WEAKLY INDUCED MODIFICATIONS OF *I*-FUZZY TOPOLOGIES

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The aim of this paper is to study weakly induced *I*-fuzzy topological spaces and weakly induced modifications of *I*-fuzzy topologies. We give two kinds of weakly induced *I*-fuzzy topologies for each *I*-fuzzy topology and prove that *I*-WIFTOP is a reflective and coreflective full subcategory of *I*-FTOP. We also discuss some relationships between several categories.

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1. Introduction and preliminaries

Since Chang [2] introduced fuzzy theory into topology, many authors have discussed various aspects of fuzzy topology. It is well known that weakly induced and induced topological spaces play an important role in *L*-topological spaces (see book [8]). According to their value ranges, *L*-topological spaces form different categories. Clearly, the investigation on their relationships is certainly important and necessary. Lowen was the first author to study the relations between *I*-topological spaces and classical topological spaces. He introduced two well-known functors— ω and ι . Later, these functors, named Lowen functors, were extended by different authors [7, 12] for various kinds of lattices studying the relations between *L*-TOP and TOP.

However, in a completely different direction, Höhle [4] created the notion of a topology being viewed as an *L*-subset of a powerset. Then Kubiak [6] and Šostak [11] independently extended Höhle's notion to *L*-subsets of L^X . From a logical point of view, Ying [13] introduced fuzzifying topological spaces (Ying's fuzzifying topology is similar to Höhle's topology). In order to discuss the relations between fuzzifying topologies and *I*-fuzzy topologies, the authors studied Lowen functors in *I*-fuzzy topological spaces in a Kubiak-Šostak sense and introduced induced *I*-fuzzy topological spaces in [15]. Zhang and Liu [17] studied weakly induced modifications of *L*-topologies. The aim of this paper is to study weakly induced *I*-fuzzy topological spaces and the weakly induced modifications of *I*-fuzzy topologies.

This paper is organized as follows. In Section 1, we give some preliminary concepts and properties. Two kinds of weakly induced modifications are introduced in Section 2.

2 Weakly induced modifications of I -fuzzy topologies

We prove that I -WIFTOP—the category of weakly induced I -fuzzy topological spaces—is a reflective and coreflective full subcategory of I -FTOP. Finally, in Section 3, we discuss the relationship between several important categories.

In this paper, X is a nonempty set and $I = [0, 1]$, $I_0 = [0, 1)$. The family of all fuzzy sets on X will be denoted by I^X . By 0_X and 1_X , we denote, respectively, the constant fuzzy set on X taking the values 0 and 1. Let $\sigma_r(A) = \{x \mid A(x) > r\}$ for $r \in I$ and $A \in I^X$. $U \in P(X)$, 1_U denotes the characteristic function of U , that is, $1_U(x) = 1$ when $x \in U$ and $1_U(x) = 0$ when $x \notin U$. For the notions about categories, please refer to [1, 5, 9].

Definition 1.1 [4, 13]. A fuzzifying topology on X is a map $\xi : P(X) \rightarrow I$ satisfying the following axioms:

(FY1) $\xi(\emptyset) = \xi(X) = 1$;

(FY2) $\xi(U \cap V) \geq \xi(U) \wedge \xi(V)$ for all $U, V \in P(X)$;

(FY3) $\xi(\bigcup_{t \in T} U_t) \geq \bigwedge_{t \in T} \xi(U_t)$ for every family $\{U_t \mid t \in T\} \subseteq P(X)$.

If ξ is a fuzzifying topology on X , the pair (X, ξ) is called a fuzzifying topological space. A fuzzifying continuous map between fuzzifying topological spaces (X, ξ) and (Y, η) is a map $f : X \rightarrow Y$ such that $\xi(f^{-1}(U)) \geq \eta(U)$ for all $U \in P(Y)$. The category of fuzzifying topological spaces and fuzzifying continuous maps is denoted by FYS. Let $\text{FYS}(X)$ denote the set of all fuzzifying topologies on X .

Definition 1.2 [6, 11]. An I -fuzzy topology on a set X is defined to be a map $\mathcal{T} : I^X \rightarrow I$ satisfying:

(FT1) $\mathcal{T}(1_X) = \mathcal{T}(0_X) = 1$;

(FT2) $\mathcal{T}(A \wedge B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B)$ for all $A, B \in I^X$;

(FT3) $\mathcal{T}(\bigvee_{t \in T} A_t) \geq \bigwedge_{t \in T} \mathcal{T}(A_t)$ for every family $\{A_t \mid t \in T\} \subseteq I^X$.

If $\mathcal{T}$ is an I -fuzzy topology on X , the pair $(I^X, \mathcal{T})$ is called an I -fuzzy topological space. An I -fuzzy continuous map between I -fuzzy topological spaces $(I^X, \mathcal{T})$ and $(I^Y, \mathcal{S})$ is a map $f : X \rightarrow Y$ such that $\mathcal{T}(f_!^{-1}(B)) \geq \mathcal{S}(B)$ for all $B \in I^Y$, where $f_!^{-1}(B)(x) = B(f(x))$ (following the notation in [10]). The category of I -fuzzy topological spaces and I -fuzzy continuous maps is denoted by I -FTOP. Let I -FTOP(X) denote the set of all I -fuzzy topologies on X .

Definition 1.3 [13]. Let ξ be a fuzzifying topology on X , $\mathcal{B} : P(X) \rightarrow I$, and $\mathcal{B} \leq \xi$. $\mathcal{B}$ is called a base of ξ if $\mathcal{B}$ satisfies the following condition:

$$\forall U \in P(X), \quad \xi(U) = \bigvee_{\bigcup_{\lambda \in \Lambda} V_\lambda = U} \bigwedge_{\lambda \in \Lambda} \mathcal{B}(V_\lambda), \quad (1.1)$$

where the expression $\bigvee_{\bigcup_{\lambda \in \Lambda} V_\lambda = U} \bigwedge_{\lambda \in \Lambda} \mathcal{B}(V_\lambda)$ will be denoted by $\mathcal{B}^{(\sqcup)}(U)$, that is, $\xi = \mathcal{B}^{(\sqcup)}$.

A map $\phi : P(X) \rightarrow I$ is called a subbase of ξ if $\phi^{(\cap)} : P(X) \rightarrow I$ defined by $\phi^{(\cap)}(U) = \bigvee_{(\cap)_{\lambda \in J} V_\lambda = U} \bigwedge_{\lambda \in J} \phi(V_\lambda)$ for all $U \in P(X)$ is a base, where $(\cap)$ stands for “finite intersection.” $\phi : P(X) \rightarrow I$ is a subbase of one fuzzifying topology if and only if $\phi^{(\sqcup)}(X) = 1$.

Definition 1.4 [14]. Let $\{(X_t, \xi_t)\}_{t \in T}$ be a family of fuzzifying topological spaces and let $P_t : \prod_{t \in T} X_t \rightarrow X_t$ be the projection. Then the fuzzifying topology whose subbase is

defined by

$$\forall W \in P\left(\prod_{t \in T} X_t\right), \quad \phi(W) = \bigvee_{t \in T} \bigvee_{P_t^-(U)=W} \xi_t(U) \quad (1.2)$$

is called the product topology of $\{\xi_t \mid t \in T\}$, denoted by $\prod_{t \in T} \xi_t$. $(\prod_{t \in T} X_t, \prod_{t \in T} \xi_t)$ is called the product space of $\{(X_t, \xi_t)\}_{t \in T}$.

Fang and Yue [3] extended the above definitions and results to I -fuzzy topological spaces. For more explicitly, we list them as follows.

(1) Let $\mathcal{T}$ be an I -fuzzy topology on X , $\mathcal{B} : I^X \rightarrow I$ s.t. $\mathcal{B} \leq \mathcal{T}$ (in a pointwise sense). Then $\mathcal{B}$ is called a base of $\mathcal{T}$ if $\mathcal{B}$ satisfies the following condition:

$$\forall A \in I^X, \quad \mathcal{T}(A) = \bigvee_{\bigvee_{\lambda \in \Lambda} B_\lambda = A} \bigwedge_{\lambda \in \Lambda} \mathcal{B}(B_\lambda), \quad (1.3)$$

where the expression $\bigvee_{\bigvee_{\lambda \in \Lambda} B_\lambda = A} \bigwedge_{\lambda \in \Lambda} \mathcal{B}(B_\lambda)$ will be denoted by $\mathcal{B}^{(\sqcup)}(A)$.

(2) Let $\phi : I^X \rightarrow I$ be a map. Then ϕ is called a subbase of $\mathcal{T}$ if $\phi^{(\cap)} : I^X \rightarrow I$ is a base, where $\phi^{(\cap)}(A) = \bigvee_{(\cap)_{\lambda \in J} B_\lambda = A} \bigwedge_{\lambda \in J} \phi(B_\lambda)$ for all $A \in I^X$ with $(\cap)$ standing for “finite intersection.” A map $\phi : I^X \rightarrow I$ is a subbase if and only if $\phi^{(\sqcup)}(1_X) = 1$.

(3) Let $\{(I^{X_t}, \mathcal{T}_t)\}_{t \in T}$ be a family of I -fuzzy topological spaces and let $P_t : \prod_{t \in T} X_t \rightarrow X_t$ be the projection. Then the I -fuzzy topology whose subbase is defined by

$$\forall A \in I^{\prod_{t \in T} X_t}, \quad \phi(A) = \bigvee_{t \in T} \bigvee_{(P_t)_I^-(B)=A} \mathcal{T}_t(B) \quad (1.4)$$

is called the product topology of $\{\mathcal{T}_t \mid t \in T\}$, denoted by $\prod_{t \in T} \mathcal{T}_t$. $(I^{\prod_{t \in T} X_t}, \prod_{t \in T} \mathcal{T}_t)$ is called the product space of $\{(I^{X_t}, \mathcal{T}_t)\}_{t \in T}$.

Definition 1.5. Let $\{(I^{X_t}, \mathcal{T}_t)\}_{t \in T}$ be a family of I -fuzzy topological spaces, let different X'_t s be disjoint and $X = \bigcup_{t \in T} X_t$, and let $\mathcal{T} : I^X \rightarrow I$ be defined as follows:

$$\forall A \in I^X, \quad \mathcal{T}(A) = \bigwedge_{t \in T} \mathcal{T}_t(A \mid X_t). \quad (1.5)$$

Then it is easy to verify that $\mathcal{T}$ is an I -fuzzy topology on X , and $\mathcal{T}$ is called the sum topology of $\{\mathcal{T}_t\}_{t \in T}$, denoted by $\bigoplus_{t \in T} \mathcal{T}_t$.

Definition 1.6. Let $(I^X, \mathcal{T})$ be an I -fuzzy topological space and let $f : X \rightarrow Y$ be a surjective map. Define the I -fuzzy quotient topology $\mathcal{T}/f_I^-$ of $\mathcal{T}$ with respect to f by

$$\forall A \in I^Y, \quad \mathcal{T}/f_I^-(A) = \mathcal{T}(f_I^-(A)). \quad (1.6)$$

It is easy to verify that $\mathcal{T}/f_I^-$ is an I -fuzzy topology on Y . $(I^Y, \mathcal{T}/f_I^-)$ is called the I -fuzzy quotient space of $(I^X, \mathcal{T})$ with respect to f and f_I^- is called an I -fuzzy quotient map.

Definition 1.7 [9]. Let $(I^X, \mathcal{T})$ be an I -fuzzy topological space and $Y \subseteq X$. $(I^Y, \mathcal{T} \mid Y)$ is called the subspace of $(I^X, \mathcal{T})$, where $\mathcal{T} \mid Y : I^Y \rightarrow I$ is defined by $\mathcal{T} \mid Y(B) = \bigvee \{\mathcal{T}(A) \mid A \in I^X, A \mid Y = B\}$ for all $B \in I^Y$.

4 Weakly induced modifications of I -fuzzy topologies

LEMMA 1.8 [5]. $I\text{-FTOP}(X)$ is a complete lattice.

Using the similar argument in [5], it is easy to show that $\text{FYS}(X)$ is also a complete lattice.

LEMMA 1.9 [15]. Let $\{\xi_t\}_{t \in T} \subseteq \text{FYS}(X)$. Then $\phi : P(X) \rightarrow I$ defined by $\phi(U) = \bigvee_{t \in T} \xi_t(U)$ is the subbase of $\bigvee_{t \in T} \xi_t$, that is, $\bigvee_{t \in T} \xi_t = (\phi^{(\cap)})^{(\cup)}$.

2. Weakly induced modifications of I -fuzzy topologies

The purpose of this section is to study weakly induced I -fuzzy topological spaces and the weakly induced modifications of I -fuzzy topologies.

Definition 2.1 [15]. Let $(I^X, \mathcal{T})$ be an I -fuzzy topological space on X . If $\mathcal{T}(A) \leq \bigwedge_{r \in I_0} \mathcal{T}(1_{\sigma_r(A)})$ for all $A \in I^X$, then $(I^X, \mathcal{T})$ is called a weakly induced I -fuzzy topological space. Let $I\text{-WIFTOP}$ denote the category of weakly induced I -fuzzy topological spaces.

Example 2.2. Let ξ be a fuzzifying topology on X . Define $\mathcal{T}_\xi : I^X \rightarrow I$ as follows:

$$\mathcal{T}_\xi(A) = \begin{cases} \xi(U) & \text{if } A \text{ is a characteristic function, that is, } A = 1_U, \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

It is easy to check that $\mathcal{T}_\xi$ is an I -fuzzy topology on X and it is weakly induced. Specially, $\mathcal{T}$ is weakly induced, where

$$\mathcal{T}(A) = \begin{cases} 1, & A = 0_X, 1_X, \\ 0, & \text{otherwise.} \end{cases} \quad (2.2)$$

Example 2.3. Let $\mathcal{T} : I^X \rightarrow I$ be defined by $\mathcal{T}(A) = 1$ for all $A \in L^X$. Then $\mathcal{T}$ is a weakly induced I -fuzzy topology on X .

In the following discussion, we will give the right adjoint functor and left adjoint functor of the inclusion functor $i : I\text{-WIFTOP} \rightarrow I\text{-FTOP}$, and show that $I\text{-WIFTOP}$ is a reflective and coreflective full subcategory of $I\text{-FTOP}$.

LEMMA 2.4. Let $(I^X, \mathcal{T})$ be an I -fuzzy topological space and let $\mathcal{T}_* : I^X \rightarrow I$ be defined by

$$\forall A \in I^X, \quad \mathcal{T}_*(A) = \bigwedge_{r \in I_0} \mathcal{T}(1_{\sigma_r(A)}) \wedge \mathcal{T}(A). \quad (2.3)$$

Then $\mathcal{T}_*$ is the biggest weakly induced I -fuzzy topology smaller than $\mathcal{T}$. Hence, if $\mathcal{T}$ is weakly induced, then $\mathcal{T} = \mathcal{T}_*$.

Proof. It is routine to prove that $\mathcal{T}_*$ is an I -fuzzy topology on X . The following computation can show that $\mathcal{T}_*$ is weakly induced:

$$\begin{aligned} \bigwedge_{r \in I_0} \mathcal{T}_*(1_{\sigma_r(A)}) &= \bigwedge_{r \in I_0} \bigwedge_{s \in I_0} \mathcal{T}(1_{\sigma_s(1_{\sigma_r(A)})}) \wedge \mathcal{T}(1_{\sigma_r(A)}) = \bigwedge_{r \in I_0} \mathcal{T}(1_{\sigma_r(A)}) \\ &\geq \bigwedge_{r \in I_0} \mathcal{T}(1_{\sigma_r(A)}) \wedge \mathcal{T}(A) = \mathcal{T}_*(A). \end{aligned} \quad (2.4)$$

Let $\mathcal{S}$ be any weakly induced I -fuzzy topology on X satisfying $\mathcal{S} \leq \mathcal{T}$. We need to prove that $\mathcal{S} \leq \mathcal{T}_*$. Since $\mathcal{S}$ is weakly induced, we have $\mathcal{S}(A) \leq \bigwedge_{r \in I_0} \mathcal{S}(1_{\sigma_r(A)})$ for all $A \in I^X$. Hence we get that

$$\mathcal{S}(A) \leq \mathcal{T}(A) \wedge \bigwedge_{r \in I_0} \mathcal{S}(1_{\sigma_r(A)}) \leq \mathcal{T}(A) \wedge \bigwedge_{r \in I_0} \mathcal{T}(1_{\sigma_r(A)}) = \mathcal{T}_*(A), \quad (2.5)$$

thus the conclusion. $\square$

LEMMA 2.5. *Let $(I^Y, \mathcal{T})$ be weakly induced and let $(I^X, \mathcal{S})$ be an I -fuzzy topological space. Then $f_I^- : (I^X, \mathcal{S}) \rightarrow (I^Y, \mathcal{T})$ is I -fuzzy continuous if and only if $f_I^- : (I^X, \mathcal{S}_*) \rightarrow (I^Y, \mathcal{T}_*) = (I^Y, \mathcal{T})$ is I -fuzzy continuous.*

Proof. The sufficiency is obvious and it needs to show the necessity. Let $f_I^- : (I^X, \mathcal{S}) \rightarrow (I^Y, \mathcal{T})$ be I -fuzzy continuous, that is, $\mathcal{T}(B) \leq \mathcal{S}(f_I^-(B))$ for all $B \in I^Y$. Since $\mathcal{T}$ is weakly induced, we have $\mathcal{T}(B) \leq \bigwedge_{r \in I_0} \mathcal{T}(1_{\sigma_r(B)})$. Hence

$$\mathcal{T}(B) \leq \mathcal{S}(f_I^-(B)) \wedge \bigwedge_{r \in I_0} \mathcal{T}(1_{\sigma_r(B)}) \leq \mathcal{S}(f_I^-(B)) \wedge \bigwedge_{r \in I_0} \mathcal{S}(1_{f_I^-(\sigma_r(B))}) = \mathcal{S}_*(f_I^-(B)). \quad (2.6)$$

Therefore, $f_I^- : (I^X, \mathcal{S}_*) \rightarrow (I^Y, \mathcal{T})$ is I -fuzzy continuous. $\square$

Remark 2.6. From Lemma 2.5, we also can get that $f_I^- : (I^X, \mathcal{S}_*) \rightarrow (I^Y, \mathcal{T}_*)$ is I -fuzzy continuous if $f_I^- : (I^X, \mathcal{S}) \rightarrow (I^Y, \mathcal{T})$ is I -fuzzy continuous. Hence we know that $(\cdot)_*$ is a functor from I -FTOP to I -WIFTOP. Furthermore, we have the following theorem.

THEOREM 2.7. $(\cdot)_*$ is the left adjoint of i .

LEMMA 2.8. *Let $(I^X, \mathcal{T})$ be an I -fuzzy topological space and let $\phi : I^X \rightarrow I$ be defined by*

$$\phi^{\mathcal{T}}(A) = \begin{cases} \bigvee_{r \in I_0} \bigvee \{ \mathcal{T}(B) \mid \sigma_r(B) = U \} & \text{if } A \text{ is a characteristic function, that is, } A = 1_U, \\ \mathcal{T}(A) & \text{otherwise.} \end{cases} \quad (2.7)$$

Then $\phi^{\mathcal{T}}$ is a subbase of one I -fuzzy topology, and denote this I -fuzzy topology by $\text{wi}(\mathcal{T})$. $\text{wi}(\mathcal{T})$ is called the weakly induced modification of $\mathcal{T}$.

Proof. It is trivial to verify that $\phi^{\mathcal{T}}$ is a subbase of one I -fuzzy topology. $\square$

THEOREM 2.9. *Let $(I^X, \mathcal{T})$ be an I -fuzzy topological space. Then $\text{wi}(\mathcal{T})$ is the smallest weakly induced I -fuzzy topology bigger than $\mathcal{T}$. Hence, if $\mathcal{T}$ is weakly induced, then $\mathcal{T} = \text{wi}(\mathcal{T})$.*

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Proof. We need to prove that $\text{wi}(\mathcal{T})(A) \leq \bigwedge_{r \in I_0} \text{wi}(\mathcal{T})(1_{\sigma_r(A)})$, that is, $\text{wi}(\mathcal{T})(A) \leq \text{wi}(\mathcal{T})(1_{\sigma_r(A)})$ for all $r \in I_0$. In fact, noting that

$$\begin{aligned} \text{wi}(\mathcal{T})(A) &= \bigvee_{\bigwedge_{\lambda \in \Lambda} B_\lambda = A} \bigwedge_{\lambda \in \Lambda} \bigvee_{(\cap)_{\beta \in \Lambda_\lambda} C_{\lambda\beta} = B_\lambda} \bigwedge_{\beta \in \Lambda_\lambda} \phi^{\mathcal{T}}(C_{\lambda\beta}), \\ \text{wi}(\mathcal{T})(1_{\sigma_r(A)}) &= \bigvee_{\bigwedge_{\lambda \in \Lambda} B_\lambda = 1_{\sigma_r(A)}} \bigwedge_{\lambda \in \Lambda} \bigvee_{(\cap)_{\beta \in \Lambda_\lambda} C_{\lambda\beta} = B_\lambda} \bigwedge_{\beta \in \Lambda_\lambda} \phi^{\mathcal{T}}(C_{\lambda\beta}), \end{aligned} \quad (2.8)$$

we have $\text{wi}(\mathcal{T})(A) \leq \text{wi}(\mathcal{T})(1_{\sigma_r(A)})$ according to $\phi^{\mathcal{T}}(C_{\lambda\beta}) \leq \phi^{\mathcal{T}}(1_{\sigma_r(C_{\lambda\beta})})$, as desired.

We now prove that $\text{wi}(\mathcal{T})$ is the smallest weakly induced I -fuzzy topology bigger than $\mathcal{T}$. Let $\mathcal{T}^*$ be any weakly induced I -fuzzy topology on X bigger than $\mathcal{T}$. We need to prove that $\text{wi}(\mathcal{T}) \leq \mathcal{T}^*$. It suffices to show that $\phi^{\mathcal{T}}(A) \leq \mathcal{T}^*(A)$ for all $A \in I^X$. Then it suffices to show that $\phi^{\mathcal{T}}(1_U) \leq \mathcal{T}^*(1_U)$ for all $U \subseteq X$, and this can be obtained by the following computation:

$$\begin{aligned} \phi^{\mathcal{T}}(1_U) &= \bigvee_{r \in I_0} \bigvee \{ \mathcal{T}(B) \mid \sigma_r(B) = U \} \leq \bigvee_{r \in I_0} \bigvee \{ \mathcal{T}^*(B) \mid \sigma_r(B) = U \} \\ &\leq \bigvee_{r \in I_0} \bigvee \left\{ \bigwedge_{s \in I_0} \mathcal{T}^*(1_{\sigma_s(B)}) \mid \sigma_r(B) = U \right\} \leq \mathcal{T}^*(1_U), \end{aligned} \quad (2.9)$$

thus the conclusion. $\square$

LEMMA 2.10. *Let $(I^Y, \mathcal{T})$ be weakly induced and let $(I^X, \mathcal{S})$ be an I -fuzzy topological space. Then $f_I^- : (I^Y, \mathcal{T}) \rightarrow (I^X, \mathcal{S})$ is I -fuzzy continuous if and only if $f_I^- : (I^Y, \mathcal{T}) \rightarrow (I^X, \text{wi}(\mathcal{S}))$ is I -fuzzy continuous.*

Proof. The sufficiency is obvious. We need to prove the necessity. It suffices to show that $\phi^{\mathcal{S}}(A) \leq \mathcal{T}(f_I^-(A))$ for all $A = 1_U \in I^X$. Since $f_I^- : (I^Y, \mathcal{T}) \rightarrow (I^X, \mathcal{S})$ is I -fuzzy continuous, we have

$$\phi^{\mathcal{S}}(1_U) = \bigvee_{r \in I_0} \bigvee \{ \mathcal{S}(B) \mid \sigma_r(B) = U \} \leq \bigvee_{r \in I_0} \bigvee \{ \mathcal{T}(f_I^-(B)) \mid \sigma_r(B) = U \} \leq \mathcal{T}(f_I^-(1_U)), \quad (2.10)$$

thus the conclusion. $\square$

Remark 2.11. From Lemma 2.10 above, we also can get that $f_I^- : (I^X, \text{wi}(\mathcal{S})) \rightarrow (I^Y, \text{wi}(\mathcal{T}))$ is I -fuzzy continuous if $f_I^- : (I^X, \mathcal{S}) \rightarrow (I^Y, \mathcal{T})$ is I -fuzzy continuous. Hence wi is another functor from I -FTOP to I -WIFTOP. Furthermore, we have the following theorem.

THEOREM 2.12. *wi is the right adjoint of i .*

From Theorems 2.7 and 2.12, we have the main theorem in this paper as follows.

THEOREM 2.13. *I -WIFTOP is a reflective and coreflective full subcategory of I -FTOP.*

By the properties of right adjoint, we have the following corollaries.

COROLLARY 2.14. Let $(I^X, \mathcal{T})$ be an I -fuzzy topological space and $Y \subseteq X$. Then $\text{wi}(\mathcal{T} \mid Y) = \text{wi}(\mathcal{T}) \mid Y$.

COROLLARY 2.15. Let $\{(I^{X_t}, \mathcal{T}_t)\}_{t \in T}$ be a family of I -fuzzy topological spaces and $X = \prod_{t \in T} X_t$. Then $\text{wi}(\prod_{t \in T} \mathcal{T}_t) = \prod_{t \in T} \text{wi}(\mathcal{T}_t)$.

THEOREM 2.16. Let $\{(I^{X_t}, \mathcal{T}_t)\}_{t \in T}$ be a family of I -fuzzy topological spaces and let different X_t 's be disjoint. Then $\text{wi}(\bigoplus_{t \in T} \mathcal{T}_t) = \bigoplus_{t \in T} \text{wi}(\mathcal{T}_t)$.

Proof. First, we have

$$\begin{aligned} \bigoplus_{t \in T} \text{wi}(\mathcal{T}_t)(A) &= \bigwedge_{t \in T} \text{wi}(\mathcal{T}_t)(A \mid X_t) \leq \bigwedge_{t \in T} \bigwedge_{r \in I_0} \text{wi}(\mathcal{T}_t)(1_{\sigma_r(A \mid X_t)}) \\ &= \bigwedge_{t \in T} \bigwedge_{r \in I_0} \text{wi}(\mathcal{T}_t)(1_{\sigma_r(A)} \mid X_t) = \bigwedge_{r \in I_0} \bigwedge_{t \in T} \text{wi}(\mathcal{T}_t)(1_{\sigma_r(A)} \mid X_t) \\ &= \bigwedge_{r \in I_0} \bigoplus_{t \in T} \text{wi}(\mathcal{T}_t)(1_{\sigma_r(A)}). \end{aligned} \quad (2.11)$$

Hence, $\bigoplus_{t \in T} \text{wi}(\mathcal{T}_t)$ is weakly induced. Therefore, $\text{wi}(\bigoplus_{t \in T} \mathcal{T}_t) \leq \bigoplus_{t \in T} \text{wi}(\mathcal{T}_t)$.

Conversely, let $\lambda < \bigoplus_{t \in T} \text{wi}(\mathcal{T}_t)(A)$, that is,

$$\lambda < \bigoplus_{t \in T} \text{wi}(\mathcal{T}_t)(A) = \bigwedge_{t \in T} \text{wi}(\mathcal{T}_t)(A \mid X_t) = \bigwedge_{t \in T} \bigvee_{\lambda \in \Lambda^t} D_\lambda^t = A \mid X_t \quad \bigwedge_{\lambda \in \Lambda^t} \bigvee_{(\cap)_{\beta \in \Lambda_\lambda^t} E_{\lambda\beta}^t = D_\lambda^t} \bigwedge_{\beta \in \Lambda_\lambda^t} \phi^{\mathcal{T}_t}(E_{\lambda\beta}^t). \quad (2.12)$$

Then, for all $t \in T$, there exists $\{D_\lambda^t\}_{\lambda \in \Lambda^t} \subseteq I^{X_t}$ such that

- (i) $\bigvee_{\lambda \in \Lambda^t} D_\lambda^t = A \mid X_t$;
- (ii) for each $\lambda \in \Lambda^t$, there exists $\{E_{\lambda\beta}^t\}_{\beta \in \Lambda_\lambda^t} \subseteq I^{X_t}$ such that $(\cap)_{\beta \in \Lambda_\lambda^t} E_{\lambda\beta}^t = D_\lambda^t$;
- (iii) for each $\beta \in \Lambda_\lambda^t$, we have $\lambda \leq \phi^{\mathcal{T}_t}(E_{\lambda\beta}^t)$.

Let $(D_\lambda^t)^* \in I^X$ and $(E_{\lambda\beta}^t)^* \in I^X$ be defined as follows:

$$\begin{aligned} (D_\lambda^t)^*(x) &= \begin{cases} D_\lambda^t(x), & x \in X_t, \\ 0, & x \notin X_t, \end{cases} \\ (E_{\lambda\beta}^t)^*(x) &= \begin{cases} E_{\lambda\beta}^t(x), & x \in X_t, \\ 0, & x \notin X_t. \end{cases} \end{aligned} \quad (2.13)$$

Then we have

$$\bigvee_{t \in T} \bigvee_{\lambda \in \Lambda^t} (D_\lambda^t)^* = A, \quad (\cap)_{\beta \in \Lambda_\lambda^t} (E_{\lambda\beta}^t)^* = (D_\lambda^t)^*, \quad \phi^{\mathcal{T}_t}(E_{\lambda\beta}^t) = \phi^{\bigoplus_{t \in T} \mathcal{T}_t}((E_{\lambda\beta}^t)^*). \quad (2.14)$$

Therefore, $\lambda \leq \phi^{\bigoplus_{t \in T} \mathcal{T}_t}((E_{\lambda\beta}^t)^*)$ due to $\lambda \leq \phi^{\mathcal{T}_t}(E_{\lambda\beta}^t)$. Note that

$$\text{wi}\left(\bigoplus_{t \in T} \mathcal{T}_t\right)(A) = \bigvee_{\lambda \in \Lambda} \bigwedge_{B_\lambda = A} \bigvee_{(\cap)_{\beta \in \Lambda_\lambda} C_{\lambda\beta} = B_\lambda} \bigwedge_{\beta \in \Lambda_\lambda} \phi^{\bigoplus_{t \in T} \mathcal{T}_t}(C_{\lambda\beta}). \quad (2.15)$$

We have $\lambda \leq \text{wi}(\bigoplus_{t \in T} \mathcal{T}_t)(A)$. Then $\bigoplus_{t \in T} \text{wi}(\mathcal{T}_t)(A) \leq \text{wi}(\bigoplus_{t \in T} \mathcal{T}_t)(A)$. This completes the proof. $\square$

The readers can easily prove the following theorem.

THEOREM 2.17. *Let $(I^X, \mathcal{T})$ be an I -fuzzy topological space and let $(I^Y, \mathcal{T}/f_1^-)$ be the I -fuzzy quotient space of $(I^X, \mathcal{T})$ with respect to $f : X \rightarrow Y$. If $(I^X, \mathcal{T})$ is weakly induced, then $(I^Y, \mathcal{T}/f_1^-)$ is weakly induced.*

3. On the relationships between several categories

In Section 2, we study weakly induced modifications of I -fuzzy topologies. Since weakly induced and induced topological spaces play an important role in L -topology, in this section, we will study induced I -fuzzy topologies and the relationships between the categories FYS, I -WIFTOP, I -SFTOP, I -IFTOP, and I -FTOP, where I -IFTOP and I -SFTOP denote the categories of induced I -fuzzy topological spaces and stratified I -fuzzy topological spaces, respectively. In the following discussion, we always assume that I -TOP denotes the category of stratified Chang-Goguen topological spaces. We know that TOP can be regarded as a full (moreover, simultaneously reflective and coreflective) subcategory of I -TOP by Lowen functors. Zhang [16] proved that TOP is a reflective and coreflective full subcategory of FYS and FYS is a reflective and coreflective full subcategory of I -TOP. From [15], we know that FYS is isomorphic to I -IFTOP. We will prove that I -IFTOP is a reflective and coreflective full subcategory of I -SFTOP and I -IFTOP is a coreflective full subcategory of I -WIFTOP.

Let $(I^X, \mathcal{T})$ be an I -fuzzy topological space and let $[\mathcal{T}] : P(X) \rightarrow I$ be defined by $[\mathcal{T}](U) = \mathcal{T}(1_U)$ for all $U \in P(X)$. Then it is easy to verify that $[\mathcal{T}]$ is a fuzzifying topology on X .

Definition 3.1 [15]. Let $(I^X, \mathcal{T})$ be an I -fuzzy topological space. $[\mathcal{T}]$ is called the background topology of $\mathcal{T}$ and $(X, [\mathcal{T}])$ is called the background space of $(I^X, \mathcal{T})$.

From the definition above, we get a functor $[\cdot]$ from I -FTOP to FYS. It is easy to verify the following two theorems.

THEOREM 3.2. *If $f_1^- : (I^X, \mathcal{T}_1) \rightarrow (I^Y, \mathcal{T}_2)$ is I -fuzzy continuous, then $f : (X, [\mathcal{T}_1]) \rightarrow (Y, [\mathcal{T}_2])$ is a fuzzifying continuous.*

THEOREM 3.3. *Let $\{(I^{X_i}, \mathcal{T}_i)\}_{i \in T}$ be a family of I -fuzzy topological spaces and let different X_i 's be disjoint. Then $[\bigoplus_{i \in T} \mathcal{T}_i] = \bigoplus_{i \in T} [\mathcal{T}_i]$.*

Definition 3.4 [15]. Let $(I^X, \mathcal{T})$ be an I -fuzzy topological space on X . If $\mathcal{T}(A) = \bigwedge_{r \in I_0} \mathcal{T}(1_{\sigma_r(A)})$ for all $A \in I^X$, then $(I^X, \mathcal{T})$ is called an induced I -fuzzy topological space. If $\mathcal{T}(\bar{\lambda}) = 1$ for all $\lambda \in I$, where $\bar{\lambda}$ is the constant function from X to I with value λ , then $(X, \mathcal{T})$ is called a stratified I -fuzzy topological space.

LEMMA 3.5 [15]. *Let $\mathcal{T}$ be an I -fuzzy topology on X and let $\phi_{\mathcal{T}} : P(X) \rightarrow I$ be defined by $\phi_{\mathcal{T}}(U) = \bigvee_{r \in I} \bigvee \{\mathcal{T}(B) \mid B \in I^X, \sigma_r(B) = U\}$ for $U \in P(X)$. Then $\phi_{\mathcal{T}}$ is the subbase of one fuzzifying topology, and let this fuzzifying topology be denoted by $\iota(\mathcal{T})$.*

Definition 3.6 [15]. Let $\mathcal{T}$ be an I -fuzzy topology on X . $\iota(\mathcal{T})$ is called a generated fuzzifying topology by $\mathcal{T}$.

We get another functor ι from I -FTOP to FYS.

LEMMA 3.7 [15]. Let (X, ξ) be a fuzzifying topological space and define $\omega(\xi) : I^X \rightarrow I$ as follows: $\omega(\xi)(A) = \bigwedge_{r \in I_0} \xi(\sigma_r(A))$ for all $A \in I^X$. Then $\omega(\xi)$ is an I -fuzzy topology on X .

From Lemma 3.7, we know that ω is a functor from FYS to I -FTOP.

LEMMA 3.8 [15]. (1) For every $\xi \in \text{FYS}(X)$, $\iota(\omega(\xi)) = \xi$.

(2) For every $\mathcal{T} \in I\text{-FTOP}(X)$, $\omega(\iota(\mathcal{T})) \geq \mathcal{T}$. If $\mathcal{T} = \omega(\xi)$, then $\omega(\iota(\mathcal{T})) = \mathcal{T}$.

COROLLARY 3.9 [15]. Both $\omega : \text{FYS}(X) \rightarrow I\text{-FTOP}(X)$ and $\iota : I\text{-FTOP}(X) \rightarrow \text{FYS}(X)$ are complete lattice isomorphisms.

COROLLARY 3.10. FYS is isomorphic to I -IFTOP.

Now we begin to study the relations between the categories FYS, I -WIFTOP, I -SFTOP, I -IFTOP, and I -FTOP. Firstly, we give the left adjoint and the right adjoint of the inclusion functor i from I -IFTOP to I -FTOP.

LEMMA 3.11. Let $(I^X, \mathcal{S})$ be a stratified I -fuzzy topological space and let $(I^Y, \mathcal{T})$ be an induced I -fuzzy topological space. Then $f_I^- : (I^X, \mathcal{S}) \rightarrow (I^Y, \mathcal{T})$ is I -fuzzy continuous if and only if $f_I^- : (I^X, \omega([\mathcal{S}])) \rightarrow (I^Y, \omega([\mathcal{T}])) = (I^Y, \mathcal{T})$ is I -fuzzy continuous.

Proof. Since $(I^X, \mathcal{S})$ is stratified, we have

$$\begin{aligned} \omega([\mathcal{S}]) (A) &= \bigwedge_{r \in I_0} \mathcal{S}(1_{\sigma_r(A)}) = \bigwedge_{r \in I_0} \mathcal{S}(\bar{r}) \wedge \mathcal{S}(1_{\sigma_r(A)}) \\ &\leq \bigwedge_{r \in I_0} \mathcal{S}(\bar{r} 1_{\sigma_r(A)}) \leq \mathcal{S}\left(\bigvee_{r \in I_0} \bar{r} 1_{\sigma_r(A)}\right) = \mathcal{S}(A). \end{aligned} \quad (3.1)$$

Hence we get that $f_I^- : (I^X, \mathcal{S}) \rightarrow (I^Y, \mathcal{T})$ is I -fuzzy continuous if $f_I^- : (I^X, \omega([\mathcal{S}])) \rightarrow (I^Y, \omega([\mathcal{T}])) = (I^Y, \mathcal{T})$ is I -fuzzy continuous. Conversely, let $f_I^- : (I^X, \mathcal{S}) \rightarrow (I^Y, \mathcal{T})$ be I -fuzzy continuous, that is, $\mathcal{T}(B) \leq \mathcal{S}(f_I^-(B))$ for all $B \in I^Y$. Since $\mathcal{T}$ is induced, we have $\mathcal{T}(B) = \bigwedge_{r \in I_0} \mathcal{T}(1_{\sigma_r(B)})$. Hence

$$\mathcal{T}(B) = \bigwedge_{r \in I_0} \mathcal{T}(1_{\sigma_r(B)}) \leq \bigwedge_{r \in I_0} \mathcal{S}(f_I^-(1_{\sigma_r(B)})) = \bigwedge_{r \in I_0} \mathcal{S}(1_{\sigma_r(f_I^-(B))}) = \omega([\mathcal{S}]) (f_I^-(B)). \quad (3.2)$$

Therefore $f_I^- : (I^X, \omega([\mathcal{S}])) \rightarrow (I^Y, \mathcal{T})$ is I -fuzzy continuous. $\square$

LEMMA 3.12. Let $(I^X, \mathcal{T})$ be an I -fuzzy topological space and let $(I^Y, \mathcal{S})$ be an induced I -fuzzy topological space. Then $f_I^- : (I^Y, \mathcal{S}) \rightarrow (I^X, \mathcal{T})$ is I -fuzzy continuous if and only if $f_I^- : (I^Y, \mathcal{S}) \rightarrow (I^X, \omega(\iota(\mathcal{T})))$ is I -fuzzy continuous.

Proof. The sufficiency is obvious. We need to prove the necessity. In fact, we have

$$\begin{aligned}
\omega(\iota(\mathcal{T}))(A) &= \bigwedge_{r \in I_0} \bigvee_{\bigcup_{\lambda \in \Lambda} V_\lambda = \sigma_r(A)} \bigwedge_{\lambda \in \Lambda} \bigvee_{(\cap)_{\beta \in \Lambda_\lambda} W_{\lambda\beta} = V_\lambda} \bigwedge_{\beta \in \Lambda_\lambda} \bigvee_{\mu \in I_0} \bigvee \{ \mathcal{T}(D) \mid \sigma_\mu(D) = W_{\lambda\beta} \} \\
&\leq \bigwedge_{r \in I_0} \bigvee_{\bigcup_{\lambda \in \Lambda} V_\lambda = \sigma_r(A)} \bigwedge_{\lambda \in \Lambda} \bigvee_{(\cap)_{\beta \in \Lambda_\lambda} W_{\lambda\beta} = V_\lambda} \bigwedge_{\beta \in \Lambda_\lambda} \bigvee_{\mu \in I_0} \bigvee \{ \mathcal{S}(f_I^-(D)) \mid \sigma_\mu(D) = W_{\lambda\beta} \} \\
&\leq \bigwedge_{r \in I_0} \bigvee_{\bigcup_{\lambda \in \Lambda} V_\lambda = \sigma_r(A)} \bigwedge_{\lambda \in \Lambda} \bigvee_{(\cap)_{\beta \in \Lambda_\lambda} W_{\lambda\beta} = V_\lambda} \bigwedge_{\beta \in \Lambda_\lambda} \mathcal{S}(1_{f^-(W_{\lambda\beta})}) \leq \mathcal{S}(f_I^-(A)),
\end{aligned} \tag{3.3}$$

thus the conclusion. $\square$

From Lemmas 3.11 and 3.12, we have the following theorems.

THEOREM 3.13. (1) $\omega \circ \iota$ is the right adjoint of the inclusion functor $i : I\text{-IFTOP} \rightarrow I\text{-FTOP}$.

(2) $\omega \circ [\cdot]$ is the left adjoint of the inclusion functor $i : I\text{-IFTOP} \rightarrow I\text{-SFTOP}$.

THEOREM 3.14. $I\text{-IFTOP}$ is a reflective and coreflective full subcategory of $I\text{-SFTOP}$ and $I\text{-IFTOP}$ is a coreflective full subcategory of $I\text{-WIFTOP}$. Hence, $I\text{-IFTOP}$ is also a coreflective full subcategory of $I\text{-FTOP}$.

COROLLARY 3.15. FYS is a reflective and coreflective full subcategory of $I\text{-SFTOP}$. Hence TOP is a reflective and coreflective full subcategory of $I\text{-SFTOP}$.

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