Let \((u_n)\) be a sequence of real numbers, \(L\) an additive limitable method with some property, and \(\mathcal{U}\) and \(\mathcal{V}\) different spaces of sequences related to each other. We prove that if the classical control modulo of the oscillatory behavior of \((u_n)\) in \(\mathcal{U}\) is a Tauberian condition for \(L\), then the general control modulo of the oscillatory behavior of integer order \(m\) of \((u_n)\) in \(\mathcal{U}\) or \(\mathcal{V}\) is also a Tauberian condition for \(L\).

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1. Introduction

In this paper, \(u_n = O(1)\) and \(u_n = o(1)\) denote \(O(1)\) as \(n \to \infty\) and \(o(1)\) as \(n \to \infty\), respectively. Let \(\mathcal{N}, \mathcal{B}, \mathcal{S},\) and \(\mathcal{M}\) denote the space of sequences converging to 0, bounded, slowly oscillating, and moderately oscillating, respectively.

The classical control modulo of the oscillatory behavior of \((u_n)\) is denoted by \(\omega_n^{(0)}(u) = n\Delta u_n\) and the general control modulo of the oscillatory behavior of order \(m\) of \((u_n)\) is defined by \(\omega_n^{(m)}(u) = \omega_n^{(m-1)}(u) - \sigma_n^{(1)}(\omega^{(m-1)}(u))\), where

\[
\Delta u_n = \begin{cases} u_n - u_{n-1}, & n \geq 1, \\ u_0, & n = 0, \end{cases} \quad \sigma_n^{(1)}(u) = \frac{1}{n+1} \sum_{k=0}^{n} u_k. \tag{1.1}
\]

Tauber [10] proved that if \((u_n)\) is Abel limitable and

\[(\omega_n^{(0)}(u)) \in \mathcal{N}, \tag{1.2}\]

then \((u_n)\) is convergent. The condition (1.2) on the sequence \((u_n)\) is called a Tauberian condition for Abel limitable method and the resulting theorem is called a Tauberian theorem.
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Tauber [10] further proved that the condition
\[(\sigma_n^{(1)}(\omega^0(u))) \in \mathcal{N}\] (1.3)
is also a Tauberian condition. It was shown by Littlewood [6] that the condition (1.2) could be replaced by
\[(\omega_n^0(u)) \in \mathcal{B}.\] (1.4)

Hardy and Littlewood [5] improved Littlewood’s theorem replacing (1.4) by onesided boundedness of \((\omega_n^0(u)).\)

Stanojević [9] reformulated the definition of slow oscillation given by Schmidt [8] in a more suitable form and then proved that the conditions (1.2) and (1.3) could be replaced by
\[(\omega_n^0(u)) \in \mathcal{F},\] (1.5)
\[(\sigma_n^{(1)}(\omega^0(u))) \in \mathcal{F},\] (1.6)
respectively.

A generalization of slow oscillation, moderate oscillation, was introduced by Stanojević and it was proved by Dik [4] that (1.5) could be replaced by
\[(\omega_n^0(u)) \in \mathcal{M},\] (1.7)
and (1.6) could not be replaced by
\[(\sigma_n^{(1)}(\omega^0(u))) \in \mathcal{M}.\] (1.8)

Recently, Čanak and Totur [3] have shown that for any nonnegative integer \(m \geq 1,\)
\[(\omega_n^m(u)) \in \mathcal{M}\] (1.9)
is a Tauberian condition for Abel limitable method.

Meyer-König and Tietz [7] proved that if (1.2) is a Tauberian conditions for an additive and regular limitability method, then (1.3) is a Tauberian condition for \(L\). Čanak et al. [1] extended and generalized Meyer-König and Tietz’s [7] result and obtained the following theorems for an additive and \((C,1)\) regular method \(L\).

**Theorem 1.1.** If \((\omega_n^0(u)) \in \mathcal{F}\) is a Tauberian condition for an additive and \((C,1)\) regular limitable method \(L\), then \((\omega_n^{(1)}(u)) \in \mathcal{F}\) is a Tauberian condition for \(L\).

**Theorem 1.2.** If \((\omega_n^0(u)) \in \mathcal{B}\) is a Tauberian condition for an additive and \((C,1)\) regular limitable method \(L\), then \((\omega_n^{(1)}(u)) \in \mathcal{B}\) is a Tauberian condition for \(L\).

Let \(\mathcal{U}\) and \(\mathcal{V}\) be distinct spaces of sequences related to each other. In this paper, we prove that if the classical control modulo of the oscillatory behavior of \((u_n)\) in \(\mathcal{U}\) is a Tauberian condition for an additive and \((C,1)\) limitable method \(L\), then the general control modulo of the oscillatory behavior of integer order \(m\) of \((u_n)\) in \(\mathcal{U}\) or \(\mathcal{V}\) is also a Tauberian condition for \(L\).
2. Notations and definitions

Throughout this paper, let \( u = (u_n) \) be a sequence of real numbers. For each integer \( m \geq 0 \) and for all nonnegative integers \( n \) denote \( \sigma^{(m)}_n(u) \) by

\[
\sigma^{(m)}_n(u) = \begin{cases} 
\frac{1}{n+1} \sum_{k=0}^{n} \sigma^{(m-1)}_k(u) = u_0 + \sum_{k=1}^{n} \frac{V^{(m-1)}_k(\Delta u)}{k}, & m \geq 1, \\
0, & m = 0,
\end{cases}
\]  

(2.1)

where

\[
V^{(m)}_n(\Delta u) = \begin{cases} 
\sigma^{(1)}(n)(V^{(m-1)}(\Delta u)), & m \geq 1, \\
\frac{1}{n+1} \sum_{k=0}^{n} k \Delta u_k, & m = 0.
\end{cases}
\]  

(2.2)

The identity

\[ u_n - \sigma^{(1)}(u) = V^{(0)}_n(\Delta u) \]  

(2.3)

is well known and will be extensively used. We define inductively for each integer \( m \geq 1 \) and for all nonnegative integers \( n \),

\[(n\Delta)_mu_n = n\Delta((n\Delta)_{m-1}u_n), \quad \text{where } (n\Delta)_0u_n = u_n.\]  

(2.4)

It is proved in [2] that for each integer \( m \geq 1 \),

\[
\omega^{(m)}_n(u) = (n\Delta)_m V^{(m-1)}_n(\Delta u).
\]  

(2.5)

**Definition 2.1.** A sequence \( u = (u_n) \) is Abel limitable to \( s \) if the limit \( \lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n = s \).

**Definition 2.2.** A sequence \( u = (u_n) \) is \( L \) limitable to \( s \) if \( L - \lim_n u_n = s \).

A limitation method \( L \) is called additive if \( L - \lim_n u_n + v_n = s + t \) imply that \( L - \lim_n (u_n + v_n) = s + t \). A limitation method \( L \) is called regular if the \( L - \) limit of every convergent sequence is equal to its limit. \( L \) is called \( (C,1) \) regular if \( L - \lim_n (\sigma^{(1)}(u)) = s \) implies \( L - \lim_n \sigma^{(1)}(u) = s \). It is clear that every regular limitable method is \( (C,1) \) regular.

**Definition 2.3.** A sequence \( u = (u_n) \) is one-sidedly bounded if for some \( C \geq 0 \) and for each nonnegative integer \( n \), \( u_n \geq -C \).

**Definition 2.4.** A sequence \( u = (u_n) \) is slowly oscillating [9] if

\[
\lim_{\lambda \to 1^+} \lim_{n \to \infty} \max_{n+1 \leq k \leq [\lambda n]} \left| \sum_{j=n+1}^{k} \Delta u_j \right| = 0, 
\]  

(2.6)

where \([\lambda n]\) denotes the integer part of \(\lambda n\).
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A sequence \( u = (u_n) \in \mathcal{S} \) if and only if \((V_n^{(0)}(\Delta u)) \in \mathcal{S}\) and \((V_n^{(0)}(\Delta u)) \in \mathcal{B}\) (see [4]).

The next definition is a generalization of slow oscillation.

**Definition 2.5.** A sequence \( u = (u_n) \) is moderately oscillating [9] if for \( \lambda > 1 \),

\[
\lim_{n} \max_{n+1 \leq k \leq [\lambda n]} \left| \sum_{j=n+1}^{k} \Delta u_j \right| < \infty. \tag{2.7}
\]

A sequence \( (u_n) \in \mathcal{M} \) if and only if \((V_n^{(0)}(\Delta u)) \in \mathcal{B}\) (see [4]).

3. Results and proofs

**Theorem 3.1.** If \((\omega_n^{(0)}(u)) \in \mathcal{M}\) is a Tauberian condition for \( L \), then for any integer \( m \geq 1 \),

\( (\omega_n^{(m)}(u)) \in \mathcal{M}\) is also a Tauberian condition for \( L \).

**Proof.** Assume that \((\omega_n^{(0)}(u)) \in \mathcal{M}\) is a Tauberian condition for \( L \). Let \( L - \lim_n u_n = s \). Since \( L \) is \((C, 1)\) regular, it follows by (2.3) that \( L - \lim_n V_n^{(0)}(\Delta u) = 0 \). It is obvious that \( L - \lim_n u_n = s \) implies \( L - \lim_n (n\Delta_{m-1}V_n^{(m-1)}(\Delta u)) = 0 \). Since

\[
(\omega_n^{(m)}(u)) = (n\Delta((n\Delta)_{m-1}V_n^{(m-1)}(\Delta u))) \in \mathcal{M}, \tag{3.1}
\]

by assumption, we have

\[
(n\Delta)_{m-1}V_n^{(m-1)}(\Delta u) = o(1). \tag{3.2}
\]

By the same reasoning, we deduce that

\[
(n\Delta)_{m-1}V_n^{(m-1)}(\Delta u) = n\Delta((n\Delta)_{m-2}V_n^{(m-1)}(\Delta u)) = o(1) \tag{3.3}
\]

and \( L - \lim_n (n\Delta)_{m-2}V_n^{(m-1)}(\Delta u) = 0 \). Again by assumption, we have

\[
(n\Delta)_{m-2}V_n^{(m-1)}(\Delta u) = o(1). \tag{3.4}
\]

From the identity

\[
(n\Delta)_{m-1}V_n^{(m-1)}(\Delta u) = (n\Delta)_{m-2}V_n^{(m-2)}(\Delta u) - (n\Delta)_{m-2}V_n^{(m-1)}(\Delta u), \tag{3.5}
\]

(3.2), and (3.4), we have

\[
(n\Delta)_{m-2}V_n^{(m-2)}(\Delta u) = o(1). \tag{3.6}
\]

Continuing in this vein, we have

\[
n\Delta V_n^{(1)}(\Delta u) = o(1). \tag{3.7}
\]

Since \( L - \lim_n V_n^{(1)}(\Delta u) = 0 \), it follows by (3.7) that

\[
V_n^{(1)}(\Delta u) = o(1). \tag{3.8}
\]
From (3.7) and (3.8), we have $V^{(0)}_n(\Delta u) = o(1)$. Let $L = \lim_n u_n = s$. Since $L = \lim_{m \to \infty} u_n$ (or $m \in \mathbb{N}$) imply that $\lim_{m \to \infty} u_n = s$. Hence, by (2.3), $(u_n)$ converges to $s$. □

Theorem 3.2. If $(\omega^{(0)}_n(u)) \in B$ is a Tauberian condition for $L$, then for any integer $m \geq 1$, $(\omega^{(m)}_n(u)) \in B$ is also a Tauberian condition for $L$.

Proof. Assume that $\omega^{(0)}_n(u) = O(1)$ is a Tauberian condition for $L$. Let $L = \lim_n u_n = s$. Since $L = \lim_{m \to \infty} u_n$ (or $m \in \mathbb{N}$) imply that $\lim_{m \to \infty} u_n = s$, $(u_n)$ converges to $s$. The rest of the proof is as in the proof of Theorem 3.1. □

Theorem 3.3. If for some $C \geq 0$, $\omega^{(0)}_n(u) \geq -C$ is a Tauberian condition for $L$, then for any integer $m \geq 1$, $\omega^{(m)}_n(u) \geq -C$ is also a Tauberian condition for $L$.

Proof. Assume that $\omega^{(0)}_n(u) \geq -C$ for some $C \geq 0$ is a Tauberian condition for $L$. Let $L = \lim_n u_n = s$. Since $L = \lim_{m \to \infty} u_n$ (or $m \in \mathbb{N}$) imply that $\lim_{m \to \infty} u_n = s$, $(u_n)$ converges to $s$. The rest of the proof is as in the proof of Theorem 3.1. □

We now prove that if $(\omega^{(0)}_n(u)) \in M$ (or $\in B$) is a Tauberian condition for $L$, then for any integer $m \geq 1$, $(\omega^{(m)}_n(u)) \in B$ (or $\in M$) is a Tauberian condition for $L$, respectively.

Theorem 3.4. If $(\omega^{(0)}_n(u)) \in M$ is a Tauberian condition for $L$, then for any integer $m \geq 1$, $(\omega^{(m)}_n(u)) \in B$ is also a Tauberian condition for $L$.

Proof. It is sufficient to note that $\omega^{(m)}_n(u) = (n\Delta)_m V^{(m-1)}_n(\Delta u) = V^{(0)}_n(\Delta \omega^{(m-1)}(u)) = O(1)$ implies $\omega^{(m-1)}_n(u) \in M$. Proof now follows from Theorem 3.1. □

Theorem 3.5. If $(\omega^{(0)}_n(u)) \in B$ is a Tauberian condition for $L$, then for any integer $m \geq 1$, $(\omega^{(m)}_n(u)) \in M$ is also a Tauberian condition for $L$.

Proof. It is sufficient to note that $\omega^{(m)}_n(u) \in M$ implies $V^{(0)}_n(\Delta \omega^{(m)}(u)) = \omega^{(m+1)}_n(u) = O(1)$. Proof now follows from Theorem 3.4. □

Remark 3.6. Because of the inclusion $\mathcal{N} \subset \mathcal{F} \subset M$, the condition “belonging to $M$” can be replaced by “belonging to $\mathcal{F}$” or “belonging to $\mathcal{N}$.”

In Theorems 3.1, 3.2, and 3.3, taking $m = 1$ and replacing $M$ by $\mathcal{F}$, we have [1, Theorems 4.1, 4.2, and 4.4] by Çanak et al.

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