AN EXTENSION AND A REFINEMENT OF VAN DER CORPUT'S INEQUALITY

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van der Corput's inequality is extended and refined by using Euler-Maclaurin formula and other analytic techniques.

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1. Introduction

Let $S_n = \sum_{k=1}^n (1/k)$ and $a_n \geq 0$ for $n \in \mathbb{N}$ such that $0 < \sum_{n=1}^{\infty} a_n < \infty$. The famous van der Corput inequality [10] reads that

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{1/S_n} < e^{1+\gamma} \sum_{n=1}^{\infty} (n+1) a_n, \quad (1.1)$$

where $\gamma = 0.57721566\dots$ stands for Euler-Mascheroni constant. The constant $e^{1+\gamma}$ in (1.1) is the best possible.

Hu [5] gave a strengthened version of (1.1) as

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{1/S_n} < e^{1+\gamma} \sum_{n=1}^{\infty} \left(n - \frac{\ln n}{4} \right) a_n. \quad (1.2)$$

Yang [14] established a relation between Carleman's inequality and van der Corput's inequality and presented the following:

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k^\alpha} \right)^{1/S_n(\alpha)} < e \sum_{n=1}^{\infty} e^{\alpha n^{\alpha-1} S_n(\alpha)} a_n, \quad (1.3)$$

where $S_n(\alpha) = \sum_{k=1}^n (1/k^\alpha)$ and $\alpha \in [0, 1]$.

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In a recent paper [15], Yang has obtained another extension of (1.1) as follows:

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/(k+\beta)} \right)^{1/S_n(\beta)} < e^{1+\gamma_1(\beta)} \sum_{n=1}^{\infty} \left(n + \frac{1}{2} + \beta \right) a_n, \quad (1.4)$$

where $\beta \in (-1, \infty)$, $S_n(\beta) = \sum_{k=1}^n (1/(k+\beta))$, and

$$\gamma_1(\beta) = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \frac{1}{k+\beta} - \ln(n+\beta) \right]. \quad (1.5)$$

Applying $\beta = 0$ in (1.4) leads to

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{1/S_n} < e^{1+\gamma} \sum_{n=1}^{\infty} \left(n + \frac{1}{2} \right) a_n, \quad (1.6)$$

which improved inequality (1.1) clearly.

For more information about van der Corput's inequality, please refer to [2, 5, 10, 14, 15] and the references therein.

The aim of this paper is to further extend and refine van der Corput's inequality by using Euler-Maclaurin formula and other analytic techniques.

Our main results are the following two theorems.

THEOREM 1.1. *Let $a_n \geq 0$ for $n \in \mathbb{N}$ such that $0 \leq \sum_{n=1}^{\infty} a_n < \infty$. Then*

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[\prod_{k=1}^n a_k^{1/\sqrt{k(k+\lambda)}} \right]^{1/S_n(\lambda)} \\ & < e^{1+(1+\lambda/3)\gamma(\lambda)} \sum_{n=1}^{\infty} (n+1)^{\lambda/3} \left[1 - \frac{\ln(n+1)}{4(n+1+\lambda/2)} \right] a_n, \end{aligned} \quad (1.7)$$

where $\lambda \in [0, \infty)$,

$$S_n(\lambda) = \sum_{k=1}^n \frac{1}{\sqrt{k(k+\lambda)}}, \quad (1.8)$$

$$\gamma(\lambda) = \lim_{n \rightarrow \infty} \left[S_n(\lambda) - 2 \ln \frac{\sqrt{n} + \sqrt{n+\lambda}}{1 + \sqrt{1+\lambda}} \right]. \quad (1.9)$$

THEOREM 1.2. *Let $a_n \geq 0$ for $n \in \mathbb{N}$ such that $0 \leq \sum_{n=1}^{\infty} a_n < \infty$. Then*

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{1/S_n} < e^{1+\gamma} \sum_{n=1}^{\infty} n \left(1 - \frac{\ln n}{3n - 1/4} \right) a_n. \quad (1.10)$$

Remark 1.3. It is easy to see that inequality (1.10) refines inequalities (1.1), (1.2), and (1.6).

2. Lemmas

To prove our main results, the following lemmas are necessary.

Recall [7, 9] that a function f is called completely monotonic on an interval I if f has derivatives of all orders on I and $0 < (-1)^k f^{(k)}(x) < \infty$ for all $k \geq 0$ on I . The background information and an extensive bibliography about the theory of completely monotonic function can be found in the recent papers [4, 7, 8].

LEMMA 2.1. *The function $f(x) = 1/\sqrt{x(x+\lambda)}$ for $\lambda \in [0, \infty)$ is completely monotonic in $(0, \infty)$ and $\lim_{x \rightarrow \infty} f^{(i)}(x) = 0$ for any nonnegative integer i .*

Proof. It is not difficult to verify that the functions $1/\sqrt{x}$ and $1/\sqrt{x+\lambda}$ are completely monotonic in $x \in (0, \infty)$. Since the product of any finite completely monotonic functions is also strictly completely monotonic (see [11]), then the function $f(x)$ is strictly completely monotonic in $(0, \infty)$.

By induction, it is easy to verify that $\lim_{x \rightarrow \infty} f^{(i)}(x) = 0$ holds for any nonnegative integer i . The proof of Lemma 2.1 is complete. $\square$

Recall that Euler-Maclaurin formula (see [1, pages 617–623] and [6, 12, 14]) states

$$\sum_{k=1}^n f(k) = \int_1^n f(x) dx + \frac{1}{2}[f(n) + f(1)] + \int_1^n \rho_1(x) f'(x) dx, \quad (2.1)$$

where $\rho_1(x) = x - [x] + 1/2$ is Bernoulli's function and $f \in C^1[1, \infty)$. Furthermore, if $(-1)^i f^{(i)}(x) > 0$ and $\lim_{x \rightarrow \infty} f^{(i)}(x) = 0$ for $i = 1, 2, 3$, then

$$\int_n^\infty \rho_1(x) f'(x) dx = -\frac{1}{12} f'(n) \epsilon, \quad 0 < \epsilon < 1. \quad (2.2)$$

LEMMA 2.2. *For $n \in \mathbb{N}$ and $\lambda \in [0, \infty)$,*

$$S_n(\lambda) < \ln(n+1) + \gamma(\lambda), \quad (2.3)$$

where $S_n(\lambda)$ and $\gamma(\lambda)$ are defined by (1.8) and (1.9), respectively.

Proof. It is clear that Lemma 2.1 allows us to apply Euler-Maclaurin formula (2.1) and formula (2.2) to $f(x) = 1/\sqrt{x(x+\lambda)}$. From this, it follows that

$$\begin{aligned} S_n(\lambda) &= 2 \ln \frac{\sqrt{n} + \sqrt{n+\lambda}}{1 + \sqrt{1+\lambda}} + \frac{1}{2} \left[\frac{1}{\sqrt{1+\lambda}} + \frac{1}{\sqrt{n(n+\lambda)}} \right] \\ &\quad + \int_1^n \rho_1(x) \left[\frac{1}{\sqrt{x(x+\lambda)}} \right]' dx, \\ \int_n^\infty \rho_1(x) \left[\frac{1}{\sqrt{x(x+\lambda)}} \right]' dx &= -\frac{1}{12} \left[\frac{1}{\sqrt{n(n+\lambda)}} \right]' \epsilon = \frac{(2n+\lambda)\epsilon}{24[n(n+\lambda)]^{3/2}}, \end{aligned} \quad (2.4)$$

where $0 < \epsilon < 1$, and

$$\gamma(\lambda) = \frac{1}{2\sqrt{1+\lambda}} + \int_1^\infty \rho_1(x) \left[\frac{1}{\sqrt{x(x+\lambda)}} \right]' dx. \quad (2.5)$$

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Therefore,

$$\begin{aligned} S_n(\lambda) &= 2 \ln \frac{\sqrt{n} + \sqrt{n+\lambda}}{1 + \sqrt{1+\lambda}} + \gamma(\lambda) - \frac{(2n+\lambda)\epsilon}{24[n(n+\lambda)]^{3/2}} + \frac{1}{2\sqrt{n(n+\lambda)}} \\ &< \ln n + \gamma(\lambda) + \frac{1}{2\sqrt{n(n+\lambda)}}, \end{aligned} \quad (2.6)$$

and then

$$\begin{aligned} S_n(\lambda) &= \sum_{k=1}^{n+1} \frac{1}{\sqrt{k(k+\lambda)}} - \frac{1}{\sqrt{(n+1)(n+1+\lambda)}} \\ &< \ln(n+1) + \gamma(\lambda) - \frac{1}{2\sqrt{(n+1)(n+1+\lambda)}} < \ln(n+1) + \gamma(\lambda). \end{aligned} \quad (2.7)$$

The proof of Lemma 2.2 is complete. $\square$

LEMMA 2.3. For $k \in \mathbb{N}$ and $\lambda \in [0, \infty)$,

$$\sqrt{\frac{k(k+\lambda)}{(k+1)(k+1+\lambda)}} \leq \frac{k+\lambda/2}{k+1+\lambda/2}, \quad (2.8)$$

$$\sqrt{(k+1)(k+1+\lambda)} - \sqrt{k(k+\lambda)} \leq 1 + \frac{\lambda}{3}. \quad (2.9)$$

Proof. Inequality (2.8) is equivalent to

$$\left(k + \frac{\lambda}{2}\right)^2 (k+1)(k+1+\lambda) \geq k(k+\lambda) \left(k+1 + \frac{\lambda}{2}\right)^2. \quad (2.10)$$

The difference between both sides of (2.10) equals

$$\begin{aligned} &\left[k^4 + 2k^3(\lambda+1) + k^2 \left(\frac{5}{4}\lambda^2 + 3\lambda + 1 \right) + k \left(\frac{\lambda^3}{4} + \frac{3}{2}\lambda^2 + \lambda \right) + \frac{\lambda+1}{4}\lambda^2 \right] \\ &- \left[k^4 + 2k^3(\lambda+1) + k^2 \left(\frac{5}{4}\lambda^2 + 3\lambda + 1 \right) + k \left(\frac{\lambda^3}{4} + \lambda^2 + \lambda \right) \right] \\ &= \frac{k\lambda^2}{2} + \frac{\lambda^2}{4} + \frac{\lambda^3}{4} \geq 0. \end{aligned} \quad (2.11)$$

Inequality (2.9) can be deduced straightforwardly from

$$\begin{aligned} &\sqrt{(k+1)(k+1+\lambda)} - \sqrt{k(k+\lambda)} \\ &= \frac{2k+\lambda+1}{\sqrt{(k+1)(k+1+\lambda)} + \sqrt{k(k+\lambda)}} \leq 1 + \frac{\lambda}{3}. \end{aligned} \quad (2.12)$$

The proof of Lemma 2.3 is complete. $\square$

LEMMA 2.4. For $x \in (0, \infty)$ and $\lambda \in [0, \infty)$,

$$\left[1 - \frac{1}{2(x+1+\lambda/2)}\right]^{\ln(x+1)} < 1 - \frac{\ln(x+1)}{4(x+1+\lambda/2)}. \quad (2.13)$$

Proof. Let $u(x, \lambda) = 2(x+1+\lambda/2) - \ln(x+1)$ for $x \in (0, \infty)$ and $\lambda \in [0, \infty)$. Then $\partial u(x, \lambda)/\partial x = 2 - 1/(x+1) > 0$ and $u(0, \lambda) = 2 + \lambda > 0$. Thus,

$$\frac{\ln^2(x+1)}{8(x+1+\lambda/2)^2} < \frac{\ln(x+1)}{4(x+1+\lambda/2)}. \quad (2.14)$$

As a result, by

$$\left(1 - \frac{1}{t}\right)^{-t} > e \quad (2.15)$$

for $t > 1$ and

$$e^t < 1 + t + \frac{t^2}{2} \quad (2.16)$$

for $t < 0$, it follows that

$$\begin{aligned} \left[1 - \frac{1}{2(x+1+\lambda/2)}\right]^{\ln(x+1)} &< \left(\frac{1}{e}\right)^{\ln(x+1)/2(x+1+\lambda/2)} \\ &< 1 - \frac{\ln(x+1)}{2(x+1+\lambda/2)} + \frac{\ln^2(x+1)}{8(x+1+\lambda/2)^2} < 1 - \frac{\ln(x+1)}{4(x+1+\lambda/2)}. \end{aligned} \quad (2.17)$$

The proof of Lemma 2.4 is complete. $\square$

LEMMA 2.5. For $k \in \mathbb{N}$ and $\lambda \in [0, \infty)$,

$$\begin{aligned} B_k(\lambda) &\triangleq \left[\frac{\sqrt{(k+1)(k+1+\lambda)} S_{k+1}(\lambda)}{\sqrt{k(k+\lambda)} S_k(\lambda)} \right]^{\sqrt{k(k+\lambda)} S_k(\lambda)} \\ &\leq e^{1+(1+\lambda/3)\gamma(\lambda)} (k+1)^{1+\lambda/3} \left[1 - \frac{\ln(k+1)}{4(k+1+\lambda/2)} \right]. \end{aligned} \quad (2.18)$$

Proof. For $k \in \mathbb{N}$,

$$B_k(\lambda) = \left\{ 1 + \frac{1 + [\sqrt{(k+1)(k+1+\lambda)} - \sqrt{k(k+\lambda)}] S_k(\lambda)}{\sqrt{k(k+\lambda)} S_k(\lambda)} \right\}^{\sqrt{k(k+\lambda)} S_k(\lambda)} \triangleq C_k^{h(k, \lambda)}, \quad (2.19)$$

where

$$\begin{aligned} C_k &= \left[1 + \frac{1}{g(k, \lambda)} \right]^{g(k, \lambda)}, \quad g(k, \lambda) = \frac{\sqrt{k(k+\lambda)} S_k(\lambda)}{h(k, \lambda)}, \\ h(k, \lambda) &= 1 + \left[\sqrt{(k+1)(k+1+\lambda)} - \sqrt{k(k+\lambda)} \right] S_k(\lambda). \end{aligned} \quad (2.20)$$

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It is easy to see that

$$\begin{aligned} g(k, \lambda) + 1 &= \frac{1 + \sqrt{(k+1)(k+1+\lambda)} S_k(\lambda)}{1 + [\sqrt{(k+1)(k+1+\lambda)} - \sqrt{k(k+\lambda)}] S_k(\lambda)} \\ &\leq \frac{\sqrt{(k+1)(k+1+\lambda)}}{\sqrt{(k+1)(k+1+\lambda)} - \sqrt{k(k+\lambda)}}. \end{aligned} \quad (2.21)$$

By using the inequality $(1 + 1/x)^x < e[1 - 1/2(x+1)]$ obtained in [13], inequalities (2.21) and (2.8) in Lemma 2.3, it is deduced that

$$\begin{aligned} C_k &= \left[1 + \frac{1}{g(k, \lambda)} \right]^{g(k, \lambda)} \leq e \left\{ 1 - \frac{1}{2[g(k, \lambda) + 1]} \right\} \\ &\leq e \left[\frac{1}{2} + \frac{\sqrt{k(k+\lambda)}}{2\sqrt{(k+1)(k+1+\lambda)}} \right] \leq e \left[1 - \frac{1}{2(k+1+\lambda/2)} \right]. \end{aligned} \quad (2.22)$$

Hence, from inequalities (2.3), (2.9), (2.22) in Lemma 2.2, and (2.13) in Lemma 2.4, it is shown that

$$\begin{aligned} B_k(\lambda) &\leq \left\{ e \left[1 - \frac{1}{2(k+1+\lambda/2)} \right] \right\}^{h(k, \lambda)} \\ &\leq \left\{ e \left[1 - \frac{1}{2(k+1+\lambda/2)} \right] \right\}^{1+(1+\lambda/3)[\ln(k+1)+\gamma(\lambda)]} \\ &\leq e^{1+(1+\lambda/3)\gamma(\lambda)} (k+1)^{1+\lambda/3} \left[1 - \frac{1}{2(k+1+\lambda/2)} \right]^{\ln(k+1)} \\ &\leq e^{1+(1+\lambda/3)\gamma(\lambda)} (k+1)^{1+\lambda/3} \left[1 - \frac{\ln(k+1)}{4(k+1+\lambda/2)} \right]. \end{aligned} \quad (2.23)$$

The proof of Lemma 2.5 is complete. □

LEMMA 2.6. For $n \in \mathbb{N}$,

$$\left(1 - \frac{1}{2n+11/6} \right)^{\ln n} \leq 1 - \frac{\ln n}{3n-1/4}. \quad (2.24)$$

Proof. For $n = 1$, inequality (2.24) holds clearly. For $n = 2$,

$$1 - \frac{\ln 2}{6-1/4} - \left(1 - \frac{1}{4+11/6} \right)^{\ln 2} = 0.0016626... > 0, \quad (2.25)$$

inequality (2.24) holds also.

For $n \geq 3$, by using (2.15) and (2.16), it is shown that

$$\left(1 - \frac{1}{2n+11/6}\right)^{\ln n} < e^{-\ln n/(2n+11/6)} < 1 - \frac{\ln n}{2n+11/6} + \frac{\ln^2 n}{2(2n+11/6)^2}. \quad (2.26)$$

So, it is sufficient to prove that

$$1 - \frac{\ln n}{2n+11/6} + \frac{\ln^2 n}{2(2n+11/6)^2} < 1 - \frac{\ln n}{3n-1/4} \quad (2.27)$$

for $n \geq 3$. For this purpose, let $m = n - 1/12$, then inequality (2.27) can be rearranged as

$$g(m) \triangleq \frac{4m}{3} - \frac{4}{3} - \frac{8}{3m} - \ln\left(m + \frac{1}{12}\right) > 0. \quad (2.28)$$

Differentiation of $g(x)$ gives

$$g'(x) = \frac{4}{3} + \frac{8}{3x^2} - \frac{1}{x+1/12} > 0. \quad (2.29)$$

This means that $g(m)$ is increasing. Further, since

$$g\left(3 - \frac{1}{12}\right) = \frac{4}{3}\left(3 - \frac{1}{12}\right) - \frac{4}{3} - \frac{8}{3(3-1/12)} - \ln 3 = 0.5426575526\dots > 0, \quad (2.30)$$

then $g(m)$ is positive for $m \geq 3$. The proof of Lemma 2.6 is complete. $\square$

3. Proofs of theorems

Proof of Theorem 1.1. Setting $c_k > 0$ for $1 \leq k \leq n$ and letting

$$\left[\prod_{k=1}^n c_k^{1/\sqrt{k(k+\lambda)}} \right]^{-1/S_n(\lambda)} = \frac{1}{\sqrt{(n+1)(n+1+\lambda)} S_{n+1}(\lambda)}, \quad (3.1)$$

then

$$c_k = \frac{[\sqrt{(k+1)(k+1+\lambda)} S_{k+1}(\lambda)]^{\sqrt{k(k+\lambda)} S_k(\lambda)}}{[\sqrt{k(k+\lambda)} S_k(\lambda)]^{\sqrt{k(k+\lambda)} S_{k-1}(\lambda)}}. \quad (3.2)$$

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Using the discrete weighted arithmetic-geometric mean inequality and (3.2) and interchanging the order of summation yield

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/\sqrt{k(k+\lambda)}} \right)^{1/S_n(\lambda)} \\
&= \sum_{n=1}^{\infty} \left(\prod_{k=1}^n (c_k a_k)^{1/\sqrt{k(k+\lambda)}} \right)^{1/S_n(\lambda)} \left(\prod_{k=1}^n c_k^{1/\sqrt{k(k+\lambda)}} \right)^{-1/S_n(\lambda)} \\
&\leq \sum_{n=1}^{\infty} \sum_{k=1}^n \frac{1}{\sqrt{k(k+\lambda)} S_n(\lambda)} c_k a_k \frac{1}{\sqrt{(n+1)(n+1+\lambda)} S_{n+1}(\lambda)} \\
&= \sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k+\lambda)}} c_k a_k \sum_{n=k}^{\infty} \frac{1}{\sqrt{(n+1)(n+1+\lambda)} S_n(\lambda) S_{n+1}(\lambda)} \\
&= \sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k+\lambda)}} c_k a_k \sum_{n=k}^{\infty} \left[\frac{1}{S_n(\lambda)} - \frac{1}{S_{n+1}(\lambda)} \right] \\
&= \sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k+\lambda)}} c_k a_k \frac{1}{S_k(\lambda)} \\
&= \sum_{k=1}^{\infty} \left[\frac{\sqrt{(k+1)(k+1+\lambda)} S_{k+1}(\lambda)}{\sqrt{k(k+\lambda)} S_k(\lambda)} \right]^{\sqrt{k(k+\lambda)} S_k(\lambda)} a_k.
\end{aligned} \tag{3.3}$$

Applying inequality (2.18) in the final line of (3.3) gives inequality (1.7). The proof of Theorem 1.1 is complete. $\square$

Proof of Theorem 1.2. It is easy to see that

$$B_1(0) = 3 < e^{1+\gamma}, \quad B_2(0) = \left(\frac{11}{6} \right)^3 < e^{1+\gamma} \cdot 2 \left(1 - \frac{\ln 2}{6 - 1/4} \right). \tag{3.4}$$

For $n \geq 3$, inequality

$$e^{1/2n} \left(1 - \frac{1}{2n + 11/6} \right)^{1+\gamma} < e^{1/2n} e^{-(1+\gamma)/(2n+11/6)} < 1 \tag{3.5}$$

follows from using an inequality

$$\left(1 + \frac{1}{x} \right)^x < e \left(1 - \frac{1}{2x + 11/6} \right) \tag{3.6}$$

in [3]. By (2.3), (3.6), Lemma 2.6, and inequality (3.5),

$$\begin{aligned}
 B_n(0) &= \left[\frac{(n+1)S_n + 1}{nS_n} \right]^{nS_n} = \left\{ \left[1 + \frac{1}{nS_n/(S_n+1)} \right]^{nS_n/(S_n+1)} \right\}^{S_n+1} \\
 &< \left\{ e \left[1 - \frac{1}{2nS_n/(S_n+1) + 11/6} \right] \right\}^{S_n+1} \\
 &< \left\{ e \left[1 - \frac{1}{2nS_n/(S_n+1) + 11/6} \right] \right\}^{1+\ln n + \gamma + 1/2n} \\
 &< \left[e \left(1 - \frac{1}{2n + 11/6} \right) \right]^{1+\ln n + \gamma + 1/2n} \\
 &< e^{1+\gamma} n e^{1/2n} \left(1 - \frac{1}{2n + 11/6} \right)^{1+\ln n + \gamma} \\
 &< e^{1+\gamma} n \left(1 - \frac{\ln n}{3n - 1/4} \right).
 \end{aligned} \tag{3.7}$$

Taking $\lambda = 0$ in inequality (3.3) yields

$$\begin{aligned}
 \sum_{n=1}^{\infty} \left(\prod_{k=1}^n a_k^{1/k} \right)^{1/S_n} &\leq \sum_{n=1}^{\infty} \left[\frac{(n+1)S_{n+1}}{nS_n} \right]^{nS_n} a_n \\
 &= \sum_{n=1}^{\infty} B_n(0) a_n < e^{1+\gamma} \sum_{n=1}^{\infty} n \left(1 - \frac{\ln n}{3n - 1/4} \right) a_n.
 \end{aligned} \tag{3.8}$$

The proof of Theorem 1.2 is complete. $\square$

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