

C-COMPACTNESS MODULO AN IDEAL

M. K. GUPTA AND T. NOIRI

Received 24 January 2006; Revised 30 March 2006; Accepted 4 April 2006

We investigate the concepts of quasi- H -closed modulo an ideal which generalizes quasi- H -closedness and C -compactness modulo an ideal which simultaneously generalizes C -compactness and compactness modulo an ideal. We obtain a characterization of maximal C -compactness modulo an ideal. Preservation of C -compactness modulo an ideal by functions is also investigated.

Copyright © 2006 Hindawi Publishing Corporation. All rights reserved.

1. Introduction

In the present paper, we consider a topological space equipped with an ideal, a theme that has been treated by Vaidyanathaswamy [15] and Kuratowski [6] in their classical texts. An ideal \mathcal{I} on a set X is a nonempty subset of $P(X)$, the power set of X , which is closed for subsets and finite unions. An ideal is also called a *dual filter*. $\{\phi\}$ and $P(X)$ are trivial examples of ideals. Some useful ideals are (i) \mathcal{I}_f , the ideal of all finite subsets of X , (ii) \mathcal{I}_c , the ideal of all countable subsets of X , (iii) \mathcal{I}_n , the ideal of all nowhere dense subsets in a topological space (X, τ) , and (iv) \mathcal{I}_s , the set of all scattered sets in (X, τ) . For an ideal \mathcal{I} on X and $A \subset X$, we denote the ideal $\{I \cap A : I \in \mathcal{I}\}$ by \mathcal{I}_A .

A topological space (X, τ) with an ideal \mathcal{I} on X is denoted by (X, τ, \mathcal{I}) . For a subset $A \subseteq X$, $A^*(\mathcal{I}, \tau)$ (called the adherence of A modulo an ideal \mathcal{I}) or $A^*(\mathcal{I})$ or just A^* is the set $\{x \in X : A \cap U \notin \mathcal{I} \text{ for every open neighborhood } U \text{ of } x\}$. $A^*(\mathcal{I}, \tau)$ has been called the *local function* of A with respect to \mathcal{I} in [6]. It is easy to see that (i) for the ideal $\{\phi\}$, A^* is the closure of A , (ii) for the ideal $P(X)$, A^* is ϕ , and (iii) for ideal \mathcal{I}_f , A^* is the set of all ω -accumulation points of A . For general properties of the operator $*$, we refer the readers to [5, 14].

Observe that the operator $\text{cl}^* : P(X) \rightarrow P(X)$ defined by $\text{cl}^*(A) = A \cup A^*$ is a Kuratowski closure operator on X and hence generates a topology $\tau^*(\mathcal{I})$ or just τ^* on X finer than τ . As has already been observed, $\tau^*(\{\phi\}) = \tau$ and $\tau^*(P(X)) =$ the discrete topology. A description of open sets in $\tau^*(\mathcal{I})$ as given in Vaidyanathaswamy [15] is given in the following.

2 C-compactness modulo an ideal

THEOREM 1.1. *If τ is a topology and \mathcal{I} is an ideal, both defined on X , then*

$$\beta = \beta(\tau, \mathcal{I}) = \{V - I : V \in \tau, I \in \mathcal{I}\} \text{ is a base for the topology } \tau^*(\mathcal{I}) \text{ on } X. \quad (1.1)$$

Ideals have been used frequently in the fields closely related to topology, such as real analysis, measure theory, and lattice theory. Some interesting illustrations of $\tau^*(\mathcal{I})$ are as follows [5].

- (1) If τ is the topology generated by the partition $\{\{2n - 1, 2n\} : n \in \mathbb{N}\}$ on the set \mathbb{N} of natural numbers, then $\tau^*(\mathcal{I}_f)$ is the discrete topology.
- (2) If τ is the indiscrete topology on a set X , then $\tau^*(\mathcal{I}_f)$ is the cofinite topology on X , and $\tau^*(\mathcal{I}_c)$ is the co-countable topology on X . If for a fixed point $p \in X$, \mathcal{I} denotes the ideal $\{A \subset X : p \notin A\}$, then $\tau^*(\mathcal{I})$ is the particular point topology on X .
- (3) For any topological space (X, τ) , $\tau^*(\mathcal{I}_n)$ is the τ^α topology of Njåstad [10].
- (4) If τ is the usual topology on the real line \mathbb{R} and \mathcal{I} is the ideal of all subsets of Lebesgue measure zero, then τ^* -Borel sets are precisely the Lebesgue measurable sets of \mathbb{R} .

2. Quasi- H -closed modulo an ideal space

The concept of compactness modulo an ideal was introduced by Newcomb [9] and has been studied among others by Rancin [11], and Hamlett and Janković [3]. A space (X, τ) is defined to be *compact modulo an ideal \mathcal{I}* on X or just (\mathcal{I}) compact space if for every open cover \mathcal{U} of X , there is a finite subfamily $\{U_1, U_2, \dots, U_n\}$ such that $X - \bigcup_{i=1}^n U_i \in \mathcal{I}$. In this section, we define *quasi- H -closedness* modulo an ideal and study some of its properties. In the process, we get some interesting characterizations of *quasi- H -closed* spaces.

Definition 2.1. Let (X, τ) be a topological space and \mathcal{I} an ideal on X . X is *quasi- H -closed modulo \mathcal{I}* or just (\mathcal{I}) QHC if for every open cover \mathcal{U} of X , there is a finite subfamily $\{U_1, U_2, \dots, U_n\}$ of \mathcal{U} such that $X - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{I}$. Such a subfamily is said to be *proximate* subcover modulo \mathcal{I} or just (\mathcal{I}) proximate subcover.

A subset A of a topological space (X, τ) is said to be *preopen* [8] if $A \subset \text{int}(\text{cl}(A))$. The collection of all preopen sets of a space (X, τ) is denoted by $\text{PO}(X)$. An ideal \mathcal{I} of subsets of a topological space (X, τ) is said to be *codense* [1] if the complement of each of its members is dense. Note that an ideal \mathcal{I} is codense if and only if $\mathcal{I} \cap \tau = \{\phi\}$. Codense ideals are called τ -boundary ideals in [9]. An ideal \mathcal{I} of subsets of a topological space (X, τ) is said to be *completely codense* [1] if $\mathcal{I} \cap \text{PO}(X) = \{\phi\}$. Obviously, every completely codense ideal is codense. Note that if (\mathbb{R}, τ) is the set \mathbb{R} of real numbers equipped with the usual topology τ , then \mathcal{I}_c is codense but not completely codense ideal. It is proved in [1] that an ideal \mathcal{I} is completely codense if and only if $\mathcal{I} \subset \mathcal{I}_n$.

From the discussion of Section 1, the proof of the following theorem is immediate.

THEOREM 2.2. *For a space (X, τ) , the following are equivalent:*

- (a) (X, τ) is quasi- H -closed;

- (b) (X, τ) is $(\{\phi\})$ QHC;
- (c) (X, τ) is (\mathcal{I}_f) QHC;
- (d) (X, τ) is (\mathcal{I}_n) QHC;
- (e) (X, τ) is (\mathcal{I}) QHC for every codense ideal \mathcal{I} .

The significance of condition in (e) may be seen by considering the set \mathbb{R} of real numbers equipped with the usual topology τ . If A is a finite subset of \mathbb{R} and \mathcal{I} is the ideal of all subsets of $\mathbb{R} - A$, then (\mathbb{R}, τ) is (\mathcal{I}) QHC, but not quasi- H -closed.

A family \mathcal{F} of subsets of X is said to have the *finite-intersection property modulo an ideal \mathcal{I} on X* or just (\mathcal{I}) FIP if the intersection of no finite subfamily of \mathcal{F} is a member of \mathcal{I} . Recall that a subset in a space is called *regular open* if it is the interior of its own closure. The complement of a regular open set is called *regular closed*. It is proved in [12] that for completely codense ideal \mathcal{I} on a space (X, τ) , the collections of regular open sets of (X, τ) and (X, τ^*) are same. The following theorem contains a number of characterizations of (\mathcal{I}) QHC spaces. Since the proof is similar to that of a theorem in the next section, we omit it.

THEOREM 2.3. *For a space (X, τ) and an ideal \mathcal{I} on X , the following are equivalent:*

- (a) (X, τ) is (\mathcal{I}) QHC;
- (b) for each family \mathcal{F} of closed sets having empty intersection, there is a finite subfamily $\{F_1, F_2, F_3, \dots, F_n\}$ such that $\bigcap_{i=1}^n \text{int}(F_i) \in \mathcal{I}$;
- (c) for each family \mathcal{F} of closed sets such that $\{\text{int}(F) : F \in \mathcal{F}\}$ has (\mathcal{I}) FIP, one has $\bigcap \{F : F \in \mathcal{F}\} \neq \phi$;
- (d) every regular open cover has a finite (\mathcal{I}) proximate subcover;
- (e) for each family \mathcal{F} of nonempty regular closed sets having empty intersection, there is a finite subfamily $\{F_1, F_2, F_3, \dots, F_n\}$ such that $\bigcap_{i=1}^n \text{int}(F_i) \in \mathcal{I}$;
- (f) for each collection \mathcal{F} of nonempty regular closed sets such that $\{\text{int}(F) : F \in \mathcal{F}\}$ has (\mathcal{I}) FIP, one has $\bigcap \{F : F \in \mathcal{F}\} \neq \phi$;
- (g) for each open filter base \mathcal{B} on $P(X) - \mathcal{I}$, $\bigcap \{\text{cl}(B) : B \in \mathcal{B}\} \neq \phi$;
- (h) every open ultrafilter on $P(X) - \mathcal{I}$ converges.

It follows from a result in [13] that τ and $\tau^*(\mathcal{I})$ have the same regular open sets, where \mathcal{I} is a completely codense ideal on (X, τ) . In particular, if $U \in \tau^*$, then $\text{cl}(U) = \text{cl}^*(U)$. Using this observation along with the previous theorem, we have the following.

THEOREM 2.4. *Let \mathcal{I} be a completely codense ideal on a space (X, τ) . Then (X, τ) is (\mathcal{I}) QHC if and only if (X, τ^*) is (\mathcal{I}) QHC.*

Combining this result with Theorem 2.2, we have the following.

COROLLARY 2.5. *Let (X, τ) be a space and \mathcal{I} a completely codense ideal on X . Then the following are equivalent:*

- (a) (X, τ) is quasi- H -closed;
- (b) (X, τ^*) is quasi- H -closed;
- (c) (X, τ^α) is quasi- H -closed.

The last equivalence follows because $\tau^\alpha = \tau^*(\mathcal{I}_n)$, where \mathcal{I}_n is the ideal of nowhere dense sets in X .

3. C-compact modulo an ideal space

In this section, we generalize the concepts of C-compactness of Viglino [16] and compactness modulo an ideal due to Newcomb [9] and Rancin [11]. A space (X, τ) is said to be C-compact if for each closed set A and each τ -open covering \mathcal{U} of A , there exists a finite subfamily $\{U_1, U_2, U_3, \dots, U_n\}$ such that $A \subset \bigcup_{i=1}^n \text{cl}(U_i)$.

Definition 3.1. Let (X, τ) be a topological space and \mathcal{F} an ideal on X . (X, τ) is said to be C-compact modulo \mathcal{F} or just $C(\mathcal{F})$ -compact if for every closed set A and every τ -open cover \mathcal{U} of A , there is a finite subcollection $\{U_1, U_2, U_3, \dots, U_n\}$ such that $A - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{F}$.

It follows from the definition that

$$\begin{array}{ccc}
 \text{compact} & \text{-----} & (\mathcal{F}) \text{ compact} \\
 \downarrow & & \downarrow \\
 \text{C-compact} & \text{-----} & C(\mathcal{F})\text{-compact} \\
 \downarrow & & \downarrow \\
 \text{quasi-}H\text{-closed} & \text{-----} & (\mathcal{F})\text{QHC}
 \end{array} \tag{3.1}$$

Also from the definition in Section 1, we have the following.

THEOREM 3.2. *For a space (X, τ) , the following are equivalent:*

- (a) (X, τ) is C-compact;
- (b) (X, τ) is $C(\{\phi\})$ -compact;
- (c) (X, τ) is $C(\mathcal{F}_f)$ -compact.

Example 3.3. For n and m in the set N of positive integers, let Y denote the subset of the plane consisting of all points of the form $(1/n, 1/m)$ and the points of the form $(1/n, 0)$. Let $X = Y \cup \{\infty\}$. Topologize X as follows: let each point of the form $(1/n, 1/m)$ be open. Partition N into infinitely many infinite-equivalence classes, $\{Z_i\}_{i=1}^\infty$. Let a neighborhood system for the point $(1/i, 0)$ be composed of all sets of the form $G \cup F$, where

$$\begin{aligned}
 G &= \left\{ \left(\frac{1}{i}, 0 \right) \right\} \cup \left\{ \left(\frac{1}{i}, \frac{1}{m} \right) : m \geq k \right\}, \\
 F &= \left\{ \left(\frac{1}{n}, \frac{1}{m} \right) : m \in Z_i, n \geq k \right\}
 \end{aligned} \tag{3.2}$$

for some $k \in N$. Let a neighborhood system for the point ∞ be composed of sets of the form $X \setminus T$, where

$$T = \left\{ \left(\frac{1}{n}, 0 \right) : n \in N \right\} \cup \bigcup_{i=1}^k \left\{ \left(\frac{1}{i}, \frac{1}{m} \right) : m \in N \right\} \cup \left\{ \left(\frac{1}{n}, \frac{1}{m} \right) : m \in Z_i, n \in N \right\} \tag{3.3}$$

for some $k \in N$. It is shown in [16] that X is a C-compact space which is not compact. In view of Theorem 3.2, such a space is $C(\mathcal{F}_f)$ -compact, but not (\mathcal{F}_f) compact.

Example 3.4. Let $X = R^+ \cup \{a\} \cup \{b\}$, where R^+ denotes the set of nonnegative real numbers and a, b are two distinct points not in R^+ . Let $W(a) = \{V \subset X : V = \{a\} \cup \bigcup_{r=m}^{\infty} (2r, 2r+1)\}$, where m is a nonnegative integer, be a neighborhood system for the point a . Let $W(b) = \{V \subset X : V = \{b\} \cup \bigcup_{r=m}^{\infty} (2r-1, 2r)\}$, where m is a nonnegative integer, be a neighborhood system for the point b . Let R^+ , with the usual topology, be imbedded in X . Viglino [16] has shown that the space X is not C -compact. If A is a finite subset of X , then (X, τ) is $C(\mathcal{F})$ -compact, where \mathcal{F} is the ideal of all subsets of $X - A$.

In view of Examples 3.3 and 3.4, it is clear that the implications shown after Definition 3.1 are, in general, irreversible.

It is proved in [3] that if (X, τ) is quasi- H -closed and \mathcal{F} is an ideal such that $\mathcal{F}_n \subset \mathcal{F}$, then (X, τ) is (\mathcal{F}) compact (and hence $C(\mathcal{F})$ -compact).

Next, if $\{U_1, U_2, \dots, U_n\}$ is a finite collection of open subsets such that $X - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{F}_n$, then $X - \bigcup_{i=1}^n \text{cl}(U_i) = \phi$ because $\tau \cap \mathcal{F}_n = \{\phi\}$. But then $\text{int}(\text{cl}(X - \bigcup_{i=1}^n U_i)) = X - \bigcup_{i=1}^n \text{cl}(U_i) = \phi$ implies that $X - \bigcup_{i=1}^n U_i \in \mathcal{F}_n$. Therefore, a space (X, τ) is (\mathcal{F}_n) compact if and only if it is $C(\mathcal{F}_n)$ -compact. In view of this discussion, we have the following.

THEOREM 3.5. *For a space (X, τ) , the following are equivalent:*

- (a) (X, τ) is quasi- H -closed;
- (b) (X, τ) is (\mathcal{F}_n) QHC;
- (c) (X, τ) is $C(\mathcal{F}_n)$ -compact;
- (d) (X, τ) is (\mathcal{F}_n) compact.

A space (X, τ) is said to be *Baire* if the intersection of every countable family of open sets in (X, τ) is dense. It is noted in [5] that a space (X, τ) is Baire if and only if $\tau \cap \mathcal{F}_m = \{\phi\}$, where \mathcal{F}_m is the ideal of meager (first category) subsets of (X, τ) . Thus, in view of the above theorem, a Baire space (X, τ) is $C(\mathcal{F}_m)$ -compact if and only if it is quasi- H -closed.

We now give some characterizations of $C(\mathcal{F})$ -compact spaces.

THEOREM 3.6. *Let (X, τ) be a space and let \mathcal{F} be an ideal on X . Then the following are equivalent:*

- (a) (X, τ) is $C(\mathcal{F})$ -compact;
- (b) for each closed subset A of X and each family \mathcal{F} of closed subsets of X such that $\bigcap \{F \cap A : F \in \mathcal{F}\} = \phi$, there exists a finite subfamily $\{F_1, F_2, F_3, \dots, F_n\}$ such that $\bigcap (\text{int}(F_i)) \cap A \in \mathcal{F}$;
- (c) for each closed set A and each family \mathcal{F} of closed sets such that $\{\text{int}(F) \cap A : F \in \mathcal{F}\}$ has (\mathcal{F}) FIP, one has $\bigcap \{F \cap A : F \in \mathcal{F}\} \neq \phi$;
- (d) for each closed set A and each regular open cover \mathcal{U} of A , there exists a finite subcollection $\{U_1, U_2, U_3, \dots, U_n\}$ such that $A - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{F}$;
- (e) for each closed set A and each family \mathcal{F} of regular closed sets such that $\bigcap \{F \cap A : F \in \mathcal{F}\} = \phi$, there is a finite subfamily $\{F_1, F_2, F_3, \dots, F_n\}$ such that $\bigcap_{i=1}^n (\text{int}(F_i)) \cap A \in \mathcal{F}$;
- (f) for each closed set A and each family \mathcal{F} of regular closed sets such that $\{\text{int}(F) \cap A : F \in \mathcal{F}\}$ has (\mathcal{F}) FIP, one has $\bigcap \{F \cap A : F \in \mathcal{F}\} \neq \phi$;
- (g) for each closed set A , each open cover \mathcal{U} of $X - A$ and each open neighborhood V of A , there exists a finite subfamily $\{U_1, U_2, U_3, \dots, U_n\}$ of \mathcal{U} such that $X - (V \cup (\bigcup_{i=1}^n \text{cl}(U_i))) \in \mathcal{F}$;

6 C-compactness modulo an ideal

(h) for each closed set A and each open filter base \mathcal{B} on X such that $\{B \cap A : B \in \mathcal{B}\} \subset P(X) - \mathcal{F}$, one has $\bigcap \{\text{cl}(B) : B \in \mathcal{B}\} \cap A \neq \phi$.

Proof. (a) \Rightarrow (b). Let (X, τ) be $C(\mathcal{F})$ -compact, A a closed subset, and \mathcal{F} a family of closed subsets with $\bigcap \{F \cap A : F \in \mathcal{F}\} = \phi$. Then $\{X - F : F \in \mathcal{F}\}$ is an open cover of A and hence admits a finite subfamily $\{X - F_i : i = 1, 2, \dots, n\}$ such that $A - \bigcup_{i=1}^n \text{cl}(X - F_i) \in \mathcal{F}$. This set in \mathcal{F} is easily seen to be $\bigcap_{i=1}^n \{\text{int}(F_i) \cap A\}$.

(b) \Rightarrow (c). This is easy to be established.

(c) \Rightarrow (a). Let A be a closed subset, let \mathcal{U} be an open cover of A with the property that for no finite subfamily $\{U_1, U_2, U_3, \dots, U_n\}$ of \mathcal{U} , one has $A - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{F}$. Then $\{X - U : U \in \mathcal{U}\}$ is a family of closed sets. Since

$$\bigcap_{i=1}^n \{X - \text{cl}(U_i)\} \cap A = \bigcap_{i=1}^n \{A - \text{cl}(U_i)\} = A - \bigcup_{i=1}^n \text{cl}(U_i), \quad (3.4)$$

the family $\{\text{int}(X - U) \cap A : U \in \mathcal{U}\}$ has (\mathcal{F}) FIP. By the hypothesis $\bigcap \{(X - U) \cap A : U \in \mathcal{U}\} \neq \phi$. But then $A - \bigcup \{U : U \in \mathcal{U}\} \neq \phi$, that is, \mathcal{U} is not a cover of A , a contradiction.

(d) \Rightarrow (a). Let A be a closed subset of X and \mathcal{U} an open cover of A . Then $\{\text{int}(\text{cl}(U)) : U \in \mathcal{U}\}$ is a regular open cover of A . Let $\{\text{int}(\text{cl}(U_i)) : i = 1, 2, \dots, n\}$ be a finite subfamily such that $A - \bigcup_{i=1}^n \text{cl}(\text{int}(\text{cl}(U_i))) \in \mathcal{F}$. Since U_i is open and for each open set U , $\text{cl}(\text{int}(\text{cl}(U))) = \text{cl}(U)$, we have $A - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{F}$, which shows that X is $C(\mathcal{F})$ -compact.

(a) \Rightarrow (d). This is obvious.

The proofs for (d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (d) are parallel to (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a), respectively.

(a) \Rightarrow (g). Let A be a closed set, V an open neighborhood of A , and \mathcal{U} an open cover of $X - A$. Since $X - V \subset X - A$, \mathcal{U} is also an open cover of the closed set $X - V$.

Let $\{U_1, U_2, U_3, \dots, U_n\}$ be a finite subcollection of \mathcal{U} such that $(X - V) - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{F}$. However, the last set is $X - (V \cup \{\bigcup_{i=1}^n \text{cl}(U_i)\})$.

(g) \Rightarrow (a). Let A be a closed subset of X and \mathcal{U} an open covering of A . If H denotes the union of members of \mathcal{U} , then $F = X - H$ is a closed set and $X - A$ is an open neighborhood of F . Also \mathcal{U} is an open cover of $X - F$. By hypothesis, there is a finite subcollection $\{U_1, U_2, U_3, \dots, U_n\}$ of \mathcal{U} such that

$$X \left((X - A) \cup \left\{ \bigcup_{i=1}^n \text{cl}(U_i) \right\} \right) \in \mathcal{F}. \quad (3.5)$$

However, this set in \mathcal{F} is nothing but $A - \bigcup_{i=1}^n \text{cl}(U_i)$.

(a) \Rightarrow (h). Suppose A is a closed set and \mathcal{B} is any open filter base on X with $\{B \cap A : B \in \mathcal{B}\} \subset P(X) - \mathcal{F}$. Suppose, if possible, $\bigcap \{\text{cl}(B) : B \in \mathcal{B}\} \cap A = \phi$. Then $\{X - \text{cl}(B) : B \in \mathcal{B}\}$ is an open cover of A . By the hypothesis, there exists a finite subfamily $\{X - \text{cl}(B_i) : i = 1, 2, 3, \dots, n\}$ such that $A - \bigcup_{i=1}^n \text{cl}(X - \text{cl}(B_i))$ is in \mathcal{F} . However, this set is $A \cap (\bigcap_{i=1}^n \text{int}(\text{cl}(B_i)))$ and $A \cap (\bigcap_{i=1}^n B_i)$ is a subset of it. Therefore, $A \cap (\bigcap_{i=1}^n B_i) \in \mathcal{F}$. Since \mathcal{B} is a filter base, we have a $B \in \mathcal{B}$ such that $B \subset \bigcap_{i=1}^n B_i$. But then $A \cap B \in \mathcal{F}$ which contradicts the fact that $\{B \cap A : B \in \mathcal{B}\} \subset P(X) - \mathcal{F}$.

(h) \Rightarrow (a). Suppose that (X, τ) is not $C(\mathcal{F})$ -compact. Then there exist a closed subset A of X and an open cover \mathcal{U} of A such that for any finite subfamily $\{U_1, U_2, U_3, \dots, U_n\}$

of \mathcal{U} , we have $A - \bigcup_{i=1}^n \text{cl}(U_i) \notin \mathcal{F}$. We may assume that \mathcal{U} is closed under finite unions. Then the family $\mathcal{B} = \{X - \text{cl}(U) : U \in \mathcal{U}\}$ is an open filter base on X such that $\{B \cap A : B \in \mathcal{B}\} \subset P(A) - \mathcal{F}$. So, by the hypothesis, $\bigcap \{\text{cl}(X - \text{cl}(U)) : U \in \mathcal{U}\} \cap A \neq \emptyset$. Let x be a point in the intersection. Then $x \in A$ and $x \in \text{cl}(X - \text{cl}(U)) = X - \text{int}(\text{cl}(U)) \subset X - U$ for each $U \in \mathcal{U}$. But this contradicts the fact that \mathcal{U} is a cover of A . Hence (X, τ) is $C(\mathcal{F})$ -compact. \square

Next we characterize $C(\mathcal{F})$ -compact spaces using some weaker forms of filter base convergence.

Definition 3.7. A filter base \mathcal{B} is said to be (\mathcal{F}) adherent convergent if for every neighborhood G of the adherent set of \mathcal{B} , there exists an element $B \in \mathcal{B}$ such that $(X - G) \cap B \in \mathcal{F}$. Clearly, every adherent convergent filter base is (\mathcal{F}) adherent convergent and a filter base is adherent convergent if and only if it is $(\{\emptyset\})$ adherent convergent.

THEOREM 3.8. *A space (X, τ) is $C(\mathcal{F})$ -compact if and only if every open filter base on $P(X) - \mathcal{F}$ is (\mathcal{F}) adherent convergent.*

Proof. Let (X, τ) be $C(\mathcal{F})$ -compact and let \mathcal{B} be an open filter base on $P(X) - \mathcal{F}$ with A as its adherent set. Let G be an open neighborhood of A . Then $A = \bigcap \{\text{cl}(B) : B \in \mathcal{B}\}$, $A \subset G$, and $X - G$ is closed. Now $\{X - \text{cl}(B) : B \in \mathcal{B}\}$ is an open cover of $X - G$ and so by the hypothesis, it admits a finite subfamily $\{X - \text{cl}(B_i) : i = 1, 2, 3, \dots, n\}$ such that $(X - G) - \bigcup_{i=1}^n \text{cl}(X - \text{cl}(B_i)) \in \mathcal{F}$. But this implies $(X - G) \cap (\bigcap_{i=1}^n \text{int}(\text{cl}(B_i))) \in \mathcal{F}$. However, $B_i \subset \text{int}(\text{cl}(B_i))$ implies $(X - G) \cap (\bigcap_{i=1}^n B_i) \in \mathcal{F}$. Since \mathcal{B} is a filter base and $B_i \in \mathcal{B}$, there is a $B \in \mathcal{B}$ such that $B \subset \bigcap_{i=1}^n B_i$. But then $(X - G) \cap B \in \mathcal{F}$ is required.

Conversely, let (X, τ) be not $C(\mathcal{F})$ -compact, and let A be a closed set, and \mathcal{U} an open cover of A such that for no finite subfamily $\{U_1, U_2, U_3, \dots, U_n\}$ of \mathcal{U} , one has $A - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{F}$. Without loss of generality, we may assume that \mathcal{U} is closed for finite unions. Therefore, $\mathcal{B} = \{X - \text{cl}(U) : U \in \mathcal{U}\}$ becomes an open filter base on $P(X) - \mathcal{F}$. If x is an adherent point of \mathcal{B} , that is, if $x \in \bigcap \{\text{cl}(X - \text{cl}(U)) : U \in \mathcal{U}\} = X - \bigcup \{\text{int}(\text{cl}(U)) : U \in \mathcal{U}\}$, then $x \notin A$, because \mathcal{U} is an open cover of A and for $U \in \mathcal{U}$, $U \subset \text{int}(\text{cl}(U))$. Therefore, the adherent set of \mathcal{B} is contained in $X - A$, which is an open set. By the hypothesis, there exists an element $B \in \mathcal{B}$ such that $(X - (X - A)) \cap B \in \mathcal{F}$, that is, $A \cap B \in \mathcal{F}$, that is, $A \cap (X - \text{cl}(U)) \in \mathcal{F}$, that is, $A - \text{cl}(U) \in \mathcal{F}$ for some $U \in \mathcal{U}$. This however contradicts our assumption. This completes the proof. \square

Herrington and Long [4] characterized C -compact spaces using r -convergence of filters and nets. We obtain similar results for $C(\mathcal{F})$ -compact spaces in the next definition.

Definition 3.9. Let X be a space, $\emptyset \neq A \subset X$, and let \mathcal{B} be a filter base on A . \mathcal{B} is said to r -converge to $a \in A$ if for each open set V in X containing a , there is $B \in \mathcal{B}$ with $B \subset \text{cl}(V)$. The filter base \mathcal{B} is said to r -accumulate to a , if for each open set V containing a , $\text{cl}(V) \cap B \neq \emptyset$ for each $B \in \mathcal{B}$.

Similarly, a net $\varphi : D \rightarrow A \subset X$ is said to r -converge to $a \in A$ if for each open set V containing a , there is a $b \in D$ such that $\varphi(c) \in \text{cl}(V)$ for all $c \geq b$. φ is said to r -accumulate to a if for each open set V containing a and each $b \in D$, there is $c \in D$ with $c \geq b$ and $\varphi(c) \in \text{cl}(V)$.

8 C-compactness modulo an ideal

It is known [4] that convergence (accumulation) for filter bases and nets implies r -convergence (r -accumulation), but the converse is not true.

THEOREM 3.10. *For a space (X, τ) and an ideal \mathcal{I} on X , the following are equivalent:*

- (a) (X, τ) is $C(\mathcal{I})$ -compact;
- (b) for each closed set A , each filter base \mathcal{B} on $P(A) - \mathcal{I}$ r -accumulates to some $a \in A$;
- (c) for each closed set A , each maximal filter base \mathcal{M} on $P(A) - \mathcal{I}$ r -converges to some $a \in A$;
- (d) for each closed set A , each net φ on $P(A) - \mathcal{I}$ r -accumulates to some $a \in A$.

Proof. (a) \Rightarrow (b). Suppose there exist a closed set A and a filter base \mathcal{B} on $P(A) - \mathcal{I}$ which does not r -accumulate to any $a \in A$. Then for each $a \in A$, there exists an open set $U(a)$ containing a and a $B(a) \in \mathcal{B}$ such that $B(a) \cap \text{cl}(U(a)) = \emptyset$. Then $\{U(a) : a \in A\}$ is an open cover of the closed set A . By (a), there exists a finite subcollection $\{U(a_i) : i = 1, 2, 3, \dots, n\}$ such that $A - \bigcup_{i=1}^n \text{cl}(U(a_i)) \in \mathcal{I}$. If $B \in \mathcal{B}$ is such that $B \subset \bigcap_{i=1}^n B(a_i)$, then $B \cap (A - \bigcup_{i=1}^n \text{cl}(U(a_i))) \in \mathcal{I}$, that is, $B - \bigcup_{i=1}^n \text{cl}(U(a_i)) \in \mathcal{I}$. But the later set is just B , because $B \subset B(a_i)$ and $B(a_i) \cap \text{cl}(U(a_i)) = \emptyset$ for each i . However, $B \in \mathcal{I}$ is a contradiction, because $B \in \mathcal{B}$ and $\mathcal{B} \subset P(A) - \mathcal{I}$.

(b) \Leftrightarrow (c). This follows in view of parts (a), (b), and (c) of [4, Theorem 1].

(b) \Rightarrow (a). If possible, let X be not $C(\mathcal{I})$ -compact. Then by Theorem 3.6(f), there exist a closed set A and a collection \mathcal{F} of regular closed sets with the property that for every finite subcollection $\{F_1, F_2, F_3, \dots, F_n\}$, $\bigcap_{i=1}^n \text{int}(F_i) \cap A \notin \mathcal{I}$, but $\bigcap \{F : F \in \mathcal{F}\} \cap A = \emptyset$. Now the collection of sets of the form $\bigcap_{i=1}^n \text{int}(F_i) \cap A$ for all possible finite subfamilies $\{F_1, F_2, F_3, \dots, F_n\}$ of \mathcal{F} forms a filter base on $P(A) - \mathcal{I}$. By (b), this filter base r -accumulates to some $a \in A$, that is, for each open set $U(a)$ containing a and for each $F \in \mathcal{F}$, $\text{cl}(U(a)) \cap (\text{int}(F) \cap A) \neq \emptyset$. However, $a \in A$ and $A \cap \{F : F \in \mathcal{F}\} = \emptyset$ imply that there is some $F = F(a) \in \mathcal{F}$ such that $a \notin F(a)$. Then $X - F(a)$ is an open set containing a such that $\text{cl}(X - F(a)) \cap (\text{int}(F(a)) \cap A) = \emptyset$. This is a contradiction.

(b) \Leftrightarrow (d). This follows using standard arguments about nets and filters. \square

If in the above theorem, A is replaced by the whole space X , we get the characterizations of (\mathcal{I}) QHC spaces. If in addition we consider completely codense ideal \mathcal{I} , we get the characterizations of quasi- H -closed spaces.

4. $C(\mathcal{I})$ -compact spaces and functions

A function $f : (X, \tau) - (Y, \zeta)$ is said to be θ -continuous [2] at a point $x \in X$ if for every open set V of Y containing $f(x)$, there exists an open set U of X containing x such that $f(\text{cl}(U)) \subseteq \text{cl}(V)$. A function $f : (X, \tau) - (Y, \zeta)$ is said to be θ -continuous if f is θ -continuous for every $x \in X$. The concept of θ -continuity is weaker than that of continuity. An important property of C -compact spaces is that a continuous function from a C -compact space to a Hausdorff space is closed. We prove the following more general results.

THEOREM 4.1. *Let $f : (X, \tau, \mathcal{I}) - (Y, \zeta, \vartheta)$ be a θ -continuous function, (X, τ, \mathcal{I}) $C(\mathcal{I})$ -compact, (Y, ζ) Hausdorff, and $f(\mathcal{I}) \subseteq \vartheta$. Then $f(A)$ is $\zeta^*(\vartheta)$ -closed for each closed set A of X .*

Proof. Let A be any closed set in X and $a \notin f(A)$. For each $x \in A$, there exists a ζ -open set V_y containing $y = f(x)$ such that $a \notin \text{cl}(V_y)$. Now because f is θ -continuous, there exists an open set U_x containing x such that $f(\text{cl}(U_x)) \subseteq \text{cl}(V_y)$. The family $\{U_x : x \in A\}$ is an open cover of A . Therefore, there exists a finite subfamily $\{U_{x_i} : i = 1, 2, \dots, n\}$ such that $A - \bigcup_{i=1}^n \text{cl}(U_{x_i}) \in \mathcal{F}$. But then $f(A - \bigcup_{i=1}^n \text{cl}(U_{x_i})) \in f(\mathcal{F}) \subseteq \mathcal{G}$, that is, $f(A) - f(\bigcup_{i=1}^n \text{cl}(U_{x_i})) \in f(\mathcal{F}) \subseteq \mathcal{G}$ because $f(\mathcal{F})$ is also an ideal. Hence $f(A) - (\bigcup_{i=1}^n \text{cl}(V_{y_i})) \in f(\mathcal{F}) \subseteq \mathcal{G}$. Now $a \notin \text{cl}(V_{y_i})$ for any i implies that $a \in Y - \bigcup_{i=1}^n \text{cl}(V_{y_i})$ which is open in (Y, ζ) and $(Y - \bigcup_{i=1}^n \text{cl}(V_{y_i})) \cap f(A) = f(A) - \bigcup_{i=1}^n \text{cl}(V_{y_i}) \in f(\mathcal{F}) \subseteq \mathcal{G}$. Hence $a \notin (f(A))^*(\sigma, \mathcal{G})$. Thus $(f(A))^*(\sigma, \mathcal{G}) \subset f(A)$ and so $f(A)$ is $\zeta^*(\mathcal{G})$ -closed. \square

COROLLARY 4.2. *Let $f : (X, \tau, \mathcal{F}) - (Y, \zeta, \mathcal{G})$ be a continuous function, (X, τ, \mathcal{F}) $C(\mathcal{F})$ -compact, (Y, ζ) Hausdorff, and $f(\mathcal{F}) \subseteq \mathcal{G}$. Then $f(A)$ is $\zeta^*(\mathcal{G})$ -closed for each closed set A of X .*

THEOREM 4.3. *Let $f : (X, \tau, \mathcal{F}) - (Y, \zeta, \mathcal{G})$ be a continuous surjection, (X, τ, \mathcal{F}) $C(\mathcal{F})$ -compact, and $f(\mathcal{F}) \subseteq \mathcal{G}$. Then (Y, ζ, \mathcal{G}) is $C(\mathcal{G})$ -compact.*

Proof. Let A be any closed subset of (Y, ζ) and $\{V_\alpha : \alpha \in \Lambda\}$ any open cover of A by open sets in Y . Then $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is an open cover of $f^{-1}(A)$ which is closed in X . Hence, by the hypothesis, there exists a finite subcollection $\{f^{-1}(V_{\alpha_i}) : i = 1, 2, \dots, n\}$ such that $f^{-1}(A) - \bigcup_{i=1}^n \text{cl}(f^{-1}(V_{\alpha_i})) \in \mathcal{F}$. Since f is continuous, $\text{cl}(f^{-1}(B)) \subset f^{-1}(\text{cl}(B))$ for every subset B of Y . Hence we have $f^{-1}(A) - \bigcup_{i=1}^n f^{-1}(\text{cl}(V_{\alpha_i})) = f^{-1}(A - \bigcup_{i=1}^n \text{cl}(V_{\alpha_i})) \in \mathcal{F}$. Since f is surjective, $A - \bigcup_{i=1}^n \text{cl}(V_{\alpha_i}) \in f(\mathcal{F}) \subset \mathcal{G}$. Hence Y is $C(\mathcal{G})$ -compact. \square

THEOREM 4.4. *If the product space $\prod X_\alpha$ of nonempty family of topological spaces (X_α, τ_α) is $C(\mathcal{F})$ -compact, then each (X_α, τ_α) is $C(p_\alpha(\mathcal{F}))$ -compact, where p_α is the projection map and \mathcal{F} is an ideal on $\prod X_\alpha$.*

Proof. This follows from Theorem 4.3. \square

5. $C(\mathcal{F})$ -compact spaces and subspaces

In this section, we introduce three types of $C(\mathcal{F})$ -compact subsets and use them to obtain new characterizations of $C(\mathcal{F})$ -compact spaces and a characterization of maximal $C(\mathcal{F})$ -compact spaces.

Definition 5.1. Let (X, τ) be a space and \mathcal{F} an ideal on X . A subset Y of X is said to be $C(\mathcal{F})$ -compact if the subspace (Y, τ_Y) is $C(\mathcal{F})$ -compact.

Some useful results about such subspaces are contained in the following theorem. The proofs are easy to establish.

THEOREM 5.2. *Let (X, τ) be a space and \mathcal{F} an ideal on X . Then*

- (a) *a subspace Y is $C(\mathcal{F})$ -compact if and only if it is $C(\mathcal{F}_Y)$ -compact;*
- (b) *a clopen subspace of a $C(\mathcal{F})$ -compact space is $C(\mathcal{F})$ -compact;*
- (c) *if Y is a regular closed subset of a $C(\mathcal{F})$ -compact space (X, τ, \mathcal{F}) and \mathcal{F} is codense, then (Y, τ_Y) is quasi- H -closed;*
- (d) *a finite union of $C(\mathcal{F})$ -compact subspaces of X is $C(\mathcal{F})$ -compact.*

Definition 5.3. A subset Y of (X, τ) is said to be $C(\mathcal{F})$ -compact relative to τ if every τ -open cover of every relatively closed subset A of Y has a finite subfamily whose τ -closures cover A except a set in \mathcal{F} .

Some useful properties of such spaces are contained in the following.

THEOREM 5.4. *Let (X, τ) be a space and \mathcal{F} an ideal on X . Then the following hold.*

- (a) *A closed subspace of a $C(\mathcal{F})$ -compact relative to τ subspace of (X, τ) is $C(\mathcal{F})$ -compact relative to τ .*
- (b) *If (X, τ) is Hausdorff and Y is $C(\mathcal{F})$ -compact relative to τ , then Y is $\tau^*(\mathcal{F})$ -closed.*
- (c) *If Y is a $C(\mathcal{F})$ -compact relative to τ subspace of (X, τ) and $f : (X, \tau) \rightarrow (Z, \zeta)$ is a continuous bijection, then $f(Y)$ is $C(f(\mathcal{F}))$ -compact relative to ζ .*
- (d) *$C(\mathcal{F})$ -compactness relative to τ is contractive.*

The following characterization of $C(\mathcal{F})$ -compact spaces is obtained using $C(\mathcal{F})$ -compact relative to τ subspaces. The proof is easy.

THEOREM 5.5. *A space (X, τ) with an ideal \mathcal{F} is $C(\mathcal{F})$ -compact if and only if every proper closed subset of X is $C(\mathcal{F})$ -compact relative to τ .*

Definition 5.6. A subset Y of a space (X, τ) is said to be *closure $C(\mathcal{F})$ -compact* if for every τ_Y -closed subset K of Y and every τ -open cover \mathcal{U} of $\text{cl}(K)$, there is a finite subcollection $\{U_1, U_2, U_3, \dots, U_n\}$ of \mathcal{U} such that $K - \bigcup_{i=1}^n \text{cl}_Y(U_i \cap Y) \in \mathcal{F}$.

It is easy to see that closure $C(\mathcal{F})$ -compactness is contractive.

Example 5.7. Since closed subsets of $C(\mathcal{F})$ -compact spaces are not necessarily (\mathcal{F}) QHC, a space (X, τ) which is $C(\mathcal{F})$ -compact relative to τ may fail to be closure $C(\mathcal{F})$ -compact. Moreover, $]0, 1[$ as a subspace of $[0, 1]$ is closure $C(\mathcal{F})$ -compact with $\mathcal{F} = \{\emptyset\}$, but not $C(\mathcal{F})$ -compact relative to the usual topology. Thus the concepts of $C(\mathcal{F})$ -compact relative to τ and closure $C(\mathcal{F})$ -compact are independent concepts.

We now have the following characterization of $C(\mathcal{F})$ -compact spaces.

THEOREM 5.8. *A space (X, τ) is $C(\mathcal{F})$ -compact for an ideal \mathcal{F} on X if and only if every open subset of X is closure $C(\mathcal{F})$ -compact.*

Proof. Let (X, τ) be $C(\mathcal{F})$ -compact and Y an open subset of X . Let K be a τ_Y -closed subset of Y , and let \mathcal{U} be a τ -open cover of $\text{cl}(K)$. Then there exists a finite subcollection $\{U_1, U_2, U_3, \dots, U_n\}$ of \mathcal{U} such that $\text{cl}K - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{F}$. Since Y is open, therefore, $\text{cl}_Y(U \cap Y) = \text{cl}(U) \cap Y$ and so, by hereditary property of \mathcal{F} , $K - \bigcup_{i=1}^n \text{cl}_Y(U_i \cap Y) \in \mathcal{F}$. Thus Y is closure $C(\mathcal{F})$ -compact.

Conversely, let all open subsets of X be closure $C(\mathcal{F})$ -compact. Let K be a closed and \mathcal{U} an open cover of K . Choose a $U_0 \in \mathcal{U}$. Then $Y = X - \text{cl}(U_0)$ is an open subset of X and $K \cap Y$ is a τ_Y -closed subset of Y . Moreover, $\mathcal{U} - \{U_0\}$ is an open cover of $\text{cl}(K \cap Y)$. By the hypothesis, there exists a finite subcollection $\{U_1, U_2, U_3, \dots, U_n\}$ of $\mathcal{U} - \{U_0\}$ such that $K \cap Y - \bigcup_{i=1}^n \text{cl}_Y(U_i \cap Y) \in \mathcal{F}$. But then $K \cap Y - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{F}$ as $\text{cl}_Y(U_i \cap Y) = \text{cl}(U_i) \cap Y$ and \mathcal{F} is hereditary. Therefore, $K - \bigcup_{i=0}^n \text{cl}(U_i) \in \mathcal{F}$. Hence (X, τ) is $C(\mathcal{F})$ -compact.

Finally, we obtain a characterization of a maximal $C(\mathcal{F})$ -compact space. Recall that a space (X, τ) with property P is said to be *maximal P* if there is no topology σ on X which has property P and is strictly finer than τ . For a topological space (X, τ) and a subset A of X , $\tau(A) = \{U \cup (V \cap A) : U, V \in \tau\}$ is a topology called *simple extension* [7] of τ by A . $\tau(A)$ is strictly finer than τ if and only if $A \notin \tau$. \square

THEOREM 5.9. *A topological space (X, τ) is maximal $C(\mathcal{F})$ -compact if and only if for every subset A of X such that A is closure $C(\mathcal{F})$ -compact and $X - A$ is $C(\mathcal{F})$ -compact relative to τ , one has $A \in \tau$.*

Proof. First we assume that (X, τ) is maximal $C(\mathcal{F})$ -compact and that A is a subset of X satisfying the given conditions. First, we show that $(X, \tau(A))$ is $C(\mathcal{F})$ -compact. Let K be a $\tau(A)$ -closed subset of X . Then $K = K_1 \cup (K_2 \cap (X - A))$, where K_1 and K_2 are τ -closed sets. Let

$$\mathcal{U} = \{U_\alpha \cup (V_\alpha \cap A) : U_\alpha, V_\alpha \in \tau, \alpha \in \Delta\} \tag{5.1}$$

be a $\tau(A)$ -open cover of K . Then $\nu = \{U_\alpha : \alpha \in \Delta\}$ is a τ -open cover of $K \cap (X - A) = (K_1 \cup K_2) \cap (X - A)$. Since, by assumption, $X - A$ is $C(\mathcal{F})$ -compact relative to τ , we have a finite subcollection $\{U_{\alpha_1}, U_{\alpha_2}, U_{\alpha_3}, \dots, U_{\alpha_n}\}$ of ν such that $K \cap (X - A) - \bigcup_{i=1}^n \text{cl}(U_{\alpha_i}) \in \mathcal{F}$. Since $\tau(A)$ is finer than τ , this subcollection is $\tau(A)$ -open and $K \cap (X - A) - \bigcup_{i=1}^n \text{cl}_{\tau(A)}(U_{\alpha_i}) \in \mathcal{F}$. Next, $\mathcal{W} = \{U_\alpha \cup V_\alpha : \alpha \in \Delta\}$ is a τ -open cover of $\text{cl}(K \cap A) = \text{cl}(K_1 \cap A) = \text{cl}_{\tau(A)}(K_1 \cap A)$ and therefore by assumption on A , there exists a finite subcollection $\{U_{\beta_i} \cup V_{\beta_i} : i = 1, 2, \dots, k\}$ of \mathcal{W} such that

$$K_1 \cap A - \bigcup_{i=1}^k \text{cl}_{\tau(A)} [(U_{\beta_i} \cup V_{\beta_i}) \cap A] \in \mathcal{F}. \tag{5.2}$$

However, τ_A , the restriction of τ to A , is nothing but $\tau(A) \upharpoonright A$, the restriction of $\tau(A)$ to A . Therefore,

$$K_1 \cap A - \bigcup_{i=1}^k \text{cl}_{\tau(A) \upharpoonright A} [(U_{\beta_i} \cup V_{\beta_i}) \cap A] \in \mathcal{F}. \tag{5.3}$$

Now $\{U_{\alpha_i} \cup (V_{\alpha_i} \cap A) : i = 1, 2, \dots, n\} \cup \{U_{\beta_i} \cup (V_{\beta_i} \cap A) : i = 1, 2, \dots, k\}$ is a finite $\tau(A)$ (\mathcal{F}) proximate cover of K which is a subcover of \mathcal{U} . Thus the topology $\tau(A)$ on X is also $C(\mathcal{F})$ -compact. However, by the maximality of τ , we have $\tau(A) = \tau$. But then $A \in \tau$ as desired.

Conversely, let (X, τ) be not maximal $C(\mathcal{F})$ -compact. Then there is a $C(\mathcal{F})$ -compact topology σ on X which is strictly finer than τ . Let $A \in \sigma - \tau$. Then A is σ -closure $C(\mathcal{F})$ -compact by Theorem 5.8. Since the property of closure $C(\mathcal{F})$ -compact is carried over to coarser topologies, A is τ -closure $C(\mathcal{F})$ -compact. Also $X - A$ is $C(\mathcal{F})$ -compact relative to σ and hence $C(\mathcal{F})$ -compact relative to τ . By the hypothesis, then $A \in \tau$, a contradiction. \square

Remark 5.10. The readers can generalize the above concepts in bitopological spaces to unify various types of compactness.

References

- [1] J. Dontchev, M. Ganster, and D. A. Rose, *Ideal resolvability*, Topology and Its Applications **93** (1999), no. 1, 1–16.
- [2] S. V. Fomin, *Extensions of topological spaces*, Annals of Mathematics. Second Series **44** (1943), 471–480.
- [3] T. R. Hamlett and D. Janković, *Compactness with respect to an ideal*, Bollettino della Unione Matematica Italiana. B. Serie VII **4** (1990), no. 4, 849–861.
- [4] L. L. Herrington and P. E. Long, *Characterizations of C -compact spaces*, Proceedings of the American Mathematical Society **52** (1975), 417–426.
- [5] D. Janković and T. R. Hamlett, *New topologies from old via ideals*, The American Mathematical Monthly **97** (1990), no. 4, 295–310.
- [6] K. Kuratowski, *Topology. Vol. I*, Academic Press, New York, 1966.
- [7] N. Levine, *Simple extensions of topologies*, The American Mathematical Monthly **71** (1964), no. 1, 22–25.
- [8] A. S. Mashhour, M. E. Abd El-Monsef, and S. N. El-Deep, *On precontinuous and weak precontinuous mappings*, Proceedings of the Mathematical and Physical Society of Egypt (1982), no. 53, 47–53 (1983).
- [9] R. L. Newcomb, *Topologies which are compact modulo an ideal*, Ph.D. thesis, University of California, Santa Barbara, California, 1967.
- [10] O. Njåstad, *On some classes of nearly open sets*, Pacific Journal of Mathematics **15** (1965), 961–970.
- [11] D. V. Rancin, *Compactness modulo an ideal*, Soviet Mathematics. Doklady **13** (1972), 193–197.
- [12] V. Renuka Devi, D. Sivaraj, and T. Tamizh Chelvam, *Codense and completely codense ideals*, Acta Mathematica Hungarica **108** (2005), no. 3, 197–205.
- [13] P. Samuels, *A topology formed from a given topology and ideal*, Journal of the London Mathematical Society. Second Series **10** (1975), no. 4, 409–416.
- [14] R. Vaidyanathaswamy, *The localisation theory in set-topology*, Proceedings of the Indian Academy of Sciences. Section A **20** (1944), 51–61.
- [15] ———, *Set Topology*, 2nd ed., Chelsea, New York, 1960.
- [16] G. Viglino, *C -compact spaces*, Duke Mathematical Journal **36** (1969), no. 4, 761–764.

M. K. Gupta: Department of Mathematics, Faculty of Science, Ch. Charan Singh University,
Meerut-250004, India
E-mail address: mkgupta2002@hotmail.com

T. Noiri: 2949-1, Shiokita-cho, Hinagu, Yatsushiro-shi, Kumamoto-ken 869-5142, Japan
E-mail address: t.noiri@nifty.com



Hindawi

Submit your manuscripts at
<http://www.hindawi.com>

