREM 4.2. *Let R be a biregular near-ring. Then $J_0^r(R) = \{0\}$.*

Proof. Let R be a biregular near-ring. Let $x \in J_0^r(R)$. Let (x) be the ideal of R generated by x . Now $(x) = (e)$, for some central idempotent $e \in R$. Therefore $e \in J_0^r(R) \subseteq J_{1/2}^r(R)$. Since e is central, e is a distributive idempotent. By Lemma 2.6, $J_{1/2}^r(R)$ contains no nonzero distributive idempotent. Therefore, $e = 0$. Hence, $J_0^r(R) = \{0\}$. □

THEOREM 4.3. *Let R be a biregular near-ring. Then R is right (left) 0-primitive if and only if R is a nonzero simple near-ring with identity.*

Proof. Let R be a biregular near-ring. Suppose that R is a right (left) 0-primitive near-ring. Let $0 \neq a \in R$. Now $(a) = Re$, for some central idempotent $e \neq 0$ in R . Since e is a central idempotent, $(0 : e)$ is an ideal of R . Now $R = Re + (0 : e)$ is a direct sum of the ideals Re and $(0 : e)$. Since R is right (left) 0-primitive, $\{0\}$ is a prime ideal of R by Theorem 3.16 (Pilz [2, Theorem 4.34]). Since R is zero-symmetric, $Re(0 : e) \subseteq Re \cap (0 : e) = \{0\}$. So either $Re = \{0\}$ or $(0 : e) = \{0\}$. Since $e \neq 0$, $Re \neq \{0\}$. Therefore, $(0 : e) = \{0\}$. Hence, $R = Re = (a)$. So R is a simple near-ring with identity e . Conversely, suppose that R is a nonzero simple near-ring with identity. Since R is a near-ring with identity, R has a maximal right (left) ideal K and is right (left) modular by the identity. Since R is simple, $\{0\}$ is the largest ideal of R contained in K . Therefore, $\{0\}$ is a right (left) 0-primitive ideal of R and hence R is right (left) 0-primitive. $\square$

COROLLARY 4.4. *A right (left) 0-primitive ideal of a biregular near-ring is a maximal ideal which is both right and left modular.*

Proof. Let R be a biregular near-ring and let P be a right (left) 0-primitive ideal of R . By Theorem 2.30, R/P is a right (left) 0-primitive near-ring. Since R/P is also biregular, by Theorem 4.3, R/P is a simple near-ring with identity. So, P is a maximal right and left modular ideal. $\square$

COROLLARY 4.5. *Let R be a biregular near-ring and let P be an ideal of R . Then the following statements are equivalent:*

- (1) P is a right 0-primitive ideal of R ;
- (2) P is a left 0-primitive ideal of R ;
- (3) P is a maximal ideal of R which is both right and left modular.

Proof. By Corollary 4.4, (1) implies (3) and (2) implies (3). Let P be a maximal right and left modular ideal of R . Since P is a maximal right (left) modular ideal of R , R/P is a simple near-ring with a left (right) identity. So R/P is a simple near-ring with identity. Since R/P is biregular, by Theorems 4.3 and 2.30, we get that (3) implies (1) and (3) implies (2). $\square$

COROLLARY 4.6. *Let $(G, +)$ be a group with more than one element. Then $M_0(G)$ is a right and left 0-primitive near-ring.*

Proof. Since $M_0(G)$ is a nonzero simple biregular near-ring, it is a right and left 0-primitive by Theorem 4.3. $\square$

Now we observe an interesting fact that unlike the left Jacobson-type radical classes, the J_0^r -radical class contains almost all the classes of near-rings with trivial multiplication.

We consider J_0^r -radical of a near-ring with trivial multiplication. Let $(R, +)$ be a group containing more than one element. Let S be a nonempty subset of R not containing 0. The trivial multiplication on $(R, +)$ determined by S is given by $a \cdot b = a$, if $b \in S$ and 0, if $b \in S^c$, where S^c is the complement of S in R . Then $(R, +, \cdot)$ is a near-ring.

Example 4.7. Let R, S , and S^c be as defined above. Then S^c is nilpotent and hence it is right quasiregular. Moreover, R is right quasiregular if and only if S^c is not a normal subgroup of $(R, +)$ of index 2.

Let $a \in R$. Let $A = \{x - ax \mid x \in R\}$, let $B = \{x - ax \mid x \in S\}$, and let $C = \{x - ax \mid x \in S^c\}$. Now $A = B \cup C$, $B = \{x - a \mid x \in S\} = S + (-a)$, and $C = \{x \mid x \in S^c\} = S^c$. So $A = [S + (-a)] \cup S^c$. Let K be the normal subgroup of $(R, +)$ generated by A . Clearly K is a right ideal of R . Hence, K is the right ideal of R generated by A .

If $a, b \in S^c$, then $ab = 0$. Therefore, S^c is nilpotent and right quasiregular. Now suppose that $a \in S$. First assume that S^c is not a normal subgroup of $(R, +)$. Since $R = S \cup S^c$, the normal subgroup of $(R, +)$ generated by S^c contains an element $s \in S$. So $s \in K$. Since $s, s - a \in K$, $a = -(s - a) + s \in K$. Therefore, a is right quasiregular.

Assume now that S^c is a normal subgroup of $(R, +)$. Since $a \in S$, a is not in S^c . So $-a$ is not an element of S^c , as S^c is a subgroup. Since $R = S^c \cup S$, $R = [S^c + (-a)] \cup [S + (-a)]$. Now S^c and $S^c + (-a)$ are two right cosets of S^c in $(R, +)$. Let $S^c, S^c + (-a), S^c + r_\alpha$, and $\alpha \in \Delta$ be the distinct right cosets of S^c in $(R, +)$. So $R = S^c \cup [S^c + (-a)] \cup [\cup S^c + r_\alpha]$. The complement of $S^c + (-a)$ in R is $S + (-a) = S^c \cup [\cup S^c + r_\alpha]$. If the index of S^c in $(R, +)$ is 2, then $S + (-a) = S^c$ and hence $A = S^c = K \neq R$. Thus a is not right quasiregular.

Now assume that the index of S^c in $(R, +)$ is not 2. So, $S + (-a) \neq S^c$. We will show that $S + (-a)$ is not a normal subgroup of $(R, +)$. Suppose that $S + (-a)$ is a normal subgroup of $(R, +)$. Since $S^c \subseteq S + (-a)$, $R = [S^c + (-a)] \cup [S + (-a)]$, and $[S^c + (-a)] \cap [S + (-a)]$ is empty, $S^c + (-a) = S + (-a) + (-a)$. Since S^c is a proper subset of $S + (-a)$, $S^c + (-a)$ is a proper subset of $S + (-a) + (-a) = S^c + (-a)$, a contradiction. Therefore, $S + (-a)$ is not a normal subgroup of $(R, +)$. So, the normal subgroup of $(R, +)$ generated by $S + (-a)$ contains an element $x + (-a)$ of $S^c + (-a)$, $x \in S^c$. Now $x + (-a), x \in K$, and $a = -(x + (-a)) + x \in K$. Therefore, a is right quasiregular.

Remark 4.8. Let R, S , and S^c be as defined above and let S^c be a normal subgroup of $(R, +)$ of index 2. Then, S^c is a nilpotent ideal of R , $S^c = P(R) = J'_0(R) = J_2(R)$. Moreover, S^c is the set of right (left) quasiregular elements of R .

We see now by an example that there is a near-ring R which has a left 0-primitive ideal, but has no right 0-primitive ideals and that the right and left Jacobson radicals of R of type-0 are different. We know that every prime ideal of a zero-symmetric near-ring R with DCC on left R -subgroups of R is left 0-primitive, but there is a finite zero-symmetric near-ring $R \neq \{0\}$ in which no (proper) prime ideal is right 0-primitive. Even though for a zero-symmetric near-ring R with DCC on left R -subgroups of R , every left quasiregular left R -subgroup of R is nilpotent, we see that there is a finite zero-symmetric right near-ring R in which no nonzero right quasiregular right R -subgroup of R is nil.

Example 4.9. Let $(R, +)$ be a group containing more than two elements. Define a trivial multiplication in R by $rs = r$ if $s \neq 0$ and 0 if $s = 0$ for all $r, s \in R$. R is a zero-symmetric right near-ring. Clearly, R is a left R -group of type-2. Moreover, R is simple. Therefore, R is 2-primitive on the left R -group R , as $RR \neq \{0\}$. So $J_2(R) = \{0\} = J_0(R)$. But each element of R is right quasiregular. Therefore, $J'_0(R) = R$. If R is finite, then obviously R has DCC on right (left) R -groups of R , but no nonzero right quasiregular right ideal of R is nilpotent. Moreover, the zero ideal of R is prime but not right 0-primitive.

12 The right Jacobson radical of type-0

We recall some of the definitions and results of [3] which are required to observe that right Jacobson radicals are relevant for the study of near-rings in terms of matrix near-rings.

Matrix near-rings were introduced in [1].

Definition 4.10. Let R be a zero-symmetric near-ring with identity. A subset $\{e_{ij} \mid 1 \leq i, j \leq n\}$ of distributive elements in R is said to be a set of *matrix units* in R if and only if $e_{11} + e_{22} + \cdots + e_{nn} = 1$ and $e_{rs}e_{pq} = \delta_{sp}e_{rq}$, where

$$\delta_{sp} = \begin{cases} 1 & \text{if } s = p, \\ 0 & \text{if } s \neq p. \end{cases} \quad (4.1)$$

PROPOSITION 4.11. *Let R be a zero-symmetric near-ring with identity. $R = K_1 \oplus K_2 \oplus \cdots \oplus K_n$, a direct sum of n pairwise isomorphic right ideals K_i of R as right R -groups if and only if R has a set of matrix units $\{e_{ij} \mid 1 \leq i, j \leq n\}$. In this case $K_i = e_{ii}R$, for all $1 \leq i \leq n$.*

As stated soon after [3, Corollary 15], we have the following.

THEOREM 4.12. *Let R be a simple and d.g. near-ring with identity. Then R is isomorphic to a matrix near-ring $M_n(S)$ if and only if R has a set of matrix units $\{e_{ij} \mid 1 \leq i, j \leq n\}$.*

THEOREM 4.13. *Let R be a simple d.g. near-ring with DCC on right ideals of R and $J_{1/2}^r = \{0\}$. Suppose that any two minimal right ideals of R are isomorphic as right R -groups. Then, $R = K_1 \oplus K_2 \oplus \cdots \oplus K_n$, a direct sum of minimal right ideals K_i and is (isomorphic to) a matrix near-ring $M_n(S)$.*

Proof. Since $J_{1/2}^r(R) = \{0\}$ and R has DCC on right ideals of R , we get that the intersection of a finite number of maximal right modular right ideals of R is zero. So, R is a direct sum of a finite number of minimal right ideals $K_1, K_2, \dots, K_n$ of R . By Proposition 2.12, R has a left identity as the intersection of a finite number of maximal right modular right ideals of R is zero. Since R is a simple near-ring with left identity, it has an identity. Also, since by our assumption any two minimal right ideals of R are isomorphic as right R -groups, by Proposition 4.11, R has a set of matrix units $\{e_{ij} \mid 1 \leq i, j \leq n\}$. Therefore, by Theorem 4.12, R is (isomorphic to) a matrix near-ring $M_n(S)$. $\square$

Example 4.14. We give an example of a nonring which satisfies the hypothesis of Theorem 4.13. Let G be a finite simple nonabelian additive group. By [3, Corollary 19], $E(G^2)$ is isomorphic to the matrix near-ring $M_2(E(G))$. As mentioned soon after [3, Corollary 19], $E(G^2) = M_0(G^2)$. So, $M_0(G^2)$ is a simple d.g. near-ring with DCC on right ideals. Let $i \in \{1, 2\}$. Let $G_1 = G \times \{0\}$ and let $G_2 = \{0\} \times G$. Since G_i is a maximal (minimal) normal subgroups of G^2 , $K_i = (G_i : G^2) = \{m \in M_0(G^2) \mid m(a) \in G_i, \text{ for all } a \in G^2\}$ is a maximal right ideal of $M_0(G^2)$. Moreover, $K_1 \cap K_2 = \{0\}$. Thus $J_{1/2}^r(M_0(G^2)) = \{0\}$. This shows that $M_0(G^2) = K_1 \oplus K_2$, where K_i is a minimal right ideal of $M_0(G^2)$. Define $e_i : G^2 \rightarrow G_i$ by $e_i((a_1, a_2)) = (b_1, b_2)$, where $b_j = a_i$ if $j = i$ and 0 if $j \neq i$. Now e_i is a group homomorphism and hence it is a distributive idempotent in $M_0(G^2)$ and $e_i M_0(G^2) \subseteq K_i$. Since e_1 and e_2 are orthogonal distributive idempotents in $M_0(G^2)$ and $e_1 + e_2 = 1$, by [3, Proposition 2], we get that $e_i M_0(G^2)$ is a right ideal of $M_0(G^2)$. Thus, $K_i = e_i M_0(G^2)$.

The mapping $e_{12} : G^2 \rightarrow G^2$ defined by $e_{12}((a_1, a_2)) = (a_2, 0)$ is an endomorphism of G^2 . So, e_{12} is a distributive element in $M_0(G^2)$. It is an easy verification that the mapping $h : e_2 M_0(G^2) \rightarrow e_1 M_0(G^2)$ defined by $h(e_2 m) = e_{12}(e_2 m)$ is an isomorphism of the right $M_0(G^2)$ -groups. So, K_1 and K_2 are isomorphic as right $M_0(G^2)$ -groups. Since a minimal right ideal K of $M_0(G^2)$ is isomorphic to K_j for some $j \in \{1, 2\}$ as right $M_0(G^2)$ -groups, we get that any two minimal right ideals of $M_0(G^2)$ are isomorphic as right $M_0(G^2)$ -groups. So, $M_0(G^2)$ satisfies the hypothesis of Theorem 4.13.

Example 4.15. Let G be a finite simple nonabelian additive group. Now by Pilz [2, Corollary 7.48], $E(G) = M_0(G)$. So, $M_0(G)$ is a finite simple d.g. near-ring with identity. Moreover, $J_2(M_0(G)) = \{0\}$ and each minimal left ideal of $M_0(G)$ is isomorphic to G as left $M_0(G)$ -groups. Since each distributive element of $M_0(G)$ is an endomorphism of $(G, +), 0$ and the automorphisms of $(G, +)$ are the only distributive elements of $M_0(G)$. Therefore, $M_0(G)$ has no nontrivial matrix units. Hence, $M_0(G)$ is not isomorphic to a matrix near-ring $M_n(S)$, where $n > 1$.

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