We proved two common fixed point theorems for four self-mappings and two set-valued mappings with \( \phi \)-contractive condition in a Menger space.

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1. Introduction and preliminaries

Probabilistic metric space was first introduced by Menger [6]. Later, there are many authors who have some detailed discussions and applications of a probabilistic metric space, for example, we may see Schweizer and Sklar [8]. Besides, there are many results about fixed point theorems in a probabilistic metric space with contractive types having appeared; we may see the papers [1–3, 9–12].

In this paper, we will prove two common fixed point theorems for four self-mappings and two set-valued mappings with \( \phi \)-contractive condition in a Menger space, which generalize some results of Dedeiç and Sarapa [4, 5], and Sehgal and Bharucha-Reid [9].

A mapping \( F : \mathbb{R} \rightarrow \mathbb{R}^+ \) is said to be a distribution if it is nondecreasing left continuous with \( \inf \{ F(t) : t \in \mathbb{R} \} = 0 \) and \( \sup \{ F(t) : t \in \mathbb{R} \} = 1 \).

We will denote by \( \mathcal{L} \) the set of all distribution functions while \( G \) will always denote the specific distribution function defined by

\[
G(t) = \begin{cases} 
0, & t \leq 0, \\
1, & t > 0.
\end{cases}
\]  

(1.1)

A probabilistic metric space (PM-space) [7] is an ordered pair \( (X, \mathcal{F}) \) consisting of a nonempty set \( X \) and a mapping \( \mathcal{F} \) from \( X \times X \) into the collections of all distribution functions on \( \mathbb{R} \). For \( x, y \in X \), we denote the distribution function \( \mathcal{F}(x, y) \) by \( F_{x,y} \) and \( F_{x,y}(u) \) represents the value of \( \mathcal{F}(x, y) \) at \( u \in \mathbb{R} \). The functions \( F_{x,y} \) are assumed to satisfy the following conditions:

1. \( F_{x,y}(u) = 1 \) for all \( u > 0 \) if and only if \( x = y \),
2. \( F_{x,y}(0) = 0 \) for all \( x, y \) in \( X \),

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(3) \( F_{x,y}(u) = F_{y,z}(u) \) for all \( x, y \) in \( X \), and
(4) if \( F_{x,y}(u) = 1 \) and \( F_{y,z}(v) = 1 \), then \( F_{x,z}(u + v) = 1 \) for all \( x, y, z \) in \( X \) and \( u, v > 0 \).

A mapping \( t : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is called a \( t \)-norm if
(1) \( t(a, 1) = a, t(0, 0) = 0 \),
(2) \( t(a, b) = t(b, a) \),
(3) \( t(c, d) \geq t(a, b) \) for \( c \geq a, d \geq b \), and
(4) \( t(t(a, b), c) = t(a, t(b, c)) \).

A Menger space is a triplet \( (X, \mathcal{F}, t) \), where \( (X, \mathcal{F}) \) is a PM-space, \( t \) is a \( T \)-norm, and the generalized triangle inequality

\[
F_{x,y}(u + v) \geq t(F_{x,y}(u), F_{y,z}(v))
\]  

holds for all \( x, y, z \) in \( X \) and \( u, v > 0 \).

The concept of neighborhoods in a Menger space was introduced by Schweizer and Sklar [8].

Let \( (X, \mathcal{F}, t) \) be a Menger space. If \( x \in X, \varepsilon > 0, \) and \( \lambda \in (0, 1) \), then an \((\varepsilon, \lambda)\)-neighborhood of \( x \), called \( U_x(\varepsilon, \lambda) \), is defined by

\[
U_x(\varepsilon, \lambda) = \{ y \in X : F_{x,y}(\varepsilon) > 1 - \lambda \}.
\]  

An \((\varepsilon, \lambda)\)-topology in \( X \) is the topology induced by the family \( \{ U_x(\varepsilon, \lambda) : x \in X, \varepsilon > 0, \lambda \in (0, 1) \} \) of neighborhood.

Remark 1.1. If \( t \) is continuous, then Menger space \( (X, \mathcal{F}, t) \) is a Hausdorff space in the \((\varepsilon, \lambda)\)-topology. (see [8]).

Let \( (X, \mathcal{F}, t) \) be a complete Menger space and \( A \subset X \). Then \( A \) is called a bounded set if

\[
\lim_{u \to \infty} \inf_{x,y \in A} F_{x,y}(u) = 1.
\]  

Throughout this paper, \( B(X) \) will denote the family of nonempty bounded subsets of a complete Menger space \( X \).

For all \( A, B \in B(X) \) and for all \( u > 0 \), we define

\[
\delta F_{A,B}(u) = \inf \{ F_{x,y}(u) : x \in A, y \in B \},
\]
\[
D F_{A,B}(u) = \sup \{ F_{x,y}(u) : x \in A, y \in B \},
\]
\[
H F_{A,B}(u) = \inf \left\{ \sup_{a \in A} \inf_{b \in B} F_{a,b}(u), \sup_{b \in B} \inf_{a \in A} F_{a,b}(u) \right\}.
\]  

Remark 1.2. It is clear that \( \delta F_{A,B}(u) = \delta F_{B,A}(u) \), \( D F_{A,B}(u) = D F_{B,A}(u) \), and \( H F_{A,B}(u) = H F_{B,A}(u) \), for all \( A, B \in B(X) \) and \( u > 0 \).
If $A = \{x\}$, we denote $\delta_{F_{\{x\},B}}(u) = \delta_{F_{x,B}}(u)$, $D_{F_{\{x\},B}}(u) = D_{F_{x,B}}(u)$, and $H_{F_{\{x\},B}}(u) = H_{F_{x,B}}(u)$.

Let $(X, \mathcal{F}, t)$ be a complete Menger space, and let $T : X \to \mathcal{B}(X)$ be a set-valued function and $I : X \to X$ a single-valued function. Then we say that $S$ and $I$ are compatible if

$$\lim_{n \to \infty} H_{F_{Sx_n,Ix_n}}(u) = 1,$$

whenever $\{x_n\}$ is a sequence in $X$ such that

$$\lim_{n \to \infty} \delta_{F_{Ix_n,Sx_n}}(u) = 1,$$ \quad $\forall u > 0.$  

Let $\{A_n\}$ be a sequence in $\mathcal{B}(X)$. We say that $\{A_n\}$ $\delta$-converges to a set $A$ in $X$ if

$$\lim_{n \to \infty} \delta_{F_{A_n,A}}(u) = 1,$$ \quad for every $u > 0,$

and it is denoted by $A_n \xrightarrow{\delta} A$.

2. Main results

In this paper, we let $\mathbb{R}^+$ denote the set of all nonnegative real numbers, let $\mathbb{N}$ denote the set of all positive integers, and let $(X, \mathcal{F}, t)$ be a Menger space with $t(x, y) = \min(x, y)$.

We first prove the following lemmas.

**Lemma 2.1.** Let $(X, \mathcal{F}, \min)$ be a Menger space. Then for $A, B, C \in \mathcal{B}(X)$ and for $u, v > 0$,

$$\delta_{F_{A,C}}(u + v) \geq \min \{ \delta_{F_{A,B}}(u), \delta_{F_{B,C}}(v) \}. \tag{2.1}$$

**Proof.** For all $u, v > 0$, we have

$$\min \{ \delta_{F_{A,B}}(u), \delta_{F_{B,C}}(v) \} \leq \min \{ F_{a,b}(u), F_{b,c}(v) \} \leq F_{a,c}(u + v) \tag{2.2}$$

for each $a \in A$, $b \in B$, and $c \in C$.

This implies that $\min \{ \delta_{F_{A,B}}(u), \delta_{F_{B,C}}(v) \} \leq \delta_{F_{A,C}}(u + v)$. \hfill $\square$

**Lemma 2.2.** Let $(X, \mathcal{F}, \min)$ be a Menger space. Then for $A, B \in \mathcal{B}(X)$, $c \in X$, and for $u, v > 0$,

$$H_{F_{A,c}}(u + v) \geq \min \{ H_{F_{A,B}}(u), H_{F_{B,c}}(v) \}. \tag{2.3}$$

**Proof.** Since for each $a, b, c \in X$ and for all $u, v > 0$,

$$F_{a,c}(u + v) \geq \min \{ F_{a,b}(u), F_{b,c}(v) \}. \tag{2.4}$$

By taking $\inf_{c \in C}$, we have

$$\inf_{c \in C} F_{a,c}(u + v) \geq \min \left\{ F_{a,b}(u), \inf_{c \in C} F_{b,c}(v) \right\}. \tag{2.5}$$
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Hence,

\[
\sup_{a \in A} \inf_{c \in C} F_{a,c}(u + v) \geq \sup_{a \in A} \min \left\{ F_{a,b}(u), \inf_{c \in C} F_{b,c}(v) \right\} \\
= \min \left\{ \sup_{a \in A} F_{a,b}(u), \inf_{c \in C} F_{b,c}(v) \right\} \\
\geq \min \left\{ \sup_{a \in A} \inf_{b \in B} F_{a,b}(u), \inf_{c \in C} \inf_{b \in B} F_{b,c}(v) \right\}. \tag{2.6}
\]

Next, by taking \( \sup_{b \in B} \), we have

\[
\sup_{a \in A} \inf_{c \in C} F_{a,c}(u + v) \geq \sup_{a \in A} \min \left\{ \inf_{a \in A} \sup_{b \in B} F_{a,b}(u), \sup_{c \in C} \inf_{b \in B} F_{b,c}(v) \right\}. \tag{2.7}
\]

Similarly, for each \( a, b, c \in X \) and for all \( u, v > 0 \),

\[
F_{a,c}(u + v) \geq \min \{ F_{a,b}(u), F_{b,c}(v) \}. \tag{2.8}
\]

By taking \( \inf_{c \in C} \), we have

\[
\inf_{a \in A} F_{a,c}(u + v) \geq \min \left\{ \inf_{a \in A} F_{a,b}(u), \inf_{c \in C} F_{b,c}(v) \right\}. \tag{2.9}
\]

Hence,

\[
\sup_{c \in C} \inf_{a \in A} F_{a,c}(u + v) \geq \sup_{c \in C} \min \left\{ \inf_{a \in A} F_{a,b}(u), \inf_{c \in C} F_{b,c}(v) \right\} \\
= \min \left\{ \inf_{a \in A} F_{a,b}(u), \sup_{c \in C} F_{b,c}(v) \right\} \tag{2.10}
\]

\[
\geq \min \left\{ \inf_{a \in A} F_{a,b}(u), \inf \inf_{b \in B} F_{b,c}(v) \right\}.
\]

Next, by taking \( \sup_{b \in B} \), we have

\[
\sup_{c \in C} \inf_{a \in A} F_{a,c}(u + v) \geq \sup_{b \in B} \min \left\{ \inf_{a \in A} F_{a,b}(u), \inf \inf_{b \in B} F_{b,c}(v) \right\} \tag{2.11}
\]

\[
\geq \min \left\{ \sup_{b \in B} \inf_{a \in A} F_{a,b}(u), \sup \inf_{b \in B} F_{b,c}(v) \right\}.
\]
Therefore, we obtain that
\[
H_{F_{A,c}}(u + v) = \min \left\{ \sup_{c \in C} \inf_{a \in A} F_{a,c}(u + v), \sup_{a \in A} c \right\} = \min \left\{ \sup_{b \in B} \inf_{a \in A} F_{a,b}(u), \sup_{b \in B} c \right\}
\]
\[
\geq \min \left\{ \sup_{b \in B} \inf_{a \in A} F_{a,b}(u), \sup_{b \in B} c \right\} = \min \left\{ \sup_{b \in B} \inf_{a \in A} F_{a,b}(u), \sup_{b \in B} c \right\}
\]
\[
\geq \min \left\{ \sup_{b \in B} \inf_{a \in A} F_{a,b}(u), \sup_{b \in B} c \right\} = \min \{ H_{F_{A,B}}(u), H_{F_{B,c}}(v) \}.
\]
(2.12)

**Lemma 2.3.** Let \((X, \mathcal{F}, \min)\) be a Menger space. If \(A, B \in B(X)\), then 
\[
\lim_{u \to \infty} F_{A,B}(u) = 1.
\]

**Proof.** For any \(x \in A\) and \(y \in B\), by Lemma 2.1, we have
\[
\delta F_{A,B}(u) \geq \min \left\{ \sup_{b \in B} \inf_{a \in A} F_{a,b}(u), \sup_{b \in B} c \right\}.
\]
Letting \(u \to \infty\), we have
\[
\lim_{u \to \infty} \delta F_{A,B}(u) \geq \min \left\{ \lim_{u \to \infty} \sup_{b \in B} \inf_{a \in A} F_{a,b}(u), \lim_{u \to \infty} c \right\}.
\]
(2.14)
Since \(x \in A\), \(y \in B\), and \(A, B \in B(X)\), we have
\[
\lim_{u \to \infty} \delta F_{A,B}(u) = 1.
\]
(2.15)
Similarly, we have
\[
\lim_{u \to \infty} \delta F_{A,B}(u) = 1.
\]
(2.16)
By the definition of the PM-space, we have that \(\lim_{u \to \infty} F_{x,y}(u/3) = 1\).
Therefore, we conclude that
\[
\lim_{u \to \infty} \delta F_{A,B}(u) = 1.
\]
(2.17)
This completes the proof. □

The following lemma which was introduced by Chang [3], will play an important role for this paper.

**Lemma 2.4.** If \(\phi : \mathbb{R}^+ \to \mathbb{R}^+\) is a strictly increasing, continuous function such that \(0 < \phi(u) < u\) for all \(u > 0\), \(\lim_{u \to \infty} \phi(u) = \infty\), and if for each \(u > 0\), \(\phi^n(u) = u\) and \(\phi^{-n}(u) = \phi^{-1}(\phi^{n+1}(u))\) for each \(n \in \mathbb{N}\) are denoted, then \(\lim_{n \to \infty} \phi^{-n}(u) = \infty\).

In the sequel, we let \(\Phi = \{ \phi : \mathbb{R}^+ \to \mathbb{R}^+ : \phi\) is a strictly increasing, continuous function with \(\phi(t) < t\) for all \(t > 0\}\}.

**Lemma 2.5.** Let \((X, \mathcal{F}, \min)\) be a Menger space and \(\{Y_n\}\) a sequence in \(B(X)\). If for each \(u > 0\) and for each \(n \in \mathbb{N}\),
\[
\delta F_{Y_{n+1},Y_{n+1}}(\phi(u)) \geq \delta F_{Y_n,Y_{n+1}}(u), \quad \phi \in \Phi,
\]
(2.18)
then
\[ \lim_{n \to \infty} \delta F_{Y_n, Y_{n+1}}(u) = 1. \] (2.19)

Proof. For \( u > 0 \), by induction, we have
\[ \delta F_{Y_{n+1}, Y_{n+2}}(u) \geq \delta F_{Y_n, Y_{n+1}}(\phi^{-1}(u)) \geq \cdots \geq \delta F_{Y_1, Y_2}(\phi^{-n}(u)), \quad \text{for each } n \in \mathbb{N}. \] (2.20)

By Lemma 2.4, we also have that \( \phi^{-n}(u) \to \infty \), as \( n \to \infty \).
Next, since \( Y_n \) is a bounded set and \( \delta F_{Y_1, Y_2}(\phi^{-n}(u)) \to 1 \) as \( n \to \infty \), hence we have
\[ \lim_{n \to \infty} \delta F_{Y_{n+1}, Y_{n+2}}(u) = 1. \] (2.21)

\[ \square \]

Lemma 2.6. Let \((X, \mathcal{P}, \min)\) be a Menger space, and let \( A, B \in B(X) \). If
\[ \delta F_{A, B}(\phi(u)) \geq \delta F_{A, B}(u), \quad \text{for } u > 0, \] (2.22)
then \( A = B = a \), for some \( a \in X \).

Proof. For \( u > 0 \), by induction, we have
\[ \delta F_{A, B}(u) \geq \delta F_{A, B}(\phi^{-1}(u)) \geq \cdots \geq \delta F_{A, B}(\phi^{-n}(u)). \] (2.23)

Since \( A, B \in B(X) \), by Lemma 2.3, we have
\[ \lim_{n \to \infty} \delta F_{A, B}(\phi^{-n}(u)) = 1, \] (2.24)
and by Lemma 2.5, we have \( \delta F_{A, B}(u) = 1 \) for \( u > 0 \). Thus we conclude that \( A = B = \{a\} \) for some \( a \in X \).

The following lemma was introduced by Schweizer and Sklar [8].

Lemma 2.7. Let \((X, \mathcal{P}, \min)\) be a Menger space. If \( a_n \to a \) and \( b_n \to b \), then for \( u > 0 \),
\[ \lim_{n \to \infty} \inf_{\delta F_{a_n, b_n}(u)}(u) = F_{a,b}(u). \] (2.25)

From Lemma 2.7, we conclude the following lemma.

Lemma 2.8. Let \((X, \mathcal{P}, \min)\) be a Menger space. If \( A_n \overset{\delta}{\to} a \) and \( B_n \overset{\delta}{\to} b \), then for \( u > 0 \),
\[ \lim_{n \to \infty} \inf_{\delta F_{A_n, B_n}(u)}(u) = F_{a,b}(u). \] (2.26)

Proof. For \( u > 0 \) and for \( \varepsilon > 0 \), Since \( F_{a,b}(u) \) is left continuous function at \( u \), there exists a positive number \( k \) with \( 0 < 2k < u \) such that \( F_{a,b}(u) - F_{a,b}(u - 2k) < \varepsilon \).

Since \( k > 0 \) and \( A_n \overset{\delta}{\to} a \), \( B_n \overset{\delta}{\to} b \), hence we may take \( m \in \mathbb{N} \) such that for \( n \geq m \),
\[ \delta F_{A_n, a}(k) \geq F_{a,b}(u - 2k), \quad \delta F_{B_n, b}(k) \geq F_{a,b}(u - 2k). \] (2.27)
Hence, for \( n > m \),

\[
\delta F_{A_n, B_n}(u) \geq \min \{ \delta F_{A_n, a}(u - k), \delta F_{b, B_n}(k) \} \\
\geq \min \{ \delta F_{a, a}(k), \delta F_{a, b}(u - 2k), \delta F_{b, B_n}(k) \} = F_{a, b}(u - 2k),
\]

and hence

\[
-\delta F_{A_n, B_n}(u) \leq -F_{a, b}(u - 2k).
\]

Therefore, we conclude that

\[
F_{a, b}(u) - \delta F_{A_n, B_n}(u) < F_{a, b}(u) - F_{a, b}(u - 2k) < \epsilon.
\]

Taking \( \lim_{n \to \infty} \inf \), we have

\[
F_{a, b}(u) - \lim_{n \to \infty} \inf \delta F_{A_n, B_n}(u) < \epsilon.
\]

For any \( a_n \in A_n, b_n \in B_n \), since \( A_n \xrightarrow{\delta} a \) and \( B_n \xrightarrow{\delta} b \), we have \( a_n \to a, b_n \to b \). Thus, for \( u > 0 \)

\[
\delta F_{A_n, B_n}(u) \leq F_{a_n, b_n}(u).
\]

Taking \( \lim_{n \to \infty} \inf \), we have

\[
\lim_{n \to \infty} \inf \delta F_{A_n, B_n}(u) \leq \lim_{n \to \infty} \inf F_{a_n, b_n}(u).
\]

By Lemma 2.7, we have

\[
\lim_{n \to \infty} \inf F_{a_n, b_n}(u) = F_{a, b}(u), \text{ and so } F_{a, b}(u) - \lim_{n \to \infty} \inf \delta F_{A_n, B_n}(u) \geq 0.
\]

Therefore, for any \( \epsilon > 0 \),

\[
\epsilon > F_{a, b}(u) - \lim_{n \to \infty} \inf \delta F_{A_n, B_n}(u) \geq 0.
\]

This implies that

\[
\lim_{n \to \infty} \inf \delta F_{A_n, B_n}(u) = F_{a, b}(u), \text{ for } u > 0.
\]

\[
\square
\]

The following two theorems are our main results for this paper.
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Theorem 2.9. Let \((X,\mathcal{F},\min)\) be a complete Menger space. Let \(f,g,\eta,\xi : X \to X\) be four single-valued functions, and let \(S,T : X \to B(X)\) two set-valued functions. If the following conditions are satisfied:

(i) \(S(X) \subset \xi g(X),\ T(X) \subset \eta f(X),\)
(ii) \(\eta f = f \eta,\ \xi g = g \xi,\ \) and \(Sf = fS,\ Tg = gT,\)
(iii) \(\eta f\) or \(\xi g\) is continuous,
(iv) \((S,\eta f)\) and \((T,\xi g)\) are compatible, and
(v) for \(u > 0,\)

\[
\delta F_{\xi g}x_{2n+1}(\phi(u)) \geq \min \{F_{\eta f}x_2,\delta F_{x2}x_{2n}(u),\delta F_{\xi g}x_{2n}(u),\delta F_{\xi g}x_{2n+1}(u),\delta F_{\xi g}x_{2n+1}x_{2n+1}((1 - \alpha)u),
\delta F_{\xi g}x_{2n+1}x_{2n+1}((1 + \alpha)u)\}
\]

(2.37)

for all \(x,y \in X,\ \beta \in (0,2),\) where \(\phi \in \Phi,\) then \(f,\ g,\ \eta,\ \xi,\ S,\ \) and \(T\) have a unique common fixed point \(z\) in \(X.\)

Proof. Let \(x_0 \in X.\) Define the sequence \(\{x_n\}\) recursively as follows:

\[
\xi gx_{2n+1} \in Sx_{2n} = Z_{2n},\quad \eta fx_{2n+2} \in Tx_{2n+1} = Z_{2n+1}.
\]

(2.38)

For \(n \in \mathbb{N}\) and for all \(u > 0,\) and \(\beta = (1 - \alpha)\) with \(\alpha \in (0,1),\)

\[
\delta F_{Z_{2n}}Z_{2n+1}(\phi(u))
\]

\[
= \delta F_{Sx_{2n}}T_{x_{2n+1}}(\phi(u))
\]

\[
\geq \min \{F_{\eta f}x_{2n}x_{2n}(u),\delta F_{\eta f}x_{2n}x_{2n}(u),\delta F_{\xi g}x_{2n+1},T_{x_{2n+1}}(u),\delta F_{\xi g}x_{2n+1}x_{2n+1}((1 - \alpha)u),
\delta F_{\xi g}x_{2n+1}x_{2n+1}((1 + \alpha)u)\}
\]

\[
\geq \min \{\delta F_{Z_{2n-1}},Z_{2n}(u),\delta F_{Z_{2n-1}},Z_{2n}(u),\delta F_{Z_{2n}},Z_{2n+1}(u),\delta F_{Z_{2n}},Z_{2n+1}((1 - \alpha)u),
\delta F_{Z_{2n}},Z_{2n+1}((1 + \alpha)u)\}
\]

\[
\geq \min \{\delta F_{Z_{2n-1}},Z_{2n}(u),\delta F_{Z_{2n-1}},Z_{2n}(u),\delta F_{Z_{2n}},Z_{2n+1}(u),\delta F_{Z_{2n}},Z_{2n+1}(u),\delta F_{Z_{2n}},Z_{2n+1}(au)\}
\]

\[
= \min \{\delta F_{Z_{2n-1}},Z_{2n}(u),\delta F_{Z_{2n}},Z_{2n+1}(u),\delta F_{Z_{2n}},Z_{2n+1}(au)\}.
\]

(2.39)

As \(t\)-norm = \(\min\) is continuous, letting \(\alpha \to 1,\) we have

\[
\delta F_{Z_{2n}},Z_{2n+1}(\phi(u)) \geq \min \{\delta F_{Z_{2n-1}},Z_{2n}(u),\delta F_{Z_{2n}},Z_{2n+1}(u)\}.
\]

(2.40)
By Lemma 2.6, we have
\[ \delta F_{Z_{2n}, Z_{2n+1}} (\phi(u)) \geq \delta F_{Z_{2n-1}, Z_{2n}} (u). \] (2.41)

Similarly, we also can prove that for \( n \in \mathbb{N} \) and for all \( u > 0 \),
\[ \delta F_{Z_{2n+1}, Z_{2n+2}} (\phi(u)) \geq \delta F_{Z_{2n}, Z_{2n+1}} (u). \] (2.42)

So, we have
\[ \delta F_{Z_{n}, Z_{n+1}} (\phi(u)) \geq \delta F_{Z_{n+1}, Z_{n+2}} (\phi(u)), \quad \forall n \in \mathbb{N}, \ u > 0. \] (2.43)

By Lemma 2.5, we conclude that
\[ \lim_{n \to \infty} \delta F_{Z_{n}, Z_{n+1}} (u) = 1, \quad \forall u > 0. \quad (\ast) \]

Now, we consider the condition (\( v \)) with \( \beta = 1 \), and then we claim that
for \( \varepsilon > 0, \ \lambda \in (0, 1) \) there is \( M(\varepsilon, \lambda) \in \mathbb{N} \) such that \( \delta F_{Z_{n}, Z_{m}} (\varepsilon) \geq 1 - \lambda \) for \( n, m \geq M \).

If it is not the case, then there exists \( \varepsilon' > 0, \ \lambda' \in (0, 1) \) such that for \( k \in \mathbb{N} \), there exist
\( n_k > m_k \geq k \) such that
1. \( n_k \) is even and \( m_k \) is odd,
2. \( \delta F_{Z_{n_k}, Z_{m_k}} (\varepsilon') < 1 - \lambda' \), and
3. \( n_k \) is the smallest even number such that (1) and (2) hold.

By (\( \ast \)), we may choose \( m_1 \in \mathbb{N} \) such that for \( n \geq m_1 \),
\[ \delta F_{Z_{n}, Z_{n+1}} \left( \min \left\{ \frac{\varepsilon'}{2}, \frac{\phi^{-1}(\varepsilon') - \varepsilon'}{2} \right\} \right) > 1 - \lambda'. \] (2.45)

So for \( k > m_1 \), \( n_k \geq m_k + 3 \), and so for \( k > m_1 \),
\[ 1 - \lambda' > \delta F_{Z_{n_k}, Z_{m_k}} (\varepsilon') = \delta F_{Z_{n_k}, Z_{m_k}} (\varepsilon') \geq \min \{ F_{\forall f_{n_k}, \xi_{g_{n_k}}} (\phi^{-1}(\varepsilon')), \delta F_{\forall f_{n_k}, S_{n_k}} (\phi^{-1}(\varepsilon')), \delta F_{\forall g_{n_k}, T_{n_k}} (\phi^{-1}(\varepsilon')) \}, \]
\[ \delta F_{\forall g_{n_k}, S_{n_k}} (\phi^{-1}(\varepsilon')), \delta F_{\forall f_{n_k}, T_{n_k}} (\phi^{-1}(\varepsilon')) \} \] (2.46)
\[ \geq \min \{ \delta F_{Z_{m_k}, Z_{n_k}} (\phi^{-1}(\varepsilon')), \delta F_{Z_{m_k-1}, Z_{n_k}} (\phi^{-1}(\varepsilon')), \delta F_{Z_{m_k-1}, Z_{n_k}} (\phi^{-1}(\varepsilon')), \}
\[ \delta F_{Z_{m_k}, Z_{n_k-1}} (\phi^{-1}(\varepsilon')), \delta F_{Z_{m_k-1}, Z_{n_k}} (\phi^{-1}(\varepsilon')) \}. \]
Since
\[
\delta F_{Z_{n_k-1}, Z_{m_k}} (\phi^{-1}(\epsilon')) \geq \min \left\{ \delta F_{Z_{n_k-1}, Z_{m_k-2}} (\phi^{-1}(\epsilon') - \epsilon'), \delta F_{Z_{n_k-2}, Z_{m_k}} (\epsilon') \right\},
\]
\[
\geq \min \left\{ \delta F_{Z_{n_k-1}, Z_{m_k-2}} \left( \frac{\phi^{-1}(\epsilon') - \epsilon'}{2} \right), \delta F_{Z_{n_k-2}, Z_{m_k-1}} (\epsilon') \right\},
\]
\[
\geq \min \left\{ \delta F_{Z_{n_k-1}, Z_{m_k-2}} \left( \frac{\phi^{-1}(\epsilon') - \epsilon'}{2} \right), \delta F_{Z_{n_k-2}, Z_{m_k-1}} \left( \frac{\phi^{-1}(\epsilon') - \epsilon'}{2} \right) \right\},
\]
\[
\geq \min \left\{ \delta F_{Z_{n_k-1}, Z_{m_k-2}} \left( \frac{\phi^{-1}(\epsilon') - \epsilon'}{2} \right), \delta F_{Z_{n_k-2}, Z_{m_k-1}} \left( \frac{\phi^{-1}(\epsilon') - \epsilon'}{2} \right) \right\},
\]
\[
\geq \min \left\{ \delta F_{Z_{n_k-1}, Z_{m_k-2}} (\epsilon'), \delta F_{Z_{n_k-2}, Z_{m_k}} (\epsilon') \right\},
\]
\[
\geq \min \left\{ \delta F_{Z_{n_k-1}, Z_{m_k-2}} (\epsilon'), \delta F_{Z_{n_k-2}, Z_{m_k}} (\epsilon') \right\},
\]
\[
\geq \min \left\{ \delta F_{Z_{n_k-1}, Z_{m_k-2}} \left( \frac{\phi^{-1}(\epsilon') - \epsilon'}{2} \right), \delta F_{Z_{n_k-2}, Z_{m_k}} \left( \frac{\phi^{-1}(\epsilon') + \epsilon'}{2} \right) \right\},
\]
\[
\geq \min \left\{ \delta F_{Z_{n_k-1}, Z_{m_k-2}} \left( \frac{\phi^{-1}(\epsilon') - \epsilon'}{2} \right), \delta F_{Z_{n_k-2}, Z_{m_k}} \left( \frac{\phi^{-1}(\epsilon') - \epsilon'}{2} \right) \right\},
\]
\[
(2.47)
\]
so for \( k > m_1 \), we have
\[
1 - \lambda' > \delta F_{Z_{n_k}, Z_{m_k}} (\epsilon') \geq 1 - \lambda',
\]
(2.48)
which is a contradiction. And, since \( X \) is complete, hence for any choice of \( z_n \) in \( Z_n \), the sequence \( \{ z_n \} \) must converge to some point, say, \( z \) in \( X \). The point \( z \) is independent of the choice of \( z_n \) and so we have
\[
\eta f x_{2n} \to z, \quad \xi g x_{2n+1} \to z, \quad S x_{2n} \to \{ z \}, \quad T x_{2n+1} \to \{ z \}.
\]
(2.49)
That is, for \( u > 0 \),
\[
F_{\eta f x_{2n}} (u) \to 1, \quad F_{\xi g x_{2n+1}} (u) \to 1, \quad \delta F_{S x_{2n}} (u) \to 1, \quad \delta F_{T x_{2n+1}} (u) \to 1 \quad \text{as} \ n \to \infty.
\]
(2.50)
Assume that the function $\eta f$ is continuous, then for $u > 0$, we have
\[
\lim_{n \to \infty} F(\eta f)^2x_{2n}\eta f_z(u) = 1, \quad \lim_{n \to \infty} \delta F_{\eta fSx_{2n}\eta f_z(u)} = 1. \quad (2.51)
\]

By $\lim_{n \to \infty} F_{\eta fSx_{2n}z}(u) = 1$ and $\lim_{n \to \infty} \delta F_{Sx_{2n}z}(u) = 1$, we obtain $\lim_{n \to \infty} \delta F_{Sx_{2n}\eta fSx_{2n}z}(u) = 1$. Since $\delta$ and $\eta f$ are compatible, and for $u > 0$, $\lim_{n \to \infty} \delta F_{Sx_{2n}\eta fSx_{2n}z}(u) = 1$, we have $\lim_{n \to \infty} H_{F_{\eta fSx_{2n}Sx_{2n}f_x}(u)} = 1$ and $\lim_{n \to \infty} \delta F_{Sx_{2n}\eta fSx_{2n}z}(u) \geq \min \{ H_{F_{\eta fSx_{2n}Sx_{2n}f_x}(u/2)}, H_{F_{\eta fSx_{2n}\eta fSx_{2n}z}(u/2)} \}$. And, since $\lim_{n \to \infty} \delta F_{Sx_{2n}\eta fSx_{2n}z}(u/2) = 1$, $\lim_{n \to \infty} H_{F_{\eta fSx_{2n}Sx_{2n}f_x}(u/2)} = 1$, we have
\[
\lim_{n \to \infty} H_{F_{\eta fSx_{2n}Sx_{2n}f_x}(u)} = \lim_{n \to \infty} \delta F_{Sx_{2n}\eta fSx_{2n}z}(u) = 1. \quad (2.52)
\]

In order to complete the proof, we will divide it into 5 steps as follows:

**Step 1.** For $u > 0$ with $\beta = 1$ in the condition (v),
\[
\delta F_{Sx_{2n}z}(\phi(u)) \geq \min \{ \delta F_{Sx_{2n}z}(\phi(u)), \delta F_{Sx_{2n}\eta fSx_{2n}z}(\phi(u)), \delta F_{Sx_{2n}\eta fSx_{2n}z}(\phi(u)), \delta F_{Sx_{2n}\eta fSx_{2n}z}(\phi(u)) \}.
\]

Taking $\lim_{n \to \infty}$, by Lemma 2.8,
\[
F_{\eta fSx_{2n}z}(\phi(u)) \geq \min \{ F_{\eta fSx_{2n}z}(\phi(u)), F_{\eta fSx_{2n}\eta fSx_{2n}z}(\phi(u)), F_{\eta fSx_{2n}\eta fSx_{2n}z}(\phi(u)), F_{\eta fSx_{2n}\eta fSx_{2n}z}(\phi(u)) \} = F_{\eta fSx_{2n}z}(u). \quad (2.54)
\]

So we get $\eta fSx_{2n}z = z$.

**Step 2.** For $u > 0$ with $\beta = 1$ in the condition (v),
\[
\delta F_{Sx_{2n}z}(\phi(u)) = \lim_{n \to \infty} \delta F_{Sx_{2n}z}(\phi(u)) \geq \lim_{n \to \infty} \min \{ F_{\eta fSx_{2n}\xi S_{2n}z}(u), \delta F_{\eta fSx_{2n}\xi S_{2n}z}(u), \delta F_{\eta fSx_{2n}\xi S_{2n}z}(u), \delta F_{\eta fSx_{2n}\xi S_{2n}z}(u), \delta F_{\eta fSx_{2n}\xi S_{2n}z}(u) \} \geq \min \{ F_{\xi S_{2n}z}(u), \delta F_{\xi S_{2n}z}(u), \delta F_{\xi S_{2n}z}(u), \delta F_{\xi S_{2n}z}(u), \delta F_{\xi S_{2n}z}(u) \} = \delta F_{\xi S_{2n}z}(u). \quad (2.55)
\]

So we get $Sx_{2n}z = \{ z \}$.

Hence, by Steps 1 and 2, we have $Sx_{2n}z = \{ z \} = \{ \eta fSx_{2n}z \}$.

**Step 3.** By the condition (i), since $SX \subset \xi SG$, there exists $z' \in X$ such that $\{ \xi S_{2n}z' \} = Sx_{2n}z = \{ z \}$.

So for any $u > 0$ with $\beta = 1$ in the condition (v)
\[
\delta F_{Sx_{2n}z}(\phi(u)) \geq \min \{ F_{\eta fSx_{2n}\xi S_{2n}z}(u), \delta F_{\eta fSx_{2n}\xi S_{2n}z}(u), \delta F_{\eta fSx_{2n}\xi S_{2n}z}(u), \delta F_{\eta fSx_{2n}\xi S_{2n}z}(u), \delta F_{\eta fSx_{2n}\xi S_{2n}z}(u) \}. \quad (2.56)
\]

Taking $\lim_{n \to \infty}$, by Lemma 2.8,
\[
\delta F_{Sx_{2n}z}(\phi(u)) \geq \min \{ F_{\eta fSx_{2n}z}(u), \delta F_{\xi S_{2n}z}(u), \delta F_{\xi S_{2n}z}(u), \delta F_{\xi S_{2n}z}(u), \delta F_{\xi S_{2n}z}(u) \} = \delta F_{\xi S_{2n}z}(u). \quad (2.57)
\]
So we get \( Tz' = \{ z \} \). Hence, \( \{ \xi gz' \} = \{ z \} = Tz' \).

By Step 2, we may let \( \{ z \} = \{ \eta fz \} = \{ Sz \} = \{ \xi gz' \} = \{ Tz' \} \).

Since \( S \) and \( \eta f \) are compatible and \( \{ \eta fz \} = Sz \), we get \( \eta f Sz = \eta fz \), that is, \( \{ \eta fz \} = Sz \).

Now,

\[
\delta F_{Sz,z}(\phi(u)) = \delta F_{Sz,Tz'}(\phi(u)) \\
\geq \min \{ F_{\eta fz,\xi gz'}(u), \delta F_{\eta fz,Sz}(u), \delta F_{\xi gz',Tz'}(u), \delta F_{\eta fz,Tz'}(u), \delta F_{Sz,\xi gz'}(u) \} \\
= \delta F_{\eta fz,z}(u) = \delta F_{Sz,z}(u).
\]  

(2.58)

This implies \( Sz = \{ z \} = \{ \eta fz \} \).

Choose \( z' \) in \( X \) such that \( \{ \xi gz' \} = Sz = \{ z \} \), then

\[
\delta F_{z,Tz'}(\phi(u)) \\
= \delta F_{Sz,Tz'}(\phi(u)) \\
\geq \min \{ F_{\eta fz,\xi gz'}, \delta F_{\eta fz,Sz}(u), \delta F_{\xi gz',Tz'}(u), \delta F_{\eta fz,Tz'}(u), \delta F_{Sz,\xi gz'}(u) \} = \delta F_{z,Tz'}(u).
\]

(2.59)

By Lemma 2.6, we get \( Tz' = \{ z \} \).

Since \( T \) and \( \xi g \) are compatible and \( \{ \xi gz' \} = Tz' \), we get \( T\xi gz' = \xi g Tz' \), that is, \( Tz = \{ \xi gz \} \).

Now, for \( u > 0 \),

\[
\delta F_{Sz,Tz}(\phi(u)) \\
\geq \min \{ F_{\eta fz,\xi gz}(u), \delta F_{\eta fz,Sz}(u), \delta F_{\xi gz,Tz}(u), \delta F_{\eta fz,Tz}(u), \delta F_{Sz,\xi gz}(u) \} \\
= F_{\eta fz,\xi gz}(u) = \delta F_{Sz,Tz}(u).
\]

(2.60)

So we have \( Sz = Tz = \{ \eta fz \} = \{ \xi gz \} = \{ z \} \).

Step 4. For \( u > 0 \) with \( \beta = 1 \) in the condition (v), we get

\[
\delta F_{Sz,Tz}(\phi(u)) \\
\geq \min \{ F_{\eta fz,\xi gz}(u), \delta F_{\eta fz,Sz}(u), \delta F_{\xi gx,Tz}(u), \delta F_{\eta fz,Tz}(u), \delta F_{Sz,\xi gz}(u) \}.
\]

(2.61)

By the condition (ii), \( \eta f = f \eta, Sf = f S \), so we have \( \eta f (fz) = f(\eta fz) = fz \) and \( S(fz) = \{ f(Sz) \} = \{ fz \} \). Taking \( \lim_{n \to \infty} \inf \), by Lemma 2.8,

\[
F_{fz,z}(\phi(u)) \geq \min \{ F_{fz,z}(u), F_{fz,fz}(u), F_{z,z}(u), F_{z,fz}(u), F_{fz,z}(u) \} = F_{fz,z}(u).
\]

(2.62)

So we get \( fz = z \).

Hence, by Steps 1 and 4, we have \( \eta fz = z \) and \( fz = z \), which implies \( \eta z = z \). Therefore, \( \{ z \} = \{ fz \} = \{ \eta z \} = Sz \).

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Step 5. For \( u > 0 \) with \( \beta = 1 \) in condition (v), we get

\[
\delta F_{Sz,Tz}(\phi(u)) \\
\geq \min \{ F_{fSz,Tz}(\phi(u)), \delta F_{Sz,Tz}(u), \delta F_{Tz}(u), \delta F_{Sz,Tz}(u), \delta F_{fSz,Tz}(u) \}.
\]

(2.63)

Since \( Tg = gT \) and \( \xi g = g\xi \), we have \( Tgz = \{ gTz \} = \{ gz \} \) and \( \xi ggz = g(\xi gz) = gz \). Taking \( \lim_{n \to \infty} \inf \), by Lemma 2.8, we get

\[
F_{z,gz}(\phi(u)) \geq \min \{ F_{z,gz}(u), F_{z,z}(u), F_{gzz}(u), F_{gz}(u), F_{z,ggz}(u) \} = F_{z,gz}(u).
\]

(2.64)

So we get \( gz = z \).

Hence, by Steps 3 and 5, we have \( \xi gz = z \) and \( gz = z \), which implies \( \xi z = z \).

So we have \( \{ z \} = \{ gz \} = \{ \xi z \} = Tz \).

Therefore, we have

\[
\{ z \} = \{ fz \} = \{ gz \} = \{ \eta z \} = \{ \xi z \} = Sz = Tz.
\]

(2.65)

Last, we want to prove the uniqueness. Let \( y \) be the another common fixed point of \( \eta \), \( f \), \( \xi \), \( g \), \( S \), and \( T \). Then for \( u > 0 \),

\[
F_{z,y}(\phi(u)) = \delta F_{Sz,Ty}(\phi(u)) \\
\geq \min \{ F_{fSz,Ty}(\phi(u)), \delta F_{Sz,Ty}(u), \delta F_{Ty}(u), \delta F_{Sz,Ty}(u), \delta F_{fSz,Ty}(u) \} \\
\geq \min \{ F_{z,y}(u), F_{z,z}(u), F_{y,y}(u), F_{y,z}(u), F_{y,gz}(u) \} = F_{z,y}(u).
\]

(2.66)

This implies \( y = z \). We complete the proof.

If we take \( f = g = I \), the identity map on \( X \) in Theorem 2.9, then we immediately have the following corollary.

**Corollary 2.10.** Let \((X, \mathcal{F}, \min)\) be a complete Menger space. Let \( \eta, \xi : X \to X \) be two single-valued functions, and let \( S, T : X \to B(X) \) be two set-valued functions. If the following conditions are satisfied:

(i) \( S(X) \subset \xi(X) \), \( T(X) \subset \eta(X) \),

(ii) \( \eta \) or \( \xi \) is continuous,

(iii) \( (S, \eta) \) and \( (T, \xi) \) are compatible,

(iv) for \( u > 0 \),

\[
\delta F_{Sx,Ty}(\phi(u)) \geq \min \{ F_{\eta x,\xi y}(u), \delta F_{\eta x,\xi x}(u), \delta F_{\xi y,Ty}(u), \delta F_{\xi y,\xi x}(\beta u), \delta F_{\eta x,Ty}((2 - \beta)u) \} \\
\]

(2.67)

for all \( x, y \in X \), \( \beta \in (0, 2) \), where \( \phi \in \Phi \), then \( \eta \), \( \xi \), \( S \), and \( T \) have a unique common fixed point \( z \) in \( X \).

By the same process of the proof of Theorem 2.9, we also get the results of Theorem 2.11.
Theorem 2.11. Let \((X, \mathcal{F}, \min)\) be a complete Menger space. Let \(f, g, \eta, \xi : X \to X\) be four single-valued functions, and let \(S, T : X \to \mathcal{B}(X)\) be two set-valued functions. If the following conditions are satisfied:

(i) \(S(X) \subset \xi g(X)\), \(T(X) \subset \eta f(X)\),
(ii) \(\eta f = f \eta, \xi g = g \xi, Sf = fS, Tg = gT\),
(iii) \(\eta f\) or \(\xi g\) is continuous,
(iv) \((S, \eta f)\) and \((T, \xi g)\) are compatible,
(v) for \(u > 0\),
\[
\delta_{FSx, Ty}(\phi(u)) \geq \min \{ F_{\eta f, \xi g}(u), \delta F_{\eta f, Sx}(u), \delta F_{\xi g, Ty}(u), DF_{\xi g, Sx}(u) + DF_{\eta f, Ty}(u) \} 
\] (2.68)

for all \(x, y \in X\), where \(\phi \in \Phi\), then \(f, g, \eta, \xi, S,\) and \(T\) have a unique common fixed point \(z\) in \(X\).

If we take \(f = g = I\), the identity map on \(X\) in Theorem 2.11, then we immediately have the following corollary.

Corollary 2.12. Let \((X, \mathcal{F}, \min)\) be a complete Menger space. Let \(\eta, \xi : X \to X\) be two single-valued functions, and let \(S, T : X \to \mathcal{B}(X)\) be two set-valued functions. If the following conditions are satisfied:

(i) \(S(X) \subset \xi(X)\), \(T(X) \subset \eta(X)\),
(ii) \(\eta\) or \(\xi\) is continuous,
(iii) \((S, \eta)\) and \((T, \xi)\) are compatible,
(iv) for \(u > 0\),
\[
\delta_{FSx, Ty}(\phi(u)) \geq \min \{ F_{\eta \xi}(u), \delta F_{\eta Sx}(u), \delta F_{\xi Ty}(u), DF_{\xi Sx}(u) + DF_{\eta Ty}(u) \} 
\] (2.69)

for all \(x, y \in X\), where \(\phi \in \Phi\), then \(\eta, \xi, S,\) and \(T\) have a unique common fixed point \(z\) in \(X\).

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