ON THE POINTWISE MULTIPLICATION IN BESOV AND LIZORKIN-TRIEBEL SPACES

DOUADI DRIHEM AND MADANI MOUSSAI

Received 1 February 2005; Revised 22 February 2006; Accepted 4 April 2006

Under some sufficient conditions satisfied by F -space of Lizorkin and Triebel and B -space of Besov, we prove some embeddings of types $F \cdot B \hookrightarrow F$, $F \cdot F \hookrightarrow F$, and $B \cdot B \hookrightarrow B$.

Copyright © 2006 Hindawi Publishing Corporation. All rights reserved.

1. Introduction and preparations

In Besov spaces and Lizorkin-Triebel spaces, this paper is concerned with proving some embeddings of the form

$$F \cdot B \hookrightarrow F, \quad F \cdot F \hookrightarrow F, \quad B \cdot B \hookrightarrow B, \quad (1.1)$$

where F and B , with three indices, will denote the Lizorkin-Triebel space $F_{p,q}^s$ and the Besov space $B_{p,q}^s$, respectively. The different embeddings obtained here are under certain restrictions on the parameters.

In this introduction, we will recall the definition of some spaces and some necessary tools. In Sections 2 and 3, we give the first contribution of this work. The theorems of Section 2 will treat the case $F \cdot B \hookrightarrow F$ where the first theorem is a generalization of the results of Franke [4, Section 3.2, Theorem 1, Section 3.4, Corollary 1] and Marschall [7]. The second theorem is in the sense of Johnsen's works (see [5]). Section 3 will contain a treatment of the embeddings of the types $F \cdot F \hookrightarrow F$ and $B \cdot B \hookrightarrow B$ which presents an improvement of [3].

In the sense of [5, Theorems 6.5, 6.11], some limit cases are considered in Section 4, which constitute the second contribution of this paper. Section 5 is an application of our results to the continuity of pseudodifferential operators on Lizorkin-Triebel spaces.

We will work on the Euclidean space $\mathbb{R}^n$. If $f \in \mathcal{S}$, the Fourier transform is defined by the formula

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx \quad (\xi \in \mathbb{R}^n) \quad (1.2)$$

2 Multiplication in Besov and Lizorkin spaces

and $\mathcal{F}^{-1}f$ denotes the inverse Fourier transform of f ; as usual $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended from $\mathcal{S}$ to $\mathcal{S}'$.

Consider a partition of unity

$$\psi(\xi) + \sum_{j=1}^{\infty} \varphi(2^{-j}\xi) = 1 \quad (\xi \in \mathbb{R}^n), \quad (1.3)$$

where $\varphi, \psi \in C^\infty$ are positive functions such that $\text{supp } \varphi \subset \{\xi \in \mathbb{R}^n : 1 \leq |\xi| \leq 3\}$ and $\text{supp } \psi \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 3\}$. We define the convolution operators Q_j and Δ_k by the following:

$$\begin{aligned} Q_j f &= \mathcal{F}^{-1}(\psi(2^{-j} \cdot)) * f \quad (j = 1, 2, \dots), \\ \Delta_k f &= \mathcal{F}^{-1}(\varphi(2^{-k} \cdot)) * f \quad (k = 0, 1, \dots), \end{aligned} \quad (1.4)$$

and we set $Q_0 = \Delta_0$. Thus we obtain the Littlewood-Paley decomposition $f = \sum_{j=0}^{\infty} \Delta_j f$ (convergence in $\mathcal{S}'$).

Let us now recall the definitions of $F_{p,q}^s$ and $B_{p,q}^s$, where the general references include [1, 9–13].

Definition 1.1. Let $\gamma > 0$, $-\infty < s < \infty$, $0 < p < \infty$ (resp., $0 < p \leq \infty$), and $0 < q \leq \infty$. The space $L_p^\gamma(\ell_q^s)$ (resp., $\ell_q^s(L_p^\gamma)$) is the set of the sequences $\{f_k\}_{k \in \mathbb{N}} \subset \mathcal{S}'$ such that $\text{supp } \hat{f}_k \subset \{\xi \in \mathbb{R}^n : |\xi| < \gamma 2^k\}$ and

$$\begin{aligned} ||\{f_k\}_{k \in \mathbb{N}}|L_p^\gamma(\ell_q^s)|| &= ||\{2^{ks} f_k\}_{k \in \mathbb{N}}|L_p(\ell_q)|| < \infty, \\ (\text{resp., } ||\{f_k\}_{k \in \mathbb{N}}|\ell_q^s(L_p^\gamma)|| &= ||\{2^{ks} f_k\}_{k \in \mathbb{N}}|\ell_q(L_p)|| < \infty). \end{aligned} \quad (1.5)$$

Definition 1.2. (i) Let $0 < p < \infty$, $0 < q \leq \infty$, and $-\infty < s < \infty$, then

$$F_{p,q}^s = \{f \in \mathcal{S}' : ||\{2^{ks} \Delta_k f\}_{k \in \mathbb{N}}|L_p(\ell_q)|| < \infty\}. \quad (1.6)$$

(ii) Let $0 < p, q \leq \infty$, and $-\infty < s < \infty$, then

$$B_{p,q}^s = \{f \in \mathcal{S}' : ||\{2^{ks} \Delta_k f\}_{k \in \mathbb{N}}|\ell_q(L_p)|| < \infty\}. \quad (1.7)$$

Remark 1.3. We introduce the maximal function

$$\Delta_k^{*,a} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\Delta_k f(x-y)|}{1 + (2^k |y|)^a} \quad (1.8)$$

for all $x \in \mathbb{R}^n$, $f \in \mathcal{S}'$, $a > 0$, and $k = 0, 1, \dots$. Then, in Definition 1.2(i) (resp., (ii)), we can replace $\Delta_k f$ by $\Delta_k^{*,a} f$ with $a > (n/\min(p, q))$ (resp., $a > n/p$), (cf. see [13, Theorem 2.3.2]).

The product $f \cdot g$ is defined by

$$f \cdot g = \lim_{j \rightarrow \infty} Q_j f \cdot Q_j g \quad (\forall f, g \in \mathcal{S}') \quad (1.9)$$

if the limit on the right-hand side exists in $\mathcal{S}'$ (see [10, Section 4.2]), and we have

$$\Delta_k(f \cdot g) = \sum_{j, \ell=0}^{\infty} \Delta_k(\Delta_j g \cdot \Delta_\ell f) = (\Pi_{k,1} + \Pi_{k,2} + \Pi_{k,3})(f, g), \quad (1.10)$$

where

$$\begin{aligned} \Pi_{k,1}(f, g) &= \Delta_k(\tilde{\Delta}_k f \cdot Q_{k+1} g), & \Pi_{k,2}(f, g) &= \Delta_k(Q_{k+1} f \cdot \tilde{\Delta}_k g), \\ \Pi_{k,3}(f, g) &= \sum_{j=k}^{\infty} \Delta_k(\Delta_j f \cdot \bar{\Delta}_j g), \end{aligned} \quad (1.11)$$

with $\tilde{\Delta}_k = \sum_{j=k-2}^{k+4} \Delta_j$ and $\bar{\Delta}_k = \sum_{j=k-1}^{k+1} \Delta_j$.

In the below proofs of the different cases of type (1.1), written as $G_1 \cdot G_2 \hookrightarrow G_3$, to see $f \cdot g$ belongs to G_3 , ($f \in G_1, g \in G_2$), it suffices to an estimate of terms of the form $\|\{\Pi_{k,i}(f, g)\}_{k \in \mathbb{N}}\|_{L_p^\gamma(\ell_q^s)}$ and $\|\{\Pi_{k,i}(f, g)\}_{k \in \mathbb{N}}\|_{\ell_q^s(L_p^\gamma)}$, $i \in \{1, 2, 3\}$.

Now we recall some lemmas which are useful for us.

LEMMA 1.4. (i) Let $-\infty < s_i < \infty$, $0 < p_i < \infty$ (resp., $0 < p_i \leq \infty$), and $0 < q_i \leq \infty$ (with $i = 0, 1$). If

$$s_0 > s_1, \quad p_0 = p_1, \quad (1.12)$$

or

$$s_0 \geq s_1, \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1} \quad (q_0 \leq q_1 \text{ for Besov space}), \quad (1.13)$$

then it holds

$$F_{p_0, q_0}^{s_0} \hookrightarrow F_{p_1, q_1}^{s_1} \quad (\text{resp., } B_{p_0, q_0}^{s_0} \hookrightarrow B_{p_1, q_1}^{s_1}). \quad (1.14)$$

(ii) Let $-\infty < s, s_i < \infty$, $0 < p, p_i < \infty$, and $0 < q, q_i \leq \infty$ (with $i = 0, 1$) such that $s_0 - n/p_0 = s - n/p = s_1 - n/p_1$. If

$$s_0 > s > s_1, \quad q_0 \leq p \leq q_1, \quad (1.15)$$

or

$$s_0 = s = s_1, \quad q_0 \leq \min(p, q), \quad q_1 \geq \max(p, q), \quad (1.16)$$

then it holds

$$B_{p_0, q_0}^{s_0} \hookrightarrow F_{p, q}^s \hookrightarrow B_{p_1, q_1}^{s_1}. \quad (1.17)$$

4 Multiplication in Besov and Lizorkin spaces

(iii) Let $-\infty < s < \infty$, $0 < p < \infty$ (resp., $0 < p \leq \infty$), and $0 < q \leq \infty$. If

$$s > \frac{n}{p}, \quad (1.18)$$

or

$$s = \frac{n}{p}, \quad 0 < p \leq 1 \text{ (resp., } 0 < q \leq 1), \quad (1.19)$$

then it holds

$$F_{p,q}^s \hookrightarrow L_\infty \quad (\text{resp., } B_{p,q}^s \hookrightarrow L_\infty). \quad (1.20)$$

LEMMA 1.5. Let $0 < \gamma < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}_{k \in \mathbb{N}}$ be a sequence of positive real numbers such that $\|\{\varepsilon_k\}_{k \in \mathbb{N}}\|_{\ell_q} = A < \infty$. Then the sequences $\delta_k = \sum_{j=0}^k \gamma^{k-j} \varepsilon_j$ and $\eta_k = \sum_{j=k}^\infty \gamma^{j-k} \varepsilon_j$ belong to ℓ_q , and the estimate

$$\|\{\delta_k\}_{k \in \mathbb{N}}\|_{\ell_q} + \|\{\eta_k\}_{k \in \mathbb{N}}\|_{\ell_q} \leq cA \quad (1.21)$$

holds. The constant c depends only on γ and q .

LEMMA 1.6. Let $0 < p \leq \infty$ and $\gamma > 0$. Let $\{f_j\}_{j \in \mathbb{N}} \subset L_p$ be a sequence of functions such that $\text{supp } \hat{f}_j \subset \{\xi \in \mathbb{R}^n : |\xi| \leq \gamma 2^j\}$. Then the estimate

$$\|\Delta_k f_j\|_{L_p} \leq c 2^{(j-k)\varrho} \|f_j\|_{L_p} \quad \left(k \leq j < \infty, \varrho = \max\left(0, \frac{n}{p} - n\right)\right) \quad (1.22)$$

holds. The constant c depends only on n , p , and γ .

LEMMA 1.7. Let $0 < p < 1$ and $\gamma > 0$. Let $\{f_j\}_{j \in \mathbb{N}} \subset L_p$ be a sequence of functions such that $\text{supp } \hat{f}_j \subset \{\xi \in \mathbb{R}^n : |\xi| \leq \gamma 2^j\}$. Then the estimate

$$\left\| \sum_{j=0}^\infty f_j \right\|_{B_{p,\infty}^\varrho} \leq c \|\{2^{j\varrho} f_j\}_{j \in \mathbb{N}}\|_{L_p(\ell_\infty)} \quad \left(\varrho = \frac{n}{p} - n\right) \quad (1.23)$$

holds. The constant c depends only on n , p , and γ .

LEMMA 1.8. Let $0 < p \leq q \leq \infty$ and $\gamma > 0$. Then there exists a constant $c = c(n, p, q) > 0$ such that for all $f \in L_p$ with $\text{supp } \hat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq \gamma\}$, one has

$$\|f\|_{L_q} \leq c \gamma^{n(1/p-1/q)} \|f\|_{L_p}. \quad (1.24)$$

For Lemma 1.4, we can see [11, Sections 2.3 and 2.8] and [12, Section 2.7]. Lemma 1.5 follows from Young's inequality in ℓ_q . The proof of Lemma 1.6 is given in [4, Section 2.4, Theorem 1(iii)] and Lemma 1.7 in [7, Lemma 3]. For the proof of Lemma 1.8, we can see [14, Proposition 2.13], $1 \leq p \leq q \leq \infty$, it is the classical inequality of Bernstein.

2. Multiplication of mixed type

The following results give an extension of the sufficient hypotheses used in [5, Theorem 6.1].

THEOREM 2.1. *Let $0 < p, p_1, p_2 < \infty$, $0 < q, q_2 \leq \infty$, $-\infty < s < \infty$, and $r > 0$ be such that*

$$-r + \max\left(0, \frac{n}{p_1} + \frac{n}{p_2} - n\right) < s < \min\left(\frac{n}{p_1}, r\right), \quad (2.1)$$

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{r}{n}, \quad \frac{1}{q_2} \geq \frac{1}{p_2} - \frac{r}{n}, \quad r < \frac{n}{p_2} \left(\text{resp.}, r = \frac{n}{p_2}\right).$$

Then it holds

$$F_{p_1, q}^s \cdot B_{p_2, q_2}^r \hookrightarrow F_{p, q}^s \quad (\text{resp.}, F_{p_1, q}^s \cdot (B_{p_2, \infty}^{n/p_2} \cap L_\infty) \hookrightarrow F_{p_1, q}^s). \quad (2.2)$$

COROLLARY 2.2. *Under the hypotheses of Theorem 2.1. If $r < n/p_2$ (resp., $r = n/p_2$) then it holds*

$$F_{p_1, q}^s \cdot F_{p_2, q_2}^r \hookrightarrow F_{p, q}^s \quad (\text{resp.}, F_{p_1, q}^s \cdot F_{p_2, q_2}^{n/p_2} \hookrightarrow F_{p_1, q}^s \text{ for } p_2 \leq 1). \quad (2.3)$$

Furthermore, in particular, if $1 < p_1 < \infty$ and $r > n/p_1 + n/p_2 - n$, can be taken $s = 0$ in (2.3).

Proof. Since $F_{p_2, q_2}^r \hookrightarrow B_{p_2, t}^r$ with $t = (1/p_2 - r/n)^{-1}$, we obtain the first embedding. However, the second embedding follows from $F_{p_2, q_2}^{n/p_2} \hookrightarrow B_{p_2, \infty}^{n/p_2} \cap L_\infty$. $\square$

Remark 2.3. In Corollary 2.2, when $r < n/p_2$ (resp., $r = n/p_2$), we obtain [10, Theorems 4.4.3/2(21) and 4.4.4/2(16) (resp., Theorems 4.4.3/2(22) and 4.4.4/2(17))]. The particular case $s = 0$ presents a complement of [10, Theorem 4.4.4/4(i)].

To prove Theorem 2.1, we need the following lemma.

LEMMA 2.4. *Let $0 < p < \infty$ and $a > n/p$. Then there exists a constant $c > 0$ such that*

$$\|\{Q_j^{*,a}g\}_{j \in \mathbb{N}}\|_{L_p(\ell_\infty)} \leq c\|g\|_{F_{p,2}^0}, \quad (2.4)$$

for any $g \in F_{p,2}^0$.

Proof. First, we define the maximal function of $Q_j g$, of Hardy-Littlewood type, by the formula

$$MQ_j g(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |Q_j g(y)| dy, \quad (2.5)$$

where $B(x, r)$ is the ball centered at x of radius r and $|B(x, r)|$ denotes its measure. Next, let $t > 0$ satisfy $n/a < t < p$. From [13, Theorem 1.3.1], we have

$$Q_j^{*,a}g(x) \leq Q_j^{*,n/t}g(x) \leq c(M|Q_j g|^t(x))^{1/t} \quad (\forall x \in \mathbb{R}^n). \quad (2.6)$$

6 Multiplication in Besov and Lizorkin spaces

Then we obtain

$$\begin{aligned} \left\| \sup_{j \in \mathbb{N}} Q_j^{*,a} g \mid L_p \right\| &\leq c \left\| \left(\sup_{j \in \mathbb{N}} M \mid Q_j g \mid^t \right)^{1/t} \mid L_p \right\| = c \left\| \sup_{j \in \mathbb{N}} M \mid Q_j g \mid^t \mid L_{p/t} \right\|^{1/t} \\ &\leq c' \left\| \sup_{j \in \mathbb{N}} \mid Q_j g \mid^t \mid L_{p/t} \right\|^{1/t}. \end{aligned} \quad (2.7)$$

A proof of the last inequality may be found in [13, Theorem 2.2.2, page 89]. Now, it is easy to see that the last member of (2.7) is bounded by

$$\left\| \sup_{j \in \mathbb{N}} \mid Q_j g \mid \mid L_p \right\| \leq c \|g \mid F_{p,2}^0\|. \quad (2.8)$$

Inequality (2.8) follows from the equality between the local Hardy spaces h_p and $F_{p,2}^0$, (cf. see [12, Section 2.2, page 37, and Theorem 2.5.8/1]). $\square$

Proof of Theorem 2.1

Case 1 ($r < n/p_2$). (i) Estimate of $\{\Pi_{k,1}(f, g)\}_{k \in \mathbb{N}}$. Since

$$\begin{aligned} \left| \Pi_{k,1}(f, g)(x) \right| &= \left| \int_{\mathbb{R}^n} (\mathcal{F}^{-1} \varphi(y)) (Q_{k+1} g \cdot \tilde{\Delta}_k f)(x - 2^{-k} y) dy \right| \\ &\leq c Q_{k+1}^{*,a_1} g(x) \tilde{\Delta}_k^{*,a_2} f(x) \quad (\forall x \in \mathbb{R}^n), \end{aligned} \quad (2.9)$$

where Q_k^{*,a_1} and $\tilde{\Delta}_k^{*,a_2}$ are defined as in Remark 1.3, we obtain

$$\| \{2^{ks} \Pi_{k,1}(f, g)\}_{k \in \mathbb{N}} \mid \ell_q \| \leq c \sup_{j \in \mathbb{N}} (Q_j^{*,a_1} g) \| \{2^{ks} \tilde{\Delta}_k^{*,a_2} f\}_{k \in \mathbb{N}} \mid \ell_q \|, \quad (2.10)$$

where a_1 and a_2 are real numbers at our disposal. We set $1/b = 1/p_2 - r/n$. The left-hand side of (2.10), in L_p -norm, is bounded by

$$c \left\| \sup_{j \in \mathbb{N}} Q_j^{*,a_1} g \mid L_b \right\| \| \{2^{ks} \tilde{\Delta}_k^{*,a_2} f\}_{k \in \mathbb{N}} \mid L_{p_1}(\ell_q) \|. \quad (2.11)$$

Choose $a_1 > n/b$ and $a_2 > n/\min(p_1, q)$, then both Lemma 2.4 and the embedding $B_{p_2, q_2}^r \hookrightarrow F_{b,2}^0$ yield that (2.11) is estimated as desired.

(ii) Estimate of $\{\Pi_{k,2}(f, g)\}_{k \in \mathbb{N}}$. Let $u \in \mathbb{R}$ such that

$$\max \left(0, \frac{1}{p_1} - \frac{r}{n} \right) < \frac{1}{u} < \min \left(\frac{1}{p_1}, \frac{1}{p_1} - \frac{s}{n} \right). \quad (2.12)$$

We set

$$\frac{1}{v} = \frac{1}{p_2} + \frac{1}{u}, \quad \sigma = s - \frac{n}{p} + \frac{n}{v}, \quad \beta = s - \frac{n}{p_1} + \frac{n}{u}. \quad (2.13)$$

We have

$$\ell_p^\sigma(L_v^\gamma) \hookrightarrow L_p^\gamma(\ell_q^s), \quad F_{p_1,q}^s \hookrightarrow B_{u,p_1}^\beta. \quad (2.14)$$

For the first embedding of (2.14), we can see [4, Section 2.3, Theorem 3]. On the other hand, the Hölder inequality yields

$$2^{k\sigma} \|\Pi_{k,2}(f, g) \mid L_v\| \leq c 2^{kr} \|\tilde{\Delta}_k g \mid L_{p_2}\| \left(2^{k\beta} \sum_{j=0}^{k+1} 2^{-j\beta} \cdot 2^{j\beta} \|\Delta_j f \mid L_u\| \right). \quad (2.15)$$

We set $1/\tilde{q}_2 = 1/p - 1/p_1$. Applying, successively, the Hölder inequality again in ℓ_p -norm and Lemma 1.5, we obtain the bound $c \|g \mid B_{p_2, \tilde{q}_2}^r\| \|f \mid B_{u, p_1}^\beta\|$. So (2.14) and $B_{p_2, q_2}^r \hookrightarrow B_{p_2, \tilde{q}_2}^r$ give

$$\|\{2^{ks} \Pi_{k,2}(f, g)\}_{k \in \mathbb{N}} \mid L_p(\ell_q)\| \leq c \|g \mid B_{p_2, q_2}^r\| \|f \mid F_{p_1, q}^s\|. \quad (2.16)$$

(iii) Estimate of $\{\Pi_{k,3}(f, g)\}_{k \in \mathbb{N}}$. We first consider $1/p_1 + 1/p_2 \leq 1$. Let $u \in \mathbb{R}$ such that

$$\max\left(0, \frac{1}{p_1} - \frac{r}{n}, \frac{1}{p_1} - \frac{r+s}{n}\right) < \frac{1}{u} < \frac{1}{p_1}. \quad (2.17)$$

We use the notations ν , σ , and β from (2.13). Lemma 1.6 provides

$$2^{k\sigma} \|\Pi_{k,3}(f, g) \mid L_v\| \leq c 2^{k(\beta+r)} \sum_{j=k}^{\infty} 2^{-j(\beta+r)} \cdot 2^{j(\beta+r)} \|\overline{\Delta}_j g \mid L_{p_2}\| \|\Delta_j f \mid L_u\|. \quad (2.18)$$

A similar argument as above yields

$$\|\{2^{k\sigma} \Pi_{k,3}(f, g)\}_{k \in \mathbb{N}} \mid \ell_p(L_v)\| \leq c \|\{2^{j(\beta+r)} \|\overline{\Delta}_j g \mid L_{p_2}\| \|\Delta_j f \mid L_u\|\}_{j \in \mathbb{N}} \mid \ell_p\|. \quad (2.19)$$

We set $1/\tilde{q}_2 = 1/p - 1/p_1$. By the Hölder inequality in ℓ_p -norm, the right-hand side of (2.19) is bounded by $c \|g \mid B_{p_2, \tilde{q}_2}^r\| \|f \mid B_{u, p_1}^\beta\|$. Then we conclude the desired estimate by (2.14).

We now study case $1/p_1 + 1/p_2 > 1$. Let $u \in \mathbb{R}$ such that

$$\max\left(0, 1 - \frac{1}{p_2}, \frac{1}{p_1} - \frac{r}{n}\right) < \frac{1}{u} < \frac{1}{p_1}. \quad (2.20)$$

We employ the notations ν and σ from (2.13). By Lemma 1.6, we obtain

$$2^{k\sigma} \|\Pi_{k,3}(f, g) \mid L_v\| \leq c 2^{k\mu} \sum_{j=k}^{\infty} 2^{-j\mu} \cdot 2^{j(r+\varrho)} \|\overline{\Delta}_j g \mid L_{p_2}\| \|\Delta_j f \mid L_u\|, \quad (2.21)$$

where $\varrho = s - n/p_1 + n/u$ and $\mu = s + r - n/p_1 - n/p_2 + n > 0$, therefore,

$$\|\{2^{k\sigma} \Pi_{k,3}(f, g)\}_{k \in \mathbb{N}} \mid \ell_p(L_v)\| \leq c \|\{2^{j(r+\varrho)} \|\overline{\Delta}_j g \mid L_{p_2}\| \|\Delta_j f \mid L_u\|\}_{j \in \mathbb{N}} \mid \ell_p\|. \quad (2.22)$$

8 Multiplication in Besov and Lizorkin spaces

On the right-hand side of (2.22), we employ the Hölder inequality in ℓ_p -norm (with $1/p = 1/p_1 + 1/\tilde{q}_2$), $F_{p_1,q}^s \hookrightarrow B_{u,p_1}^s$, and $B_{p_2,q_2}^r \hookrightarrow B_{p_2,\tilde{q}_2}^r$ successively. Since $\sigma > s$ and $v < p$, we can finish the proof of this case by applying, in the left-hand side of (2.22), embeddings (2.14).

Case 2 ($r = n/p_2$). We only estimate $\{\Pi_{k,1}(f,g)\}_{k \in \mathbb{N}}$. It is sufficient to see that

$$2^{ks} |\Pi_{k,1}(f,g)| \leq c \|g\|_{L_\infty} \|(2^{ks} \tilde{\Delta}_k^{*,a} f)\| \quad (2.23)$$

with $a > n/\min(p_1, q)$ and to take the $L_{p_1}(\ell_q)$ -norm. $\square$

THEOREM 2.5. *Let $0 < p, p_1 < \infty, 0 < p_2, q \leq \infty, -\infty < s < \infty$, and $r > 0$ be such that*

$$-r + \max\left(0, \frac{n}{p_1} + \frac{n}{p_2} - n\right) < s < r. \quad (2.24)$$

If either of the following assertions is satisfied:

(i) $1/p = 1/p_1 + 1/p_2$,

(ii) $\max(1/p_1, s/n) + \max(0, 1/p_2 - r/n) < 1/p < 1/p_1 + 1/p_2$,

then it holds

$$F_{p_1,q}^s \cdot B_{p_2,\infty}^r \hookrightarrow F_{p,q}^s. \quad (2.25)$$

COROLLARY 2.6. *Let p, p_1, q, r, s be as in Theorem 2.5 and $0 < p_2 < \infty$. If (i) or (ii) of Theorem 2.5 is satisfied, then the embedding $F_{p_1,q}^s \cdot F_{p_2,\infty}^r \hookrightarrow F_{p,q}^s$ holds.*

The proof of Corollary 2.6 is immediate because $F_{p_2,\infty}^r \hookrightarrow B_{p_2,\infty}^r$.

Remark 2.7. We note that Theorem 2.5(i) when $p_2 = \infty$ is given in [4, Section 3.2, Theorem 1]. Also, we note that Corollary 2.6 is given in both [5, Theorem 6.1 with $r = p$ in formula (6.6)] and [10, Theorems 4.4.3/1(7) and 4.4.4/1(7)].

Proof of Theorem 2.5(i). Noting Remark 2.7, we only need to treat the part $0 < p_2 < \infty$.

(i) Estimate of $\{\Pi_{k,1}(f,g)\}_{k \in \mathbb{N}}$. From (2.9) and Lemma 2.4, we have

$$\|\{2^{ks} \Pi_{k,1}(f,g)\}_{k \in \mathbb{N}}\|_{L_p(\ell_q)} \leq c \|g\|_{F_{p_2,2}^0} \|f\|_{F_{p_1,q}^s}. \quad (2.26)$$

By embeddings $B_{p_2,\infty}^r \hookrightarrow B_{p_2,\min(p_2,2)}^0 \hookrightarrow F_{p_2,2}^0$, we obtain that the last term of (2.26) is bounded by the desired quantity.

(ii) Estimate of $\{\Pi_{k,2}(f,g)\}_{k \in \mathbb{N}}$. The Hölder inequality provides

$$\|\Pi_{k,2}(f,g)\|_{L_p} \leq c \left(2^{-kr} \sum_{j=0}^{k+1} 2^{-j(s-r)} \cdot 2^{-jr} \right) \|g\|_{B_{p_2,\infty}^r} \|f\|_{B_{p_1,\infty}^s}. \quad (2.27)$$

The hypothesis $s < r$ yields

$$\|\{2^{ks} \Pi_{k,2}(f,g)\}_{k \in \mathbb{N}}\|_{\ell_{\min(p,q)}(L_p)} \leq c \|g\|_{B_{p_2,\infty}^r} \|f\|_{B_{p_1,\infty}^s}. \quad (2.28)$$

Using embeddings

$$\ell_{\min(p,q)}^s(L_p^\gamma) \hookrightarrow L_p^\gamma(\ell_q^s), \quad F_{p_1,q}^s \hookrightarrow B_{p_1,\infty}^s, \quad (2.29)$$

we obtain the desired result.

(iii) Estimate of $\{\Pi_{k,3}(f,g)\}_{k \in \mathbb{N}}$. We set $\varrho = s + r - \max(0, n/p - n)$. Using Lemma 1.6, we obtain

$$\begin{aligned} 2^{ks} \|\Pi_{k,3}(f,g) \mid L_p\| &\leq c 2^{-kr} \cdot 2^{k\varrho} \sum_{j=k}^{\infty} 2^{-j\varrho} (2^{jr} \|\bar{\Delta}_j g \mid L_{p_2}\|) (2^{js} \|\Delta_j f \mid L_{p_1}\|) \\ &\leq c 2^{-kr} \|g \mid B_{p_2,\infty}^r\| \|f \mid B_{p_1,\infty}^s\|. \end{aligned} \quad (2.30)$$

□

In this inequality, we take $\ell_{\min(p,q)}$ -norm and we conclude the desired estimate using (2.29).

Proof of Theorem 2.5(ii). (1) *Estimate of $\{\Pi_{k,1}(f,g)\}_{k \in \mathbb{N}}$.* We set $1/u = 1/p - 1/p_1$. As in (2.26), we have the bound $c \|g \mid F_{u,2}^0\| \|f \mid F_{p_1,q}^s\|$ which, by the embeddings $B_{p_2,\infty}^r \hookrightarrow B_{p_2,u}^{n/p_2-n/u} \hookrightarrow F_{u,2}^0$, is estimated as desired.

(2) *Estimate of $\{\Pi_{k,2}(f,g)\}_{k \in \mathbb{N}}$.* In part, for technical reasons, we prove this in three separate cases:

Case 1 ($s < 0$). By Lemma 1.6, the Hölder inequality, and Lemma 1.8, we have

$$\|\Pi_{k,2}(f,g) \mid L_p\| \leq c 2^{(n/p_1+n/p_2-n/p-r-s)k} \|g \mid B_{p_2,\infty}^r\| \|f \mid B_{p_1,\infty}^s\|. \quad (2.31)$$

Since $n/p_1 + n/p_2 - n/p - r < 0$, we obtain an inequality of type (2.28) and finish the proof of this case using (2.29).

Case 2 ($0 \leq s < n/p_1$). We set $1/b = 1/p_2 + 1/p_1 - s/n$. We continue with the following subcases.

Subcase 2.1 ($r \leq n/p_2$ and $p \leq b$ (or $s \leq n/p_2 < r$ and $p \leq b$)). As in Case 1, we have

$$\|\Pi_{k,2}(f,g) \mid L_p\| \leq c \gamma_k 2^{-kr} \|g \mid B_{p_2,\infty}^r\| \|f \mid B_{p_1,\infty}^s\|, \quad (2.32)$$

where

$$\gamma_k = \begin{cases} k+2 & \text{if } p = b, \\ (1 - 2^{n/b-(n/p)-s})^{-1} & \text{if } p < b. \end{cases} \quad (2.33)$$

Now since $\{2^{k(s-r)} \gamma_k\}_{k \in \mathbb{N}} \in \ell_{\min(p,q)}$, we conclude the desired conclusion using (2.28) and (2.29).

Subcase 2.2 ($r \leq n/p_2$ and $p > b$ (or $s \leq n/p_2 < r$ and $p > b$)). Let $u > 0$ satisfy

$$\max\left(0, \frac{1}{p} - \frac{1}{p_2}\right) < \frac{1}{u} < \frac{1}{p_1} - \frac{s}{n}. \quad (2.34)$$

We employ the notations ν , σ , and β from (2.13). We have

$$\|\Pi_{k,2}(f, g) \mid L_\nu\| \leq c \|g \mid B_{p_2, \infty}^r\| \|f \mid B_{u, \infty}^\beta\| (2^{-k(\beta+r)}). \quad (2.35)$$

Since $\{2^{-k(\beta+r-\sigma)}\}_{k \in \mathbb{N}} \in \ell_p$, we can finish the proof of this case using (2.14).

Subcase 2.3 ($n/p_2 < s < r$). We have only case $p < b$ needs to be verified. As in (2.32), we immediately obtain the result.

Case 3 ($s \geq n/p_1$). We have the following subcases.

Subcase 3.1 ($p < p_2$). We set $1/\nu = 1/p - 1/p_2$. Observe that

$$\begin{aligned} 2^{ks} \|\Pi_{k,2}(f, g) \mid L_p\| &\leq c 2^{kr} \|\tilde{\Delta}_k g \mid L_{p_2}\| \left(2^{k(s-r)} \sum_{j=0}^{k+1} 2^{j(n/p_1 - n/\nu)} \|\Delta_j f \mid L_{p_1}\| \right) \\ &\leq c \|g \mid B_{p_2, \infty}^r\| \|f \mid B_{p_1, \infty}^{n/p_1}\| \left(2^{k(s-r)} \sum_{j=0}^{k+1} 2^{-jn/\nu} \right). \end{aligned} \quad (2.36)$$

Then, we calculate $\ell_{\min(p, q)}$ -norm and conclude the desired estimate by the fact that

$$\left\{ 2^{k(s-r)} \sum_{j=0}^{k+1} 2^{-jn/\nu} \right\}_{k \in \mathbb{N}} \in \ell_{\min(p, q)}. \quad (2.37)$$

Subcase 3.2 ($s > n/p_1$ and $p \geq p_2$). It suffices to apply both embedding $B_{p_1, \infty}^s \hookrightarrow B_{p_1, 1}^{n/p_1}$ and (2.29) to

$$\begin{aligned} \|\Pi_{k,2}(f, g) \mid L_p\| &\leq c \|\tilde{\Delta}_k g \mid L_p\| \|Q_{k+1} f \mid L_\infty\| \\ &\leq c 2^{k(n/p_2 - r - n/p)} \|g \mid B_{p_2, \infty}^r\| \|f \mid B_{p_1, 1}^{n/p_1}\|. \end{aligned} \quad (2.38)$$

Subcase 3.3 ($s = n/p_1$ and $p \geq p_2$). We choose $\alpha > 0$ such that $\varepsilon = \alpha - n/p + n/p_1 + n/p_2 - r < 0$, then it suffices to apply (2.29) to

$$2^{kn/p_1} \|\Pi_{k,2}(f, g) \mid L_p\| \leq c 2^{k\varepsilon} \|g \mid B_{p_2, \infty}^r\| \|f \mid B_{p_1, \infty}^{n/p_1 - \alpha}\|. \quad (2.39)$$

(3) *Estimate of $\{\Pi_{k,3}(f, g)\}_{k \in \mathbb{N}}$* . The proof of this case is obtained similarly to the proof of Theorem 2.1 just by replacing (2.17) and (2.20) with

$$\begin{aligned} \max \left(0, \frac{1}{p} - \frac{1}{p_2}, \frac{1}{p_1} - \frac{r+s}{n} \right) &< \frac{1}{u} < \frac{1}{p_1}, \\ \max \left(0, 1 - \frac{1}{p_2}, \frac{1}{p} - \frac{1}{p_2} \right) &< \frac{1}{u} < \frac{1}{p_1}, \end{aligned} \quad (2.40)$$

respectively. □

3. Multiplication of types $F \cdot B$ and $B \cdot B$

The next theorem presents a continuation of [3], [5, Theorem 6.1], [6], and [7, Section 5].

THEOREM 3.1. *Let $0 < p_1 < \infty$ (resp., $0 < p_1 \leq \infty$), $1 \leq p_2 \leq \infty$, $0 < q \leq \infty$, and $n/p_1 - n < s < \min(n/p_1, n/p_2)$. Then it holds*

$$F_{p_1,q}^s \cdot (B_{p_2,\infty}^{n/p_2} \cap L_\infty) \hookrightarrow F_{p_1,q}^s \quad (\text{resp., } B_{p_1,q}^s \cdot (B_{p_2,\infty}^{n/p_2} \cap L_\infty) \hookrightarrow B_{p_1,q}^s). \quad (3.1)$$

Remark 3.2. We note that (3.1), in the F -case, was proved by Franke [4, Section 3.4, Corollary 1] but only in the particular case

$$p_2 = \begin{cases} p_1 & \text{if } 0 < p_1 < 2, \\ p_1(p_1 - 1)^{-1} & \text{if } 2 \leq p_1 < \infty. \end{cases} \quad (3.2)$$

Also this case yields

$$F_{p_1,q}^s \cdot (F_{p_2,\infty}^{n/p_2} \cap L_\infty) \hookrightarrow F_{p_1,q}^s. \quad (3.3)$$

Remark 3.3. Theorem 3.1, when $1 \leq p_1 \leq p_2 < \infty$, was proved in [3].

Proof of Theorem 3.1. The estimates of $\{\Pi_{k,1}(f,g)\}_{k \in \mathbb{N}}$ and $\{\Pi_{k,3}(f,g)\}_{k \in \mathbb{N}}$ are similar to Theorem 2.1, see also [3]. For $\{\Pi_{k,2}(f,g)\}_{k \in \mathbb{N}}$ we take, in (2.16), $r = n/p_2$, and $q_2 = \tilde{q}_2 = \infty$, we obtain (3.1) in the F -case. In the B -case, we will employ the notations u, v, σ , and β from (2.12) and (2.13) with the modifications $r = n/p_2$ and $\sigma = s - n/p_1 + n/v$. One has

$$2^{k\sigma} \|\Pi_{k,2}(f,g) \mid L_v\| \leq c \|g \mid B_{p_2,\infty}^{n/p_2}\| \left(2^{k\beta} \sum_{j=0}^{k+1} 2^{-j\beta} \cdot 2^{j\beta} \|\Delta_j f \mid L_u\| \right). \quad (3.4)$$

Since $\beta < 0$, the last inequality, in the ℓ_q -norm, is bounded by the expression $c \|g \mid B_{p_2,\infty}^{n/p_2}\| \|f \mid B_{u,q}^\beta\|$. At the end, it suffices to use

$$\ell_q^\sigma(L_v^\gamma) \hookrightarrow \ell_q^s(L_{p_1}^\gamma), \quad B_{p_1,q}^s \hookrightarrow B_{u,q}^\beta. \quad (3.5)$$

□

4. Some limit cases

We will prove results of independent interest concerning the limit case for the parameters $s + r$, see [5, Theorems 6.5 and 6.11].

THEOREM 4.1. *Let $0 < p, q, p_i, q_i \leq \infty$, ($i = 1, 2$), $-\infty < s < \infty$, and $r > 0$ such that*

$$\frac{1}{q_1} + \frac{1}{q_2} \geq 1, \quad s + r = \frac{n}{p_1} + \frac{n}{p_2} - n > 0. \quad (4.1)$$

12 Multiplication in Besov and Lizorkin spaces

If either of the following assertions is satisfied:

(i)

$$\begin{aligned} r < \frac{n}{p_2}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{r}{n}, \quad s < \min\left(\frac{n}{p_1}, r\right), \\ \frac{1}{q_2} \geq \frac{1}{p_2} - \frac{r}{n}, \quad q = \infty, \quad p_2 \neq \infty, \end{aligned} \quad (4.2)$$

(ii)

$$\max\left(\frac{1}{p_1}, \frac{s}{n}\right) + \max\left(0, \frac{1}{p_2} - \frac{r}{n}\right) < \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \quad (4.3)$$

and either of the following cases is satisfied:

(1) $s < r$, $q_1 \leq q$,

(2) $s = r$, $\max(q_1, q_2) \leq q$,

then it holds

$$B_{p_1, q_1}^s \cdot B_{p_2, q_2}^r \hookrightarrow B_{p, q}^s. \quad (4.4)$$

Remark 4.2. In Theorem 4.1(i), when $r = n/p_2$, we have

$$B_{p_1, q_1}^s \cdot (B_{p_2, q_2}^{n/p_2} \cap L_\infty) \hookrightarrow B_{p_1, q}^s. \quad (4.5)$$

THEOREM 4.3. Let $0 < p$, $p_1, p_2 < \infty$, $0 < q \leq \infty$, $-\infty < s < \infty$, and $r > 0$ such that

$$s + r = \frac{n}{p_1} + \frac{n}{p_2} - n > 0. \quad (4.6)$$

If either of the following assertions is satisfied:

(i)

$$r < \frac{n}{p_2}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{r}{n}, \quad s < \min\left(\frac{n}{p_1}, r\right), \quad (4.7)$$

(ii)

$$\max\left(\frac{1}{p_1}, \frac{s}{n}\right) + \max\left(0, \frac{1}{p_2} - \frac{r}{n}\right) < \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}, \quad s \leq r, \quad (4.8)$$

then it holds

$$F_{p_1, q}^s \cdot F_{p_2, \infty}^r \hookrightarrow B_{p, \infty}^s. \quad (4.9)$$

Remark 4.4. In Theorem 4.3(i), when $r = n/p_2$, we have

$$F_{p_1, q}^s \cdot (F_{p_2, \infty}^{n/p_2} \cap L_\infty) \hookrightarrow B_{p_1, \infty}^s. \quad (4.10)$$

THEOREM 4.5. Let $0 < p, p_1 < \infty, 0 < p_2, q_1, q_2 \leq \infty, -\infty < s < \infty$, and $r > 0$ such that

$$q_1 \geq p_1, \quad \frac{1}{q_1} + \frac{1}{q_2} \geq 1, \quad s < r, \quad s + r = \max\left(0, \frac{n}{p_1} + \frac{n}{p_2} - n\right). \quad (4.11)$$

If either of the following assertions is satisfied:

(i)

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad (4.12)$$

(ii)

$$\max\left(\frac{1}{p_1}, \frac{s}{n}\right) + \max\left(0, \frac{1}{p_2} - \frac{r}{n}\right) < \frac{1}{p} < \frac{1}{p_1} + \frac{1}{p_2}, \quad (4.13)$$

then it holds

$$F_{p_1, q_1}^s \cdot B_{p_2, q_2}^r \hookrightarrow F_{p, q_1}^s. \quad (4.14)$$

Proof of Theorem 4.1(i). (i) *Estimate of $\{\Pi_{k,1}(f, g)\}_{k \in \mathbb{N}}$.* We set $1/u = 1/p_2 - r/n$. The Hölder inequality and Lemma 2.4 give

$$2^{ks} \|\Pi_{k,1}(f, g) \mid L_p\| \leq c \|g \mid F_{u,2}^0\| (2^{ks} \|\tilde{\Delta}_k f \mid L_{p_1}\|). \quad (4.15)$$

The embedding $B_{p_2, q_2}^r \hookrightarrow F_{u,2}^0$ together with the ℓ_∞ -norm of (4.15) and $\ell_{q_1} \hookrightarrow \ell_\infty$ give the desired estimate.

(ii) *Estimate of $\{\Pi_{k,2}(f, g)\}_{k \in \mathbb{N}}$.* Using the notations u, v, σ , and β from (2.12) and (2.13), we have, as in (2.15),

$$2^{k\sigma} \|\{\Pi_{k,2}(f, g)\}_{k \in \mathbb{N}} \mid \ell_\infty(L_v)\| \leq c \|g \mid B_{p_2, \infty}^r\| \|f \mid B_{u, \infty}^\beta\|, \quad (4.16)$$

and the conclusion is obtained by (3.5).

(iii) *Estimate of $\{\Pi_{k,3}(f, g)\}_{k \in \mathbb{N}}$.* We set $1/b = 1/p_1 + 1/p_2$. By Lemma 1.6 and the Hölder inequality, we obtain

$$\|\Pi_{k,3}(f, g) \mid L_b\| \leq c 2^{-k(s+r)} \sum_{j=k}^{\infty} (2^{jr} \|\overline{\Delta}_j g \mid L_{p_2}\|) (2^{js} \|\Delta_j f \mid L_{p_1}\|). \quad (4.17)$$

Using $\ell_d \hookrightarrow \ell_1$ (with $1/d = 1/q_1 + 1/q_2$) we employ the Hölder inequality again to conclude that the last term of (4.17) is bounded by $c \|g \mid B_{p_2, q_2}^r\| \|f \mid B_{p_1, q_1}^s\|$. We finish the proof of this case by applying the embedding $\ell_\infty^{s+r}(L_b^\gamma) \hookrightarrow \ell_\infty^s(L_p^\gamma)$. $\square$

Proof of Theorem 4.1(ii). For $\{\Pi_{k,1}(f, g)\}_{k \in \mathbb{N}}$ and $\{\Pi_{k,2}(f, g)\}_{k \in \mathbb{N}}$, we can use the same methods in Theorem 2.5, see also [10, Sections 4.4.3 and 4.4.4].

14 Multiplication in Besov and Lizorkin spaces

Estimate of $\{\Pi_{k,3}(f, g)\}_{k \in \mathbb{N}}$. By Lemmas 1.6, 1.8 and the Hölder inequality, we have

$$2^{ks} \|\Pi_{k,3}(f, g) \mid L_p\| \leq c 2^{k(n/p_1 + n/p_2 - n/p - r)} \sum_{j=k}^{\infty} (2^{jr} \|\bar{\Delta}_j g \mid L_{p_2}\|) (2^{js} \|\Delta_j f \mid L_{p_1}\|). \quad (4.18)$$

Since $n/p_1 + n/p_2 - n/p - r < 0$, we conclude the desired estimate using $\ell_d \hookrightarrow \ell_1$ (with $1/d = 1/q_1 + 1/q_2$). $\square$

Proof of Theorem 4.3(i). (i) *Estimate of $\{\Pi_{k,1}(f, g)\}_{k \in \mathbb{N}}$.* We set $1/u = 1/p_2 - r/n$, (i.e., $1/p = 1/p_1 - 1/u$). As in (2.9), the choice of $a_1 > n/u$ and $a_2 > n/p_1$ leads to

$$\begin{aligned} \|\{2^{ks} \Pi_{k,1}(f, g)\}_{k \in \mathbb{N}} \mid \ell_{\infty}(L_p)\| &\leq c \left\| \sup_{j \in \mathbb{N}} Q_j^{*, a_1} g \mid L_u \right\| \|\{2^{ks} \tilde{\Delta}_k^{*, a_2} f\}_{k \in \mathbb{N}} \mid \ell_{\infty}(L_{p_1})\| \\ &\leq c \|g \mid F_{u,2}^0\| \|f \mid B_{p_1, \infty}^s\|. \end{aligned} \quad (4.19)$$

We conclude the desired estimate by applying both $F_{p_2, \infty}^r \hookrightarrow F_{u,2}^0$ and $F_{p_1, \infty}^s \hookrightarrow B_{p_1, \infty}^s$.

(ii) *Estimate of $\{\Pi_{k,2}(f, g)\}_{k \in \mathbb{N}}$.* Using the notations u , v , σ , and β from (2.12) and (2.13), we have

$$2^{ks} \|\Pi_{k,2}(f, g)\| \leq c \sup_{\ell \in \mathbb{N}} (2^{\ell r} \Delta_{\ell}^{*, a_1} g) \left(2^{k\beta} \sum_{j=0}^{k+1} 2^{-j\beta} (2^{j\beta} \Delta_j^{*, a_2} f) \right). \quad (4.20)$$

Since $\beta < 0$, then

$$\|\{2^{ks} \Pi_{k,2}(f, g)\}_{k \in \mathbb{N}} \mid \ell_{\infty}\| \leq c \sup_{\ell \in \mathbb{N}} (2^{\ell r} \bar{\Delta}_{\ell}^{*, a_1} g) \|\{2^{j\beta} \Delta_j^{*, a_2} f\}_{j \in \mathbb{N}} \mid \ell_{\infty}\|. \quad (4.21)$$

We choose $a_1 > n/p_2$ and $a_2 > n/u$. We obtain the desired result by applying the Hölder inequality, the embeddings $L_v^{\gamma}(\ell_{\infty}^{\sigma}) \hookrightarrow L_p^{\gamma}(\ell_{\infty}^s) \hookrightarrow \ell_{\infty}^s(L_p^{\gamma})$ and $F_{p_1, q}^s \hookrightarrow F_{u, \infty}^{\beta}$.

(iii) *Estimate of $\{\Pi_{k,3}(f, g)\}_{k \in \mathbb{N}}$.* We set $1/u = 1/p_2 + 1/p_1$. We begin by the inequality

$$\left\| \sum_{k=0}^{\infty} \Pi_{k,3}(f, g) \mid B_{p, \infty}^s \right\| \leq c \left\| \sum_{j=0}^{\infty} Q_j(\bar{\Delta}_j g \cdot \Delta_j f) \mid B_{u, \infty}^{s+r} \right\|. \quad (4.22)$$

We can write

$$|Q_j(\bar{\Delta}_j g \cdot \Delta_j f)| \leq c \sup_{j \in \mathbb{N}} (\Delta_j^{*, a_1} g \cdot \Delta_j^{*, a_2} f). \quad (4.23)$$

We choose $a_1 > n/p_2$ and $a_2 > n/\min(p_1, q)$. Then Lemma 1.7 gives the correct bound for (4.22). $\square$

The same method works for the proofs of Theorems 4.3(ii) and 4.5. We omit the details.

Remark 4.6. Theorems 4.1(i) and 4.3(i), when $1 \leq p \leq \infty$, were proved by Johnsen in [5, Theorems 6.11 and 6.5], respectively.

5. Application

We consider $S_{1,0}^0(E)$ (E a Banach space), the class of symbols $(x, \xi) \rightarrow a(x, \xi)$ satisfying

$$\|\partial_\xi^\beta a(\cdot, \xi) - E\| \leq c_\beta (1 + |\xi|)^{-|\beta|} \quad (\forall \beta \in \mathbb{N}^n), \quad (5.1)$$

and we define the pseudodifferential operator by the formula

$$\text{Op}_a f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} a(x, \xi) \hat{f}(\xi) d\xi \quad (\forall f \in \mathcal{S}, \forall x \in \mathbb{R}^n). \quad (5.2)$$

As mentioned in the introduction, the theorems of this section present an application of the previous results in this paper.

THEOREM 5.1. *Let $1 \leq p, p_1, p_2, q, q_2 \leq \infty$, $-\infty < s < \infty$, and $r > 0$. Under the hypotheses of Theorem 2.1 (with $p_2 \neq \infty$ and $r < n/p_2$) or Theorem 2.5, the operator Op_a is bounded from $F_{p_1, q}^s$ to $F_{p, q}^s$ for all $a \in S_{1,0}^0(B_{p_2, q_2}^r)$.*

The proof of Theorem 5.1 is based on the following almost-orthogonality lemma.

LEMMA 5.2. *Let $\gamma > 1$ and let $p, p_1, p_2, q, q_2, r, s$ be the same as in Theorem 2.1 (with $p_2 \neq \infty$ and $r < n/p_2$) or Theorem 2.5. For all sequences $\{m_j\}_{j \in \mathbb{N}} \subset B_{p_2, q_2}^r$ and all sequences $\{f_j\}_{j \in \mathbb{N}}$ of functions such that $\text{supp } \hat{f}_j \subset \{\xi \in \mathbb{R}^n : \gamma^{-1}2^j \leq |\xi| \leq \gamma 2^j\}$, the estimate*

$$\left\| \sum_{j=0}^{\infty} m_j \cdot f_j \right\|_{F_{p, q}^s} \leq c \|\{f_j\}_{j \in \mathbb{N}}\|_{L_{p_1}^\gamma(\ell_q^\gamma)} \quad (5.3)$$

holds with $c = c' \sup_{j \geq 0} \|m_j\|_{B_{p_2, q_2}^r}$.

Proof. Observe that $\Delta_k f_j \neq 0$ and $Q_{k+1} f_j \neq 0$ if $k - N \leq j \leq k + N + 2$ and $j \leq k + N + 2$, respectively, where $N = [\log_2 \gamma]$; (here $[x]$ denotes the greatest integer less than or equal to x). Then it suffices to apply Theorem 2.1 (and/or Theorem 2.5) to the following decomposition:

$$\Delta_k \left(\sum_{j=0}^{\infty} m_j \cdot f_j \right) = \sum_{\ell=-N}^{N+2} \Pi_{k,1}(m_{k+\ell}, f_{k+\ell}) + \sum_{j=0}^{k+N+2} \Pi_{k,2}(m_j, f_j) + \sum_{\ell=-N}^{N+2} \tilde{\Pi}_{k,3}(m_{\cdot+\ell}, f_{\cdot+\ell}), \quad (5.4)$$

where $\tilde{\Pi}_{k,3}(m_{\cdot+\ell}, f_{\cdot+\ell}) = \sum_{j=k}^{\infty} \Delta_k(\bar{\Delta}_j m_{j+\ell} \cdot \Delta_j f_{j+\ell})$ (see also (2.7)). □

Proof of Theorem 5.1. We begin by writing

$$a(x, \xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} (1 + |u|^2)^{-(n+1)/2} a_u(x, \xi) du + \lambda(x, \xi), \quad (5.5)$$

where $\lambda(x, \xi) = 0$ for $|\xi| \geq 3$,

$$\begin{aligned} \left\| \partial_\xi^\beta \lambda(\cdot, \xi) \mid B_{p_2, q_2}^r \right\| &\leq c_\beta (1 + |\xi|)^{-|\beta|} \quad (\forall \beta \in \mathbb{N}^n), \\ a_u(x, \xi) &= \sum_{j=0}^{\infty} m_{j,u}(x) \theta_u(2^{-j}\xi), \\ \sup_{j \in \mathbb{N}, u \in \mathbb{R}^n} \|m_{j,u} \mid B_{p_2, q_2}^r\| &\leq c, \end{aligned} \tag{5.6}$$

$$\theta_u(\xi) = (2\pi)^{-n} (1 + |u|^2)^{(n+1-L)/2} e^{iu \cdot \xi} \theta(\xi),$$

θ is a C^∞ function with $\text{supp } \theta \subset \{\xi \in \mathbb{R}^n : 1 \leq |\xi| \leq 3\}$, and

$$\|\theta_u^{(\beta)} \mid L_\infty\| \leq c \quad (\forall u \in \mathbb{R}^n, |\beta| \leq L - n - 1). \tag{5.7}$$

For the decomposition (5.5), we refer the reader to [2] or [8].

Now, by Lemma 5.2, we have

$$\|\text{Op}_{a_u} f \mid F_{p,q}^s\| \leq c \|\{2^{js} f_{u,j}\}_{j \in \mathbb{N}} \mid L_{p_1}(\ell_q)\| \leq c' \|f \mid F_{p_1,q}^s\|, \tag{5.8}$$

where $\mathcal{F}(f_{u,j})(\xi) = \theta_u(2^{-j}\xi) \hat{f}(\xi)$ and c' is independent of u . Next, we can write

$$\text{Op}_\lambda f(x) = \int_{\mathbb{R}^n} (1 + |u|^2)^{-(n+1)/2} b_u(x) f(x+u) du, \tag{5.9}$$

where

$$b_u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-iu \cdot \xi} (I - \Delta)^{2n} \lambda(x, \xi) d\xi. \tag{5.10}$$

Theorems 2.1 and 2.5 immediately give

$$\begin{aligned} \|\text{Op}_\lambda f \mid F_{p,q}^s\| &\leq \sup_{u \in \mathbb{R}^n} \|b_u \cdot f(\cdot + u) \mid F_{p,q}^s\| \\ &\leq \left(\sup_{u \in \mathbb{R}^n} \|b_u \mid B_{p_2, q_2}^r\| \right) \|f \mid F_{p_1, q}^s\| \leq c \|f \mid F_{p_1, q}^s\|. \end{aligned} \tag{5.11}$$

□

THEOREM 5.3. *Let $1 \leq p, p_1, q, q_1 \leq \infty$, $r > 0$, and*

$$-r + \frac{n}{p} + \frac{n}{p_1} - n < s < \min\left(\frac{n}{p}, r\right). \tag{5.12}$$

Suppose that $a \in S_{1,0}^0(B_{p_1, q_1}^r)$ if $r > n/p_1$ and $a \in S_{1,0}^0(L_\infty) \cap S_{1,0}^0(B_{p_1, \infty}^{n/p_1})$ if $r = n/p_1$. Then the operator Op_a is bounded on $F_{p,q}^s$ and $B_{p,q}^s$.

For the proof, we apply Theorem 3.1 and proceed as in Theorem 5.1, however, we need an almost-orthogonality estimate of the type in Lemma 5.2, that is, the following lemma.

LEMMA 5.4. *Let $\gamma > 1$, $0 < p, p_1, q, q_1 \leq \infty$, $r \geq n/p_1$, and s be as in Theorem 5.3. For all sequences of functions $\{f_j\}_{j \in \mathbb{N}}$ such that $\text{supp } \hat{f}_j \subset \{\xi \in \mathbb{R}^n : \gamma^{-1}2^j \leq |\xi| \leq \gamma 2^j\}$ and all sequences $\{m_j\}_{j \in \mathbb{N}} \subset B_{p_1, q_1}^r$ (or $\{m_j\}_{j \in \mathbb{N}} \subset B_{p_1, q_1}^{n/p_1} \cap L_\infty$), the estimates*

$$\left\| \sum_{j=0}^{\infty} m_j \cdot f_j \mid F_{p, q}^s \right\| \leq c \|\{f_j\}_{j \in \mathbb{N}} \mid L_p^\gamma(\ell_q^s)\|,$$

$$\left\| \sum_{j=0}^{\infty} m_j \cdot f_j \mid B_{p, q}^s \right\| \leq c' \|\{f_j\}_{j \in \mathbb{N}} \mid \ell_q^s(L_p^\gamma)\|,$$
(5.13)

hold. The constants c and c' are of the form $c'' \sup_{j \in \mathbb{N}} (\|m_j \mid L_\infty\| + \|m_j \mid B_{p_1, q_1}^r\|)$.

Acknowledgment

We would like to thank the referee(s) for the most helpful remarks and corrections which led to the improvements, the results, and the presentation of this paper.

References

- [1] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, Berlin, 1976.
- [2] G. Bourdaud and M. Moussai, *Continuité des commutateurs d'intégrales singulières sur les espaces de Besov*, Bulletin des Sciences Mathématiques **118** (1994), no. 2, 117–130.
- [3] D. Drihem and M. Moussai, *Some embeddings into the multiplier spaces associated to Besov and Lizorkin-Triebel spaces*, Zeitschrift für Analysis und ihre Anwendungen **21** (2002), no. 1, 179–184.
- [4] J. Franke, *On the spaces $F_{p, q}^s$ of Triebel-Lizorkin type: pointwise multipliers and spaces on domains*, Mathematische Nachrichten **125** (1986), 29–68.
- [5] J. Johnsen, *Pointwise multiplication of Besov and Triebel-Lizorkin spaces*, Mathematische Nachrichten **175** (1995), 85–133.
- [6] J. Marschall, *On the boundedness and compactness of nonregular pseudo-differential operators*, Mathematische Nachrichten **175** (1995), 231–262.
- [7] ———, *Nonregular pseudo-differential operators*, Zeitschrift für Analysis und ihre Anwendungen **15** (1996), no. 1, 109–148.
- [8] M. Moussai, *Continuity of pseudo-differential operators on Bessel and Besov spaces*, Serdica. Mathematical Journal **27** (2001), no. 3, 249–262.
- [9] J. Peetre, *New Thoughts on Besov Spaces*, Duke University Mathematics Series, no. 1, Mathematics Department, Duke University, North Carolina, 1976.
- [10] T. Runst and W. Sickel, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*, de Gruyter Series in Nonlinear Analysis and Applications, vol. 3, Walter de Gruyter, Berlin, 1996.
- [11] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland Mathematical Library, vol. 18, North-Holland, Amsterdam, 1978.
- [12] ———, *Theory of Function Spaces*, Monographs in Mathematics, vol. 78, Birkhäuser, Basel, 1983.

- [13] ———, *Theory of Function Spaces. II*, Monographs in Mathematics, vol. 84, Birkhäuser, Basel, 1992.
- [14] M. Yamazaki, *A quasi-homogeneous version of paradifferential operators. I. Boundedness on spaces of Besov type*, Journal of the Faculty of Science. University of Tokyo. Section IA. Mathematics **33** (1986), no. 1, 131–174.

Douadi Drihem: Department of Mathematics, Laboratory of Mathematics Pure and Applied,
M'Sila University, P.O. Box 166, M'Sila 28000, Algeria
E-mail address: douadidr@yahoo.fr

Madani Moussai: Department of Mathematics, Laboratory of Mathematics Pure and Applied,
M'Sila University, P.O. Box 166, M'Sila 28000, Algeria
E-mail address: mmoussai@yahoo.fr