MATRIX TRANSFORMATIONS BETWEEN THE SPACES OF CESÀRO SEQUENCES AND INVARIANT MEANS

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Received 6 March 2005; Revised 16 October 2005; Accepted 18 December 2005

The main purpose of this paper is to characterize the classes of matrices $(\text{ces}[(p), (q)], c^\sigma)$ and $(\text{ces}[(p), (q)], l_\infty^\sigma)$, where c^σ is the space of all bounded sequences all of whose σ -means are equal, l_∞^σ is the space of σ -bounded sequences, and $\text{ces}[(p), (q)]$ is the generalized Cesàro sequence space.

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1. Introduction

Let ω be the space of all sequences, real or complex, and let l_∞ and c , respectively, be the Banach spaces of bounded and convergent sequences $x = (x_n)$ with norm $\|x\| = \sup_{k \geq 0} |x_k|$. Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or a σ -mean if and only if (i) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for each n ; (ii) $\phi(e) = 1$, where $e = (1, 1, 1, \dots)$; and (iii) $\phi((x_{\sigma(n)})) = \phi(x)$, $x \in l_\infty$.

For certain kinds of mappings, every σ -mean extends the limit functional ϕ on c in the sense that $\phi(x) = \lim x$ for $x \in c$ (see [2, 15]). Consequently, $c \subset c^\sigma$, where c^σ is the set of bounded sequences, all of whose invariant means are equal (see [1, 9, 10]). When σ is translation, the σ -means are classical Banach limits on l_∞ (see [2]) and c^σ is the set of almost convergent sequences $\hat{c}$ (see [7]). Almost convergence for double sequences was introduced and studied by Móricz and Rhoades [8] and further by Mursaleen and Savaş [13], Mursaleen and Edely [12], and Mursaleen [11].

If $x = (x_n)$, write $Tx = (Tx_n) = (x_{\sigma(n)})$, then

$$c^\sigma = \left\{ x \in l_\infty : \lim_{m \rightarrow \infty} t_{m,n}(x) = L, \text{ uniformly in } n, L = \sigma - \lim x \right\}, \quad (1.1)$$

where

$$t_{m,n}(x) = \frac{1}{m+1} \sum_{j=0}^m T^j x_n \text{ with } T^j x_n = x_{\sigma^j(n)}, t_{-1,n}(x) = 0. \quad (1.2)$$

We define l_∞^σ the space of σ -bounded sequences (Ahmad et al. [2]) in the following way.

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Let $x_n = z_0 + z_1 + z_2 + \cdots + z_n$ and

$$l_\infty^\sigma = \left\{ z \in \omega : \sup_{m,n} |\psi_{m,n}(z)| < \infty \right\}, \quad (1.3)$$

where

$$\begin{aligned} \psi_{m,n}(z) &= t_{m,n}(x) - t_{m-1,n}(x) \\ &= \frac{1}{m(m+1)} \sum_{j=1}^m j \sum_{i=h_{j-1}+1}^{h_j} z_i, \quad h_j = \sigma^j(n). \end{aligned} \quad (1.4)$$

If $\sigma(n) = (n+1)$, then l_∞^σ is the set of almost bounded sequences $\widehat{l}_\infty$ (see [14]).

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} ($n, k = 1, 2, \dots$) and X, Y two subsets of ω . We say that the matrix A defines a matrix transformation from X into Y if for every sequence $x = (x_k) \in X$ the sequence $A(x) = (A_n(x)) \in Y$, where $A_n(x) = \sum_k a_{nk} x_k$ converges for each n . We denote the class of matrix transformations from X into Y by (X, Y) .

The main purpose of this paper is to characterize the classes $(\text{ces}[(p), (q)], c^\sigma)$ and $(\text{ces}[(p), (q)], l_\infty^\sigma)$ and deduce some known and unknown interesting results as corollaries.

The classes $(\text{ces}[(p), (q)], c^\sigma)$ and $(\text{ces}[(p), (q)], l_\infty^\sigma)$ are due to Khan and Rahman [4].

If $\{q_n\}$ is a sequence of positive real numbers, then for $p = (p_r)$ with $\inf p_r > 0$, we define the space $\text{ces}[(p), (q)]$ by

$$\text{ces}[(p), (q)] = \left\{ x \in \omega : \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_k |x_k| \right)^{p_r} < \infty \right\}, \quad (1.5)$$

where $Q_{2^r} = q_{2^r} + q_{2^r+1} + \cdots + q_{2^{r+1}-1}$ and $\sum_r$ denotes a sum over the range $2^r \leq k < 2^{r+1}$.

Remark 1.1. If $q_n = 1$ for all n , then $\text{ces}[(p), (q)]$ reduces to $\text{ces}(p)$ studied by Lim [6]. Also, if $p_n = p$ for all n and $q_n = 1$ for all n , then $\text{ces}[(p), (q)]$ reduces to ces_p studied by Lim [5].

For any bounded sequence p , the space $\text{ces}[(p), (q)]$ is a paranormed space with the paranorm given by (see [4])

$$g(x) = \left(\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_k |x_k| \right)^{p_r} \right)^{1/M} \quad (1.6)$$

if $H = \sup_r p_r < \infty$ and $M = \max(1, H)$.

2. Sequence-to-sequence transformations

In this section, we characterize the classes $(\text{ces}[(p), (q)], c^\sigma)$ and $(\text{ces}[(p), (q)], l_\infty^\sigma)$.

We write $a(n, k)$ to denote the elements a_{nk} of the matrix A , and for all integers $n, m \geq 1$, we write

$$\begin{aligned} t_{mn}(Ax) &= \frac{Ax_n + TAx_n + \cdots + T^m Ax_n}{m+1} \\ &= \sum_k t(n, k, m)x_k, \end{aligned} \quad (2.1)$$

where $t(n, k, m) = 1/(m+1) \sum_{j=0}^m a(\sigma^j(n), k)$.

We also define the spaces of σ -convergent series and σ -bounded series, respectively, as follows:

$$\begin{aligned} c_s^\sigma &= \left\{ x : \sum_{i=1}^m \left(\frac{1}{i+1} \sum_{j=0}^i x_{\sigma^j(n)} \right) \text{ is convergent uniformly in } n, \text{ as } m \rightarrow \infty \right\}, \\ b_s^\sigma &= \left\{ x : \sup_{n,m} \sum_{i=1}^m \left(\frac{1}{i+1} \sum_{j=0}^i x_{\sigma^j(n)} \right) < \infty \right\}. \end{aligned} \quad (2.2)$$

If we take $\sigma(n) = n+1$, c_s^σ and b_s^σ reduce to $\hat{c}_s$ and $\hat{b}_s$, as defined below:

$$\begin{aligned} \hat{c}_s &= \left\{ x : \sum_{i=1}^m \left(\frac{1}{i+1} \sum_{j=0}^i x_{j+n} \right) \text{ is convergent uniformly in } n, \text{ as } m \rightarrow \infty \right\}, \\ \hat{b}_s &= \left\{ x : \sup_{n,m} \sum_{i=1}^m \left(\frac{1}{i+1} \sum_{j=0}^i x_{j+n} \right) < \infty \right\}. \end{aligned} \quad (2.3)$$

Now we prove the following theorem.

THEOREM 2.1. *Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (\text{ces}[(p), (q)], c^\sigma)$ if and only if*

(i) *there exists an integer $E > 1$ such that for all n ,*

$$U(E) = \sup_m \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r \left(\frac{|t(n, k, m)|}{q_k} \right) \right)^{t_r} E^{-t_r} < \infty, \quad (2.4)$$

where $1/p_r + 1/t_r = 1$, $r = 0, 1, 2, \dots$, and $\max_r$ means maximum over $2^r \leq k < 2^{r+1}$;

(ii) $a_{(k)} = (a_{nk})_{n=1}^\infty \in c^\sigma$ for each k , that is, $\lim_m t(n, k, m) = u_k$ uniformly in n , for each k .

In this case, σ -limit of Ax is $\sum_{k=1}^\infty u_k x_k$.

Proof

Necessity. Suppose that $A \in (\text{ces}[(p), (q)], c^\sigma)$. Now $\sum_{k=1}^\infty t(n, k, m)x_k$ exists for each m and n and $x \in \text{ces}[(p), (q)]$, whence $\{t(n, k, m)\}_k \in \text{ces}^*[(p), (q)]$ for each m and n , (see F. M. Khan and M. A. Khan [3] for Köthe-Toeplitz and continuous duals of $\text{ces}[(p), (q)]$).

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Therefore, it follows that each $\{f_{m,n}\}_m$ defined by

$$f_{m,n}(x) = t_{m,n}(Ax) \quad (2.5)$$

is an element of $\text{ces}^*[(p), (q)]$. Since $\text{ces}[(p), (q)]$ is complete and further for each n , $\sup_m |t_{m,n}(Ax)| < \infty$ on $\text{ces}[(p), (q)]$. Now arguing with the uniform boundedness principle, we have condition (i). Since $e_k \in \text{ces}[(p), (q)]$, condition (ii) follows.

Sufficiency. Suppose that the conditions hold. Fix $n \in \mathbb{N}$. For every integer $s \geq 1$, from (i) we have

$$\sum_{r=0}^s \left(Q_{2^r} \max_r (q_k^{-1} |t(n, k, m)|) \right)^{t_r} E^{-t_r} \leq \sup_m \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r (q_k^{-1} |t(n, k, m)|) \right)^{t_r} E^{-t_r}. \quad (2.6)$$

Now letting $s \rightarrow \infty$, we obtain

$$\lim_{m \rightarrow \infty} \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r (q_k^{-1} |t(n, k, m)|) \right)^{t_r} E^{-t_r} \leq \sup_m \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r (q_k^{-1} |t(n, k, m)|) \right)^{t_r} E^{-t_r}. \quad (2.7)$$

Therefore, from (ii) we have

$$\sum_{r=0}^{\infty} \left(Q_{2^r} \max_r (q_k^{-1} |u_k|) \right)^{t_r} E^{-t_r} \leq \sup_m \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r (q_k^{-1} |t(n, k, m)|) \right)^{t_r} E^{-t_r} < \infty. \quad (2.8)$$

Hence $(u_k)_k$ and $\{t(n, k, m)\}_k \in \text{ces}^*[(p), (q)]$, therefore the series $\sum_{k=1}^{\infty} t(n, k, m)x_k$ and $\sum_{k=1}^{\infty} u_k x_k$ converge for each m and n and $x \in \text{ces}[(p), (q)]$. For given $\epsilon > 0$ and $x \in \text{ces}[(p), (q)]$, choose s such that

$$\left(\sum_{r=s+1}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_k |x_k| \right)^{p_r} \right)^{1/M} < \epsilon. \quad (2.9)$$

Since (ii) holds, there exists m_0 such that

$$\left| \sum_{k=1}^s t(n, k, m) - u_k \right| < \epsilon \quad \forall m > m_0. \quad (2.10)$$

Since (i) holds, it follows that

$$\left| \sum_{k=s+1}^{\infty} t(n, k, m) - u_k \right| \text{ is arbitrary small.} \quad (2.11)$$

Therefore,

$$\lim_m \sum_{k=1}^{\infty} t(n, k, m) x_k = \sum_{k=1}^{\infty} u_k x_k, \quad \text{uniformly in } n. \quad (2.12)$$

This completes the proof. $\square$

Remark 2.1. For different choices of p, q , and σ , we can deduce many corollaries from the above theorem to characterize the matrix classes, for example, $(ces(p), c^\sigma)$, (ces_p, c^σ) , $(ces_p(q), c^\sigma)$, $(ces[(p), (q)], \hat{c})$, and so forth. The class $(ces(p), \hat{c})$ was characterized by F. M. Khan and M. A. Khan [3] which we can obtain directly from our theorem by taking $q_n = 1$ for all n and $\sigma(n) = n + 1$.

We write (see [2])

$$\begin{aligned} x_0 &= z_0 + z_1 + \cdots + z_n, \\ \psi_{m,n}(Az) &= \sum_k \alpha(n, k, m) z_k, \end{aligned} \quad (2.13)$$

where

$$\alpha(n, k, m) = \frac{1}{m(m+1)} \sum_{j=1}^m j \left[\sum_{i=h_{j-1}+1}^{h_j} a_{ik} \right], \quad h_j = \sigma^j(n). \quad (2.14)$$

Now we prove the following theorem.

THEOREM 2.2. *Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (ces[(p), (q)], l_\infty^\sigma)$ if and only if*

$$\sup_{m,n} \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r (q_k^{-1} |\alpha(n, k, m)|) \right)^{t_r} E^{-t_r} < \infty, \quad (2.15)$$

where E is an integer greater than 1 and $1/p_r + 1/t_r = 1$, $r = 0, 1, 2, \dots$

Proof

Necessity. Suppose that $A \in (ces[(p), (q)], l_\infty^\sigma)$. Now $\sum_{k=1}^{\infty} \alpha(n, k, m) z_k$ exists for each m and n and $z \in ces[(p), (q)]$, whence $\{\alpha(n, k, m)\}_k \in ces^*[(p), (q)]$ for each m and n . Therefore, it follows that $\{f_{m,n}\}$ defined by

$$f_{m,n}(x) = \psi_{m,n}(Az) \quad (2.16)$$

is an element of $ces^*[(p), (q)]$. Since $ces[(p), (q)]$ is complete and further $\sup_{m,n} |\psi_{m,n}(Az)| < \infty$ on $ces[(p), (q)]$, so by arguing with uniform boundedness principle, we have the condition.

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Sufficiency. Suppose that condition (2.15) holds. Fix $n \in \mathbb{N}$. For every integer $s \geq 1$ we have

$$\sum_{r=0}^s \left(Q_{2^r} \max_r (q_k^{-1} |\alpha(n, k, m)|) \right)^{t_r} E^{-t_r} \leq \sup_{m, n} \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r (q_k^{-1} |\alpha(n, k, m)|) \right)^{t_r} E^{-t_r}. \quad (2.17)$$

So

$$\begin{aligned} \lim_{s \rightarrow \infty} \sum_{r=0}^s \left(Q_{2^r} \max_r (q_k^{-1} |\alpha(n, k, m)|) \right)^{t_r} E^{-t_r} \\ \leq \sup_{m, n} \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r (q_k^{-1} |\alpha(n, k, m)|) \right)^{t_r} E^{-t_r} < \infty. \end{aligned} \quad (2.18)$$

Hence $\{\alpha(n, k, m)\} \in \text{ces}^*[(p), (q)]$. Therefore, the series $\sum_{k=1}^{\infty} \alpha(n, k, m) z_k$ converges for each m and n and $z \in \text{ces}[(p), (q)]$.

This completes the proof. $\square$

Remark 2.2. The matrix class $(\text{ces}(p), \widehat{l_{\infty}})$, was characterized by F. M. Khan and M. A. Khan [3] which we can obtain directly from the above theorem by letting $q_n = 1$ for all n and $\sigma(n) = n + 1$. Besides, we can further deduce many corollaries for different choices of p , q , and σ .

3. Sequence-to-series transformations

For all integers $m, n \geq 1$, we write

$$t_{mn}^*(Ax) = \sum_{i=1}^m t_{in}(Ax) = \sum_k \sum_{i=1}^m \frac{1}{i+1} \sum_{j=0}^i a(\sigma^j(n), k) x_k = \sum_k t^*(m, n, k) x_k, \quad (3.1)$$

where

$$t^*(m, n, k) = \sum_{i=1}^m \frac{1}{i+1} \sum_{j=0}^i a(\sigma^j(n), k). \quad (3.2)$$

THEOREM 3.1. *Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (\text{ces}[(p), (q)], c_s^{\sigma})$ if and only if*

(i) *there exists an integer $E > 1$ such that for all n ,*

$$U(E) = \sup_m \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r \left(\frac{|t^*(n, k, m)|}{q_k} \right) \right)^{t_r} E^{-t_r} < \infty, \quad (3.3)$$

where $1/p_r + 1/t_r = 1$, $r = 0, 1, 2, \dots$, and $\max_r$ means maximum over $2^r \leq k \leq 2^{r+1}$;

(ii) $a_{(k)} = \{a_{nk}\}_{n=1}^{\infty} \in c_s^{\sigma}$ for each k , that is, $\lim_m t^*(n, k, m) = u_k$ uniformly in n , for each k .

In this case, the σ -limit of Ax is $\sum_{k=1}^{\infty} u_k x_k$.

THEOREM 3.2. Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (\text{ces}[(p), (q)], b_{\infty}^{\sigma})$ if and only if

$$\sup_{m,n} \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r (q_k^{-1} |t^*(n, k, m)|) \right)^{t_r} E^{-t_r} < \infty, \quad (3.4)$$

where E is an integer greater than 1 and $1/p_r + 1/t_r = 1$, $r = 0, 1, 2, \dots$.

Proofs of Theorems 3.1 and 3.2 are similar to those of Theorems 2.1 and 2.2, respectively.

Remark 3.1. If σ is translation, then Theorems 3.1 and 3.2 give the characterization for the classes $(\text{ces}[(p), (q)], \hat{c}_s)$ and $(\text{ces}[(p), (q)], \hat{b}_s)$. As Remarks 2.1 and 2.2, for different choices of p , q , and σ , we can deduce many corollaries.

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