INVERSION FORMULAS FOR RIEMANN-LIOUVILLE TRANSFORM AND ITS DUAL ASSOCIATED WITH SINGULAR PARTIAL DIFFERENTIAL OPERATORS

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We define Riemann-Liouville transform $\mathcal{R}_\alpha$ and its dual $^{t}\mathcal{R}_\alpha$ associated with two singular partial differential operators. We establish some results of harmonic analysis for the Fourier transform connected with $\mathcal{R}_\alpha$. Next, we prove inversion formulas for the operators $\mathcal{R}_\alpha$, $^{t}\mathcal{R}_\alpha$ and a Plancherel theorem for $^{t}\mathcal{R}_\alpha$.

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1. Introduction

The mean operator is defined for a continuous function $f$ on $\mathbb{R}^2$, even with respect to the first variable by

$$
\mathcal{R}_0(f)(r,x) = \frac{1}{2\pi} \int_0^{2\pi} f(r \sin \theta, x + r \cos \theta) d\theta,
$$

which means that $\mathcal{R}_0(f)(r,x)$ is the mean value of $f$ on the circle centered at $(0,x)$ and radius $r$. The dual of the mean operator $^{t}\mathcal{R}_0$ is defined by

$$
^{t}\mathcal{R}_0(f)(r,x) = \frac{1}{\pi} \int_\mathbb{R} f\left(\sqrt{r^2 + (x-y)^2}, y\right) dy.
$$

The mean operator $\mathcal{R}_0$ and its dual $^{t}\mathcal{R}_0$ play an important role and have many applications, for example, in image processing of the so-called synthetic aperture radar (SAR) data [11, 12] or in the linearized inverse scattering problem in acoustics [6].

Our purpose in this work is to define and study integral transforms which generalize the operators $\mathcal{R}_0$ and $^{t}\mathcal{R}_0$. More precisely, we consider the following singular partial differential operators:

$$
\Delta_1 = \frac{\partial}{\partial x},
$$

$$
\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{2\alpha + 1}{r} \frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2}, \quad (r,x) \in ]0, +\infty[ \times \mathbb{R}, \alpha \geq 0.
$$
2 Inversion formulas for Riemann-Liouville transform

We associate to $\Delta_1$ and $\Delta_2$ the Riemann-Liouville transform $R_\alpha$, defined on $C_*(\mathbb{R}^2)$ (the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable) by

$$R_\alpha(f)(r,x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^{1} f\left(r\sqrt{1-t^2}, x + rt\right) \times (1-t^2)^{\frac{\alpha}{2}}(1-s^2)^{\alpha-1} \, dt \, ds, & \text{if } \alpha > 0, \\
\frac{1}{\pi} \int_{-1}^{1} f\left(r\sqrt{1-t^2}, x + rt\right) \frac{dt}{\sqrt{1-t^2}}, & \text{if } \alpha = 0. \end{cases} \tag{1.4}$$

The dual operator $t R_\alpha$ is defined on the space $L_*(\mathbb{R}^2)$ (the space of infinitely differentiable functions on $\mathbb{R}^2$, rapidly decreasing together with all their derivatives, even with respect to the first variable) by

$$t R_\alpha(f)(r,x) = \begin{cases} \frac{2\alpha}{\pi} \int_{r}^{+\infty} \int_{-\sqrt{u^2-r^2}}^{\sqrt{u^2-r^2}} f(u,x+v)(u^2-v^2-r^2)^{\alpha-1} \, u \, du \, dv, & \text{if } \alpha > 0, \\
\frac{1}{\pi} \int_{\mathbb{R}} f\left(\sqrt{r^2 + (x-y)^2}, y\right) \, dy, & \text{if } \alpha = 0. \end{cases} \tag{1.5}$$

For more general fractional integrals and fractional differential equations, we can see the works of Debnath [3, 4] and Debnath with Bhatta [5].

We establish for the operators $R_\alpha$ and $t R_\alpha$ the same results given by Helgason, Ludwig, and Solmon for the classical Radon transform on $\mathbb{R}^2$ [10, 14, 17] and we find the results given in [15] for the spherical mean operator. Especially

(i) we give some harmonic analysis results related to the Fourier transform associated with the Riemann-Liouville transform $R_\alpha$;

(ii) we define and characterize some spaces of the functions on which $R_\alpha$ and $t R_\alpha$ are isomorphisms;

(iii) we give the following inversion formulas for $R_\alpha$ and $t R_\alpha$:

$$f = R_\alpha K_1^1 R_\alpha(f), \quad f = R_\alpha K_2^2 R_\alpha(f),$$

$$f = t R_\alpha K_1^1 t R_\alpha(f), \quad f = t R_\alpha K_2^2 t R_\alpha(f), \tag{1.6}$$

where $K_1^1$ and $K_2^2$ are integro-differential operators;

(iv) we establish a Plancherel theorem for $t R_\alpha$;

(v) we show that $R_\alpha$ and $t R_\alpha$ are transmutation operators.

This paper is organized as follows. In Section 2, we show that for $(\mu, \lambda) \in \mathbb{C}^2$, the differential system

$$\Delta_1 u(r,x) = -i\lambda u(r,x),$$

$$\Delta_2 u(r,x) = -\mu^2 u(r,x),$$

$$u(0,0) = 1, \quad \frac{\partial u}{\partial r}(0,x) = 0, \quad \forall x \in \mathbb{R}, \tag{1.7}$$
admits a unique solution \( \varphi_{\mu, \lambda} \) given by

\[
\varphi_{\mu, \lambda}(r, x) = j_\alpha \left( r \sqrt{\mu^2 + \lambda^2} \right) \exp(-i\lambda x),
\]

where \( j_\alpha \) is the modified Bessel function defined by

\[
j_\alpha(s) = 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(s)}{s^\alpha},
\]

and \( J_\alpha \) is the Bessel function of first kind and index \( \alpha \). Next, we prove a Mehler integral representation of \( \varphi_{\mu, \lambda} \) and give some properties of \( R_\alpha \).

In Section 3, we define the Fourier transform \( \mathcal{F}_\alpha \) connected with \( R_\alpha \), and we establish some harmonic analysis results (inversion formula, Plancherel theorem, Paley-Wiener theorem) which lead to new properties of the operator \( R_\alpha \) and its dual \( \mathcal{F}_\alpha \).

In Section 4, we characterize some subspaces of \( \mathcal{S}_+(\mathbb{R}^2) \) on which \( R_\alpha \) and \( \mathcal{F}_\alpha \) are isomorphisms, and we prove the inversion formulas cited below where the operators \( K_\alpha^1 \) and \( K_\alpha^2 \) are given in terms of Fourier transforms. Next, we introduce fractional powers of the Bessel operator,

\[
\ell_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha + 1}{r} \frac{\partial}{\partial r},
\]

and the Laplacian operator,

\[
\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial x^2},
\]

that we use to simplify \( K_\alpha^1 \) and \( K_\alpha^2 \).

Finally, we prove the following Plancherel theorem for \( \mathcal{F}_\alpha \):

\[
\int_\mathbb{R} \int_0^{+\infty} |f(r, x)|^2 r^{2\alpha+1} \, dr \, dx = \int_\mathbb{R} \int_0^{+\infty} \left| K_\alpha^3(\mathcal{F}_\alpha(f))(r, x) \right|^2 \, dr \, dx,
\]

where \( K_\alpha^3 \) is an integro-differential operator.

In Section 5, we show that \( R_\alpha \) and \( \mathcal{F}_\alpha \) satisfy the following relations of permutation:

\[
\mathcal{F}_\alpha(\Delta_2 f) = \frac{\partial^2}{\partial r^2} \mathcal{F}_\alpha(f), \quad \mathcal{F}_\alpha(\Delta_1 f) = \Delta_1 \mathcal{F}_\alpha(f),
\]

\[
\Delta_2 R_\alpha(f) = R_\alpha \left( \frac{\partial^2 f}{\partial r^2} \right), \quad \Delta_1 R_\alpha(f) = R_\alpha(\Delta_1 f).
\]

2. Riemann-Liouville transform and its dual associated with the operators \( \Delta_1 \) and \( \Delta_2 \)

In this section, we define the Riemann-Liouville transform \( R_\alpha \) and its dual \( \mathcal{F}_\alpha \), and we give some properties of these operators. It is well known [21] that for every \( \lambda \in \mathbb{C} \), the system

\[
\ell_\alpha v(r) = -\lambda^2 v(r); \quad v(0) = 1; \quad v'(0) = 0,
\]

\[
(2.1)
\]
where $\ell_\alpha$ is the Bessel operator, admits a unique solution, that is, the modified Bessel function $r \mapsto j_\alpha(r\lambda)$. Thus, for all $(\mu, \lambda) \in \mathbb{C} \times \mathbb{C}$, the system

\[
\Delta_1 u(r, x) = -i\lambda u(r, x), \\
\Delta_2 u(r, x) = -\mu^2 u(r, x),
\]

admits the unique solution given by

\[
\varphi_{\mu, \lambda}(r, x) = j_\alpha(r\sqrt{\mu^2 + \lambda^2}) \exp(-i\lambda x).
\]

(2.3)

The modified Bessel function $j_\alpha$ has the Mehler integral representation, (we refer to [13, 21])

\[
j_\alpha(s) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + 1/2)} \int_{-1}^{1} (1 - t^2)^{\alpha - 1/2} \exp(-ist) dt.
\]

(2.4)

In particular,

\[
\forall k \in \mathbb{N}, \forall s \in \mathbb{R}, \quad |j^{(k)}_\alpha(s)| \leq 1.
\]

(2.5)

On the other hand,

\[
\sup_{r \in \mathbb{R}} |j_\alpha(r\lambda)| = 1 \quad \text{iff } \lambda \in \mathbb{R}.
\]

(2.6)

This involves that

\[
\sup_{(r, x) \in \mathbb{R}^2} |\varphi_{\mu, \lambda}(r, x)| = 1 \quad \text{iff } (\mu, \lambda) \in \Gamma,
\]

(2.7)

where $\Gamma$ is the set defined by

\[
\Gamma = \mathbb{R}^2 \cup \{ (i\mu, \lambda); (\mu, \lambda) \in \mathbb{R}^2, |\mu| \leq |\lambda| \}.
\]

(2.8)

**Proposition 2.1.** The eigenfunction $\varphi_{\mu, \lambda}$ given by (2.3) has the following Mehler integral representation:

\[
\varphi_{\mu, \lambda}(r, x) = \begin{cases}
\frac{\alpha}{\pi} \int_{-1}^{1} \cos (\mu rs \sqrt{1-t^2}) \exp (-i\lambda (x + rt)) (1 - t^2)^{\alpha - 1/2} (1 - s^2)^{\alpha - 1} dt ds, & \text{if } \alpha > 0, \\
\frac{1}{\pi} \int_{-1}^{1} \cos (r \mu \sqrt{1-t^2}) \exp (-i\lambda (x + rt)) \frac{dt}{\sqrt{1-t^2}}, & \text{if } \alpha = 0.
\end{cases}
\]

(2.9)
Proof. From the following expansion of the function $j_\alpha$:

\[ j_\alpha(s) = 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(s)}{s^\alpha} = \Gamma(\alpha + 1) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma(\alpha + k + 1)} \left( \frac{s}{2} \right)^{2k}, \quad (2.10) \]

we deduce that

\[ j_\alpha(r \sqrt{\mu^2 + \lambda^2}) = \Gamma(\alpha + 1) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma(k + \alpha + 1)} \left( \frac{r \mu}{2} \right)^{2k} j_{\alpha+k}(r\lambda), \quad (2.11) \]

and from the equality (2.4), we obtain

\[ j_\alpha(r \sqrt{\mu^2 + \lambda^2}) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \, \Gamma(\alpha + 1/2)} \int_{-1}^{1} j_{\alpha-1/2}(r \mu \sqrt{1 - t^2}) \exp(-i\lambda t)(1 - t^2)^{\alpha-1/2} dt. \quad (2.12) \]

Then, the results follow by using again the relation (2.4) for $\alpha > 0$, and from the fact that

\[ j_{-1/2}(s) = \cos s, \quad \text{for } \alpha = 0. \quad (2.13) \]

Definition 2.2. The Riemann-Liouville transform $\mathcal{R}_\alpha$ associated with the operators $\Delta_1$ and $\Delta_2$ is the mapping defined on $C_* (\mathbb{R}^2)$ by the following. For all $(r, x) \in \mathbb{R}^2$,

\[ \mathcal{R}_\alpha(f)(r, x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^{1} f(r \sqrt{1 - t^2}, x + rt) \times (1 - t^2)^{\alpha-1/2} (1 - s^2)^{\alpha-1} dt ds, & \text{if } \alpha > 0, \\ \frac{1}{\pi} \int_{-1}^{1} f(r \sqrt{1 - t^2}, x + rt) \frac{dt}{\sqrt{1 - t^2}}, & \text{if } \alpha = 0. \end{cases} \quad (2.14) \]

Remark 2.3. (i) From Proposition 2.1 and Definition 2.2, we have

\[ \varphi_{\mu, \lambda}(r, x) = \mathcal{R}_\alpha(\cos(\mu \cdot) \exp(-i\lambda \cdot))(r, x). \quad (2.15) \]

(ii) We can easily see, as in [2], that the transform $\mathcal{R}_\alpha$ is continuous and injective from $C_* (\mathbb{R}^2)$ (the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to the first variable) into itself.
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Lemma 2.4. For \( f \in \mathcal{C}_*(\mathbb{R}^2) \), \( f \) bounded, and \( g \in \mathcal{L}_*(\mathbb{R}^2) \),

\[
\int_{\mathbb{R}} \int_{0}^{+\infty} \mathcal{R}_a(f)(r,x)g(r,x)r^{2\alpha+1}dr \, dx = \int_{\mathbb{R}} \int_{0}^{+\infty} f(r,x)\, ^t\mathcal{R}_a(g)(r,x)dr \, dx,
\]

(2.16)

where \(^t\mathcal{R}_a\) is the dual transform defined by

\[
^t\mathcal{R}_a(g)(r,x) = \begin{cases}
\frac{2\alpha}{\pi} \int_{r}^{+\infty} \int_{\sqrt{u^2-r^2}}^{\sqrt{u^2-r^2}} g(u, x + v)(u^2 - v^2 - r^2)^{\alpha-1} u \, du \, dv, & \text{if } \alpha > 0, \\
\frac{1}{\pi} \int_{\mathbb{R}} g\left(\sqrt{r^2 + (x - y)^2}, y\right) \, dy, & \text{if } \alpha = 0.
\end{cases}
\]

(2.17)

To obtain this lemma, we use Fubini’s theorem and an adequate change of variables.

Remark 2.5. By a simple change of variables, we have

\[
\mathcal{R}_0(f)(r,x) = \frac{1}{2\pi} \int_{0}^{2\pi} f(r \sin \theta, x + r \cos \theta) d\theta.
\]

(2.18)

3. Fourier transform associated with Riemann-Liouville operator

In this section, we define the Fourier transform associated with the operator \( \mathcal{R}_a \), and we give some results of harmonic analysis that we use in the next sections.

We denote by

(i) \( dv(r,x) \) the measure defined on \([0, +\infty[ \times \mathbb{R}\) by

\[
dv(r,x) = \frac{1}{\sqrt{2\pi} 2^a \Gamma(a+1)} r^{2\alpha+1} \, dr \otimes dx,
\]

(3.1)

(ii) \( L^1(dv) \) the space of measurable functions \( f \) on \([0, +\infty[ \times \mathbb{R}\) satisfying

\[
\|f\|_{1,v} = \int_{\mathbb{R}} \int_{0}^{+\infty} |f(r,x)| \, dv(r,x) < +\infty.
\]

(3.2)

Definition 3.1. (i) The translation operator associated with Riemann-Liouville transform is defined on \( L^1(dv) \) by the following. For all \((r,x), (s,y) \in [0, +\infty[ \times \mathbb{R}\),

\[
\mathcal{T}_{(r,x)}f(s,y) = \frac{\Gamma(a+1)}{\sqrt{\pi} \Gamma(a+1/2)} \int_{0}^{\pi} f\left(\sqrt{r^2 + s^2 + 2rs \cos \theta}, x + y\right) \sin^{2\alpha} \theta d\theta.
\]

(3.3)

(ii) The convolution product associated with the Riemann-Liouville transform of \( f \), \( g \in L^1(dv) \) is defined by the following. For all \((r,x) \in [0, +\infty[ \times \mathbb{R}\),

\[
f \# g(r,x) = \int_{\mathbb{R}} \int_{0}^{+\infty} \mathcal{T}_{(r,-x)} \hat{f}(s,y) g(s,y) dv(s,y),
\]

(3.4)

where \( \hat{f}(s,y) = f(s,-y) \).
We have the following properties.

(i) Since

$$\forall r, s \geq 0, \quad j_\alpha(r\lambda) \bar{j}_\alpha(s\lambda) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 1/2)} \int_0^\infty j_\alpha(\lambda \sqrt{r^2 + s^2 + 2rs \cos \theta}) \sin^2 \alpha \theta \, d\theta,$$

(3.5)

(we refer to [21]) we deduce that the eigenfunction $\varphi_{\mu, \lambda}$ defined by the relation (2.3) satisfies the product formula

$$\mathcal{T}_{(r, x)} \varphi_{\mu, \lambda}(s, y) = \varphi_{\mu, \lambda}(r, x) \varphi_{\mu, \lambda}(s, y).$$

(3.6)

(ii) If $f \in L^1(d\nu)$, then for all $(r, x) \in [0, +\infty[ \times \mathbb{R}$, $\mathcal{T}_{(r, x)} f$ belongs to $L^1(d\nu)$, and we have

$$\| \mathcal{T}_{(r, x)} f \|_{1, \nu} \leq \| f \|_{1, \nu}.$$

(3.7)

(iii) For $f, g \in L^1(d\nu)$, $f \# g$ belongs to $L^1(d\nu)$, and the convolution product is commutative and associative.

(iv) For $f, g \in L^1(d\nu)$,

$$\| f \# g \|_{1, \nu} \leq \| f \|_{1, \nu} \| g \|_{1, \nu}.$$

(3.8)

**Definition 3.2.** The Fourier transform associated with the Riemann-Liouville operator is defined by

$$\forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}_\alpha(f)(\mu, \lambda) = \int_\mathbb{R} \int_0^{+\infty} f(r, x) \varphi_{\mu, \lambda}(r, x) d\nu(r, x),$$

(3.9)

where $\Gamma$ is the set defined by the relation (2.8).

We have the following properties.

(i) Let $f$ be in $L^1(d\nu)$. For all $(r, x) \in [0, +\infty[ \times \mathbb{R}$, we have

$$\forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}_\alpha(\mathcal{T}_{(r, x)} f)(\mu, \lambda) = \varphi_{\mu, \lambda}(r, x) \mathcal{F}_\alpha(f)(\mu, \lambda).$$

(3.10)

(ii) For $f, g \in L^1(d\nu)$, we have

$$\forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}_\alpha(f \# g)(\mu, \lambda) = \mathcal{F}_\alpha(f)(\mu, \lambda) \mathcal{F}_\alpha(g)(\mu, \lambda).$$

(3.11)

(iii) For $f \in L^1(d\nu)$, we have

$$\forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}_\alpha(f)(\mu, \lambda) = B \circ \mathcal{F}_\alpha(f)(\mu, \lambda),$$

(3.12)

where

$$\forall (\mu, \lambda) \in \mathbb{R}^2, \quad \mathcal{F}_\alpha(f)(\mu, \lambda) = \int_\mathbb{R} \int_0^{+\infty} f(r, x) j_\alpha(r\mu) \exp(-i\lambda x) d\nu(r, x),$$

$$\forall (\mu, \lambda) \in \Gamma, \quad B f(\mu, \lambda) = f(\sqrt{\mu^2 + \lambda^2}, \lambda).$$

(3.13)
3.1. Inversion formula and Plancherel theorem for $\mathcal{F}_a$. We denote by (see [15])

(i) $\mathcal{S}_f(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$ rapidly decreasing together with all their derivatives, even with respect to the first variable;

(ii) $\mathcal{S}_f(\Gamma)$ the space of functions $f: \Gamma \to \mathbb{C}$ infinitely differentiable, even with respect to the first variable and rapidly decreasing together with all their derivatives, that is, for all $k_1, k_2, k_3 \in \mathbb{N}$,

$$
\sup_{(\mu, \lambda) \in \Gamma} (1 + |\mu|^2 + |\lambda|^2)^{k_1} \left| \left( \frac{\partial}{\partial \mu} \right)^{k_2} \left( \frac{\partial}{\partial \lambda} \right)^{k_3} f(\mu, \lambda) \right| < +\infty, \quad (3.14)
$$

where

$$
\frac{\partial f}{\partial \mu}(\mu, \lambda) = \begin{cases} 
\frac{\partial}{\partial r}(f(r, \lambda)), & \text{if } \mu = r \in \mathbb{R}, \\
\frac{1}{i} \frac{\partial}{\partial t}(f(it, \lambda)), & \text{if } \mu = it, |t| \leq |\lambda|.
\end{cases} \quad (3.15)
$$

Each of these spaces is equipped with its usual topology:

(i) $L^2(d\nu)$ the space of measurable functions on $[0, +\infty[ \times \mathbb{R}$ such that

$$
\|f\|_{2, \nu} = \left( \int \int_0^{+\infty} |f(r, x)|^2 d\nu(r, x) \right)^{1/2} < +\infty; \quad (3.16)
$$

(ii) $d\gamma(\mu, \lambda)$ the measure defined on $\Gamma$ by

$$
\int_\Gamma f(\mu, \lambda) d\gamma(\mu, \lambda) = \frac{1}{\sqrt{2\pi}^{2\alpha}} \Gamma(\alpha + 1) \left\{ \int \int_0^{+\infty} f(\mu, \lambda)(\mu^2 + \lambda^2)^{\alpha} \mu d\mu d\lambda + \int_\mathbb{R} \int_0^{+\lambda} f(i\mu, \lambda)(\lambda^2 - \mu^2)^{\alpha} \mu d\mu d\lambda \right\}; \quad (3.17)
$$

(iii) $L^p(d\gamma)$, $p = 1, p = 2$, the space of measurable functions on $\Gamma$ satisfying

$$
\|f\|_{p, \gamma} = \left( \int_\Gamma |f(\mu, \lambda)|^p d\gamma(\mu, \lambda) \right)^{1/p} < +\infty. \quad (3.18)
$$

Remark 3.3. It is clear that a function $f$ belongs to $L^1(d\nu)$ if, and only if, the function $Bf$ belongs to $L^1(d\gamma)$, and we have

$$
\int_\Gamma Bf(\mu, \lambda) d\gamma(\mu, \lambda) = \int_\mathbb{R} \int_0^{+\lambda} f(r, x) d\nu(r, x). \quad (3.19)
$$

Proposition 3.4 (inversion formula for $\mathcal{F}_a$). Let $f \in L^1(d\nu)$ such that $\mathcal{F}_a(f)$ belongs to $L^1(d\gamma)$, then for almost every $(r, x) \in [0, +\infty[ \times \mathbb{R}$,

$$
f(r, x) = \int_\Gamma \mathcal{F}_a(f)(\mu, \lambda) \mathcal{F}_{\mu, \lambda}(r, x) d\gamma(\mu, \lambda). \quad (3.20)
$$
Proof. From \([9, 19]\), one can see that if \(f \in L^1(d\nu)\) is such that \(\tilde{\mathcal{F}}_\alpha(f) \in L^1(d\nu)\), then for almost every \((r, x) \in [0, +\infty) \times \mathbb{R}\),

\[
f(r, x) = \int_\mathbb{R} \int_0^{+\infty} \tilde{\mathcal{F}}_\alpha(f)(\mu, \lambda) j_a(r\mu) \exp(i\lambda x) d\nu(\mu, \lambda).
\]  

(3.21)

Then, the result follows from the relation (3.12) and Remark 3.3. \(\square\)

Theorem 3.5. (i) The Fourier transform \(\mathcal{F}_\alpha\) is an isomorphism from \(\mathcal{S}_*(\mathbb{R}^2)\) onto \(\mathcal{S}_*(\Gamma)\).

(ii) (Plancherel formula) for \(f \in \mathcal{S}_*(\mathbb{R}^2)\),

\[
\|\mathcal{F}_\alpha(f)\|_{2, \nu} = \|f\|_{2, \nu}.
\]  

(3.22)

(iii) (Plancherel theorem) the transform \(\mathcal{F}_\alpha\) can be extended to an isometric isomorphism from \(L^2(d\nu)\) onto \(L^2(d\gamma)\).

Proof. This theorem follows from the relation (3.12), Remark 3.3, and the fact that \(\tilde{\mathcal{F}}_\alpha\) is an isomorphism from \(\mathcal{S}_*(\mathbb{R}^2)\) onto itself, satisfying that for all \(f \in \mathcal{S}_*(\mathbb{R}^2)\),

\[
\|\tilde{\mathcal{F}}_\alpha(f)\|_{2, \nu} = \|f\|_{2, \nu}.
\]  

(3.23) \(\square\)

Lemma 3.6. For \(f \in \mathcal{S}_*(\mathbb{R}^2)\),

\[
\forall (\mu, \lambda) \in \mathbb{R}^2, \quad \mathcal{F}_\alpha(f)(\mu, \lambda) = \Lambda_\alpha \circ \mathcal{R}_\alpha(f)(\mu, \lambda),
\]  

(3.24)

where \(\mathcal{R}_\alpha\) is the dual transform of the Riemann-Liouville operator, and \(\Lambda_\alpha\) is a constant multiple of the classical Fourier transform on \(\mathbb{R}^2\) defined by

\[
\Lambda_\alpha(f)(\mu, \lambda) = \int_\mathbb{R} \int_0^{+\infty} f(r, x) \cos(r\mu) \exp(-i\lambda x) dm(r, x),
\]  

(3.25)

where \(dm(r, x)\) is the measure defined on \([0, +\infty) \times \mathbb{R}\) by

\[
dm(r, x) = \frac{1}{\sqrt{2\pi} 2^\alpha \Gamma(\alpha + 1)} dr \otimes dx.
\]  

(3.26)

This lemma follows from the relation (2.15) and Lemma 2.4.

Using the relation (3.12) and the fact that the mapping \(B\) is continuous from \(\mathcal{S}_*(\mathbb{R}^2)\) into itself, we deduce that the Fourier transform \(\mathcal{F}_\alpha\) is continuous from \(\mathcal{S}_*(\mathbb{R}^2)\) into itself. On the other hand, \(\Lambda_\alpha\) is an isomorphism from \(\mathcal{S}_*(\mathbb{R}^2)\) onto itself. Then, Lemma 3.6 implies that the dual transform \(\mathcal{R}_\alpha\) maps continuously \(\mathcal{S}_*(\mathbb{R}^2)\) into itself.

Proposition 3.7. (i) \(\mathcal{R}_\alpha\) is not injective when applied to \(\mathcal{S}_*(\mathbb{R}^2)\).

(ii) \(\mathcal{R}_\alpha(\mathcal{S}_*(\mathbb{R}^2)) = \mathcal{S}_*(\mathbb{R}^2)\).
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Proof. (i) Let \( g \in \mathcal{F}(\mathbb{R}^2) \) such that \( \text{supp} \, g \subset \{(r, x) \in \mathbb{R}^2, \, |r| \leq |x|\} \), \( g \neq 0 \).

Since \( \hat{x}_\alpha \) is an isomorphism from \( \mathcal{F}(\mathbb{R}^2) \) onto itself, there exists \( f \in \mathcal{F}(\mathbb{R}^2) \) such that \( \hat{x}_\alpha(f) = g \). From the relation (3.12) and Lemma 3.6, we deduce that \( \hat{x}_\alpha(f) = 0 \).

(ii) We obtain the result by the same way as in [1]. □

3.2. Paley-Wiener theorem. We denote by

(i) \( \mathcal{H}_\alpha(\mathbb{R}^2) \) the space of infinitely differentiable functions on \( \mathbb{R}^2 \), even with respect to the first variable, and with compact support;

(ii) \( \mathcal{H}_\alpha(\mathbb{C}^2) \) the space of entire functions \( f : \mathbb{C}^2 \to \mathbb{C} \), even with respect to the first variable rapidly decreasing of exponential type, that is, there exists a positive constant \( M \), such that for all \( k \in \mathbb{N} \),

\[
\sup_{(\mu, \lambda) \in \mathbb{C}^2} (1 + |\mu|^2 + |\lambda|^2)^{k} |f(\mu, \lambda)| \exp(-M(|\text{Im}\mu| + |\text{Im}\lambda|)) < +\infty; \tag{3.27}
\]

(iii) \( \mathcal{H}_{\alpha,0}(\mathbb{C}^2) \) the subspace of \( \mathcal{H}_\alpha(\mathbb{C}^2) \), consisting of functions \( f : \mathbb{C}^2 \to \mathbb{C} \), such that for all \( k \in \mathbb{N} \),

\[
\sup_{(\mu, \lambda) \in \mathbb{R}^2} (1 - \mu^2 + 2\lambda^2)^{k} |f(i\mu, \lambda)| < +\infty; \tag{3.28}
\]

(iv) \( \mathcal{E}_\alpha(\mathbb{R}^2) \) the space of distributions on \( \mathbb{R}^2 \), even with respect to the first variable, and with compact support;

(v) \( \mathcal{H}_\alpha(\mathbb{C}^2) \) the space of entire functions \( f : \mathbb{C}^2 \to \mathbb{C} \), even with respect to the first variable, slowly increasing of exponential type, that is, there exist a positive constant \( M \) and an integer \( k \), such that

\[
\sup_{(\mu, \lambda) \in \mathbb{C}^2} (1 + |\mu|^2 + |\lambda|^2)^{-k} |f(\mu, \lambda)| \exp(-M(|\text{Im}\mu| + |\text{Im}\lambda|)) < +\infty; \tag{3.29}
\]

(vi) \( \mathcal{H}_{\alpha,0}(\mathbb{C}^2) \) the subspace of \( \mathcal{H}_\alpha(\mathbb{C}^2) \), consisting of functions \( f : \mathbb{C}^2 \to \mathbb{C} \), such that there exists an integer \( k \), satisfying

\[
\sup_{(\mu, \lambda) \in \mathbb{R}^2} (1 - \mu^2 + 2\lambda^2)^{-k} |f(i\mu, \lambda)| < +\infty. \tag{3.30}
\]

Each of these spaces is equipped with its usual topology.

Definition 3.8. The Fourier transform associated with the Riemann-Liouville operator is defined on \( \mathcal{E}_\alpha(\mathbb{R}^2) \) by

\[
\forall (\mu, \lambda) \in \mathbb{C}^2, \quad \hat{x}_\alpha(T)(\mu, \lambda) = \langle T, \varphi_{\mu, \lambda} \rangle. \tag{3.31}
\]
Proposition 3.9. For every $T \in \mathcal{E}'_a(\mathbb{R}^2)$,

$$\forall (\mu, \lambda) \in \mathbb{C}^2, \quad \tilde{\mathcal{F}}(T)(\mu, \lambda) = B \circ \tilde{\mathcal{F}}(T)(\mu, \lambda),$$  \hspace{1cm} (3.32)

where

$$\forall (\mu, \lambda) \in \mathbb{C}^2, \quad \tilde{\mathcal{F}}(T)(\mu, \lambda) = \langle T, j_\alpha(\mu) \exp(-i\lambda) \rangle,$$  \hspace{1cm} (3.33)

and $B$ is the transform defined by the relation (3.12).

Using [7, Lemma 2] (see also [15]) and the fact that $\tilde{\mathcal{F}}_\alpha$ is an isomorphism from $\mathcal{D}_*(\mathbb{R}^2)$ (resp., $\mathcal{E}'_*(\mathbb{R}^2)$) onto $\mathcal{H}_*(\mathbb{C}^2)$ (resp., $\mathcal{H}_*(\mathbb{C}^2)$), we deduce the following theorem.

Theorem 3.10 (of Paley-Wiener). The Fourier transform $\tilde{\mathcal{F}}_\alpha$ is an isomorphism

(i) from $\mathcal{D}_*(\mathbb{R}^2)$ onto $\mathcal{H}_*,0(\mathbb{C}^2)$;

(ii) from $\mathcal{E}'_*(\mathbb{R}^2)$ onto $\mathcal{H}_*,0(\mathbb{C}^2)$.

From Lemma 3.6, Theorem 3.10, and the fact that $\Lambda_\alpha$ is an isomorphism from $\mathcal{D}_*(\mathbb{R}^2)$ onto $\mathcal{H}_*(\mathbb{C}^2)$, we have the following corollary.

Corollary 3.11. (i) $^t\mathcal{R}_\alpha$ maps injectively $\mathcal{D}_*(\mathbb{R}^2)$ into itself.

(ii) $^t\mathcal{R}_\alpha(\mathcal{D}_*(\mathbb{R}^2)) \neq \mathcal{D}_*(\mathbb{R}^2)$.

4. Inversion formulas for $\mathcal{R}_\alpha$ and $^t\mathcal{R}_\alpha$ and Plancherel theorem for $^t\mathcal{R}_\alpha$

In this section, we will define some subspaces of $\mathcal{S}_*(\mathbb{R}^2)$ on which $\mathcal{R}_\alpha$ and $^t\mathcal{R}_\alpha$ are isomorphisms, and we give their inverse transforms in terms of integro-differential operators. Next, we establish Plancherel theorem for $^t\mathcal{R}_\alpha$.

We denote by

(i) $\mathcal{N}$ the subspace of $\mathcal{S}_*(\mathbb{R}^2)$, consisting of functions $f$ satisfying

$$\forall k \in \mathbb{N}, \forall x \in \mathbb{R}, \quad \left( \frac{\partial}{\partial x^2} \right)^k f(0,x) = 0,$$  \hspace{1cm} (4.1)

where

$$\frac{\partial}{\partial r^2} = \frac{1}{r} \frac{\partial}{\partial r};$$  \hspace{1cm} (4.2)

(ii) $\mathcal{S}_{*,0}(\mathbb{R}^2)$ the subspace of $\mathcal{S}_*(\mathbb{R}^2)$, consisting of functions $f$, such that

$$\forall k \in \mathbb{N}, \forall x \in \mathbb{R}, \quad \int_0^{+\infty} f(r,x) r^{2k} dr = 0;$$  \hspace{1cm} (4.3)

(iii) $\mathcal{S}^0_*(\mathbb{R}^2)$ the subspace of $\mathcal{S}_*(\mathbb{R}^2)$, consisting of functions $f$, such that

$$\text{supp} \tilde{\mathcal{F}}_\alpha(f) \subset \{ (\mu, \lambda) \in \mathbb{R}^2; |\mu| \geq |\lambda| \}. $$  \hspace{1cm} (4.4)
Lemma 4.1. (i) The mapping $\Lambda_\alpha$ is an isomorphism from $\mathcal{F}_{*,0}(\mathbb{R}^2)$ onto $\mathcal{N}$.
(ii) The subspace $\mathcal{N}$ can be written as

$$
\mathcal{N} = \left\{ f \in \mathcal{F}_*(\mathbb{R}^2) ; \quad \forall k \in \mathbb{N}, \forall x \in \mathbb{R} ; \quad \left( \frac{\partial}{\partial r} \right)^{2k} f(0,x) = 0 \right\}.
$$

(4.5)

Proof. Let $f \in \mathcal{F}_{*,0}(\mathbb{R}^2)$.
(i) For $\nu > -1$, we have

$$
\left( \frac{\partial}{\partial \mu^2} \right)^k (j_\nu(r\mu)) = \frac{\Gamma(\nu + 1)}{2^k \Gamma(\nu + k + 1)} (-r^2)^k j_{\nu+k}(r\mu),
$$

thus, from the expression of $\Lambda_\alpha$, given in Lemma 3.6, and the fact that $j_{-1/2}(s) = \cos s$, we obtain

$$
\left( \frac{\partial}{\partial \mu^2} \right)^k (\Lambda_\alpha(f))(0,\lambda) = \frac{\sqrt{\pi}}{2^k \Gamma(k+1/2)} (-1)^k \int_0^{+\infty} f(r,x) r^{2k} \exp(-i\lambda x) dm(r,x),
$$

(4.7)

which gives the result.
(ii) The proof of (ii) is immediate.

Theorem 4.2. (i) For all real numbers $\gamma$, the mappings

- (i) $f \mapsto (r^2 + x^2)^\gamma f$
- (ii) $f \mapsto |r|^\gamma f$

are isomorphisms from $\mathcal{N}$ onto itself.
(ii) For $f \in \mathcal{N}$, the function $g$ defined by

$$
g(r,x) = \begin{cases} 
  f \left( \sqrt{r^2 - x^2}, x \right) & \text{if } |r| \geq |x|, \\
  0 & \text{otherwise},
\end{cases}
$$

(4.8)

belongs to $\mathcal{F}_*(\mathbb{R}^2)$.

Proof. (i) Let $f \in \mathcal{N}$, by Leibnitz formula, we have

$$
\left( \frac{\partial}{\partial r} \right)^{k_1} \left( \frac{\partial}{\partial x} \right)^{k_2} [(r^2 + x^2)^\gamma f](r,x)
$$

$$
= \sum_{j=0}^{k_1} \sum_{i=0}^{k_2} C_{k_1}^j C_{k_2}^i P_j(r) P_i(x) (r^2 + x^2)^{\gamma - i - j} \frac{\partial^{k_1+k_2-i-j}}{\partial r^{k_1-j} \partial x^{k_2-i-j}} f(r,x),
$$

(4.9)

where $P_i$ and $P_j$ are real polynomials.
Let $n \in \mathbb{N}$ such that $\gamma - k_1 - k_2 + n > 0$. By Taylor formula and the fact that $f \in \mathcal{N}$, we have

$$\left( \frac{\partial}{\partial r} \right)^{k_1-j} \left( \frac{\partial}{\partial x} \right)^i (f)(r,x) = \frac{r^{2n}}{(2n-1)!} \int_0^1 (1-t)^{2n-1} \left( \frac{\partial}{\partial r} \right)^{k_1-j+2n} (f)(rt,x) dt$$

$$= -\frac{r^{2n}}{(2n-1)!} \int_1^{+\infty} (1-t)^{2n-1} \left( \frac{\partial}{\partial r} \right)^{k_1-j+2n} (f)(rt,x) dt, \quad (4.10)$$

$$\left( \frac{\partial}{\partial x} \right)^{k_2-i} \left( \frac{\partial}{\partial r} \right)^{k_1-j} f(r,x) = -\frac{r^{2n}}{(2n-1)!} \int_1^{+\infty} (1-t)^{2n-1} \left( \frac{\partial}{\partial x} \right)^{k_2-i} \left( \frac{\partial}{\partial r} \right)^{k_1-j+2n} f(rt,x) dt$$

$$= -\frac{r^{2n}}{(2n-1)!} \int_1^{+\infty} (1-t)^{2n-1} \left( \frac{\partial}{\partial x} \right)^{k_2-i} \left( \frac{\partial}{\partial r} \right)^{k_1-j+2n} f(rt,x) dt. \quad (4.11)$$

The relations (4.9) and (4.11) imply that the function

$$(r,x) \mapsto (r^2 + x^2)^{\gamma} f(r,x) \quad (4.12)$$

belongs to $\mathcal{N}$ and that the mapping

$$f \mapsto (r^2 + x^2)^{\gamma} f \quad (4.13)$$

is continuous from $\mathcal{N}$ onto itself. The inverse mapping is given by

$$f \mapsto (r^2 + x^2)^{-\gamma} f. \quad (4.14)$$

By the same way, we show that the mapping

$$f \mapsto |r|^\gamma f \quad (4.15)$$

is an isomorphism from $\mathcal{N}$ onto itself.

(ii) Let $f \in \mathcal{N}$, and

$$g(r,x) = \begin{cases} f(\sqrt{r^2 - x^2},x) & \text{if } |r| \geq |x|, \\ 0 & \text{if } |r| \leq |x|, \end{cases} \quad (4.16)$$

we have

$$\left( \frac{\partial}{\partial x} \right)^{k_2} \left( \frac{\partial}{\partial r} \right)^{k_1} (g)(r,x) = \sum_{j=0}^{k_1} P_j(r) \left( \sum_{p,q=0}^{k_2} Q_{p,q}(x) \left( \frac{\partial}{\partial x} \right)^p \left( \frac{\partial}{\partial r} \right)^{q+j} (f)(\sqrt{r^2 - x^2},x) \right), \quad (4.17)$$

where $P_j$ and $Q_{p,q}$ are real polynomials. This equality, together with the fact that $f$ belongs to $\mathcal{N}$, implies that $g$ belongs to $\mathcal{S}_*(\mathbb{R}^2)$. □
Theorem 4.3. The Fourier transform $\mathcal{F}_\alpha$ associated with Riemann-Liouville transform is an isomorphism from $\mathcal{F}_*^0(\mathbb{R}^2)$ onto $\mathcal{N}$.

Proof. Let $f \in \mathcal{F}_*^0(\mathbb{R}^2)$. From the relation (3.12), we get
\[
\left(\frac{\partial}{\partial \mu^2}\right)^k \tilde{\mathcal{F}}_\alpha(f)(0,\lambda) = \left(\frac{\partial}{\partial \mu^2}\right)^k (B \circ \tilde{\mathcal{F}}_\alpha(f))(0,\lambda) = 0,
\]
because $\text{supp} \tilde{\mathcal{F}}_\alpha(f) \subset \{(\mu,\lambda) \in \mathbb{R}^2, |\mu| \geq |\lambda|\}$, this shows that $\mathcal{F}_\alpha$ maps injectively $\mathcal{F}_*^0(\mathbb{R}^2)$ into $\mathcal{N}$. On the other hand, let $h \in \mathcal{N}$ and
\[
g(r,x) = \begin{cases} h(\sqrt{r^2 - x^2},x) & \text{if } |r| \geq |x|, \\ 0 & \text{if } |r| \leq |x|. \end{cases}
\]
From Theorem 4.2(ii), $g$ belongs to $\mathcal{F}_*^0(\mathbb{R}^2)$, so there exists $f \in \mathcal{F}_*^0(\mathbb{R}^2)$ satisfying $\tilde{\mathcal{F}}_\alpha(f) = h$. Consequently, $f \in \mathcal{F}_*^0(\mathbb{R}^2)$ and $\mathcal{F}_\alpha(f) = h$. □

From Lemmas 3.6, 4.1, and Theorem 4.3, we deduce the following result.

Corollary 4.4. The dual transform $\mathcal{I} \mathcal{R}_\alpha$ is an isomorphism from $\mathcal{F}_*^0(\mathbb{R}^2)$ onto $\mathcal{F}_{*,0}^0(\mathbb{R}^2)$.

4.1. Inversion formula for $\mathcal{R}_\alpha$ and $\mathcal{I} \mathcal{R}_\alpha$

Theorem 4.5. (i) The operator $K^1_\alpha$ defined by
\[
K^1_\alpha(f)(r,x) = \Lambda_\alpha^{-1} \left(\frac{\pi}{2^{2\alpha+1} \Gamma^2(\alpha + 1)} (\mu^2 + \lambda^2)^\alpha |\mu| \Lambda_\alpha(f)\right)(r,x)
\]
is an isomorphism from $\mathcal{F}_{*,0}^0(\mathbb{R}^2)$ onto itself.

(ii) The operator $K^2_\alpha$ defined by
\[
K^2_\alpha(g)(r,x) = \tilde{\mathcal{F}}_\alpha^{-1} \left(\frac{\pi}{2^{2\alpha+1} \Gamma^2(\alpha + 1)} (\mu^2 + \lambda^2)^\alpha |\mu| \tilde{\mathcal{F}}_\alpha(g)\right)(r,x)
\]
is an isomorphism from $\mathcal{F}_*^0(\mathbb{R}^2)$ onto itself.

This theorem follows from Lemma 4.1, Theorems 4.2 and 4.3.

Theorem 4.6. (i) For $f \in \mathcal{F}_{*,0}^0(\mathbb{R}^2)$ and $g \in \mathcal{F}_*^0(\mathbb{R}^2)$, there exists the inversion formula for $\mathcal{R}_\alpha$:
\[
g = \mathcal{R}_\alpha K^1_\alpha \mathcal{I} \mathcal{R}_\alpha(g), \quad f = K^1_\alpha \mathcal{I} \mathcal{R}_\alpha \mathcal{R}_\alpha(f).
\]

(ii) For $f \in \mathcal{F}_{*,0}^0(\mathbb{R}^2)$ and $g \in \mathcal{F}_*^0(\mathbb{R}^2)$, there exists the inversion formula for $\mathcal{I} \mathcal{R}_\alpha$:
\[
f = \mathcal{I} \mathcal{R}_\alpha K^2_\alpha \mathcal{R}_\alpha(f), \quad g = K^2_\alpha \mathcal{R}_\alpha \mathcal{I} \mathcal{R}_\alpha(g).
\]
Proof. (i) Let \( g \in \mathcal{F}_a^0(\mathbb{R}^2) \). From the relation (2.15), Proposition 3.4, Lemma 3.6, and Theorem 4.3, we have
\[
\begin{align*}
g(r,x) &= \int_{\mathbb{R}} \int_0^{+\infty} (\mu^2 + \lambda^2) \alpha \mu \Lambda_a (g)(\mu,\lambda) \mathcal{R}_a (\cos(\mu \cdot) \exp(i \lambda \cdot))(r,x) \, d\mu \, d\lambda \\
&= \mathcal{R}_a \left( \int_{\mathbb{R}} \int_0^{+\infty} (\mu^2 + \lambda^2) \alpha \mu \Lambda_a (g)(\mu,\lambda) \mathcal{R}_a (\cos(\mu \cdot) \exp(i \lambda \cdot))(r,x) \right) \\
&= \mathcal{R}_a \left( \Lambda_a^{-1} \left( \frac{\pi}{2^{2\alpha+1} \Gamma^2(\alpha+1)} (\mu^2 + \lambda^2)^{\alpha} \mu |\Lambda_a (g)| \right) \right) (r,x) \\
&= \mathcal{R}_a K^1_{\alpha} \mathcal{R}_a (g)(r,x).
\end{align*}
\]
(4.24)

This relation, together with Corollary 4.4 and Theorem 4.5(i), implies that \( \mathcal{R}_a \) is an isomorphism from \( \mathcal{F}_a^0(\mathbb{R}^2) \) onto \( \mathcal{F}_a^0(\mathbb{R}^2) \), and that \( K^1_{\alpha} \mathcal{R}_a \) is its inverse; in particular for \( f \in \mathcal{F}_a^0(\mathbb{R}^2) \), we have
\[
K^1_{\alpha} \mathcal{R}_a \mathcal{R}_a (f) = f.
\]
(4.25)

(ii) Let \( f \in \mathcal{F}_a^0(\mathbb{R}^2) \). From (i), we have
\[
K^1_{\alpha} \mathcal{R}_a \mathcal{R}_a (f) = f.
\]
(4.26)

Let us put \( g = \mathcal{R}_a (f) \), then \( g \in \mathcal{F}_a^0(\mathbb{R}^2) \), and we have
\[
\mathcal{R}_a^{-1} (g) = K^1_{\alpha} \mathcal{R}_a (g),
\]
(4.27)

and from Lemma 3.6, it follows that
\[
\begin{align*}
\mathcal{R}_a^{-1} (g) &= \Lambda_a^{-1} \left( \frac{\pi}{2^{2\alpha+1} \Gamma^2(\alpha+1)} (\mu^2 + \lambda^2)^{\alpha} \mu |\mathcal{R}_a (g)| \right), \\
\mathcal{R}_a^{-1} (g) &= \mathcal{R}_a^{-1} \left( \frac{\pi}{2^{2\alpha+1} \Gamma^2(\alpha+1)} (\mu^2 + \lambda^2)^{\alpha} \mu |\mathcal{R}_a (g)| \right) = K^2_{\alpha} (g),
\end{align*}
\]
(4.28)

which gives
\[
f = K^2_{\alpha} K^1_{\alpha} \mathcal{R}_a (f). \tag{4.29}
\]
\[\square\]

4.2. The expressions of the operators \( K^1_{\alpha} \) and \( K^2_{\alpha} \). In the previous subsection, we have defined the operators \( K^1_{\alpha} \) and \( K^2_{\alpha} \) in terms of Fourier transforms \( \Lambda_a \) and \( \mathcal{F}_a \). Here, we will give nice expressions of these operators using fractional powers of partial differential operators. For this, we need the following inevitable notations.

(i) \( \mathcal{E}_a(\mathbb{R}) \) is the space of even infinitely differentiable functions on \( \mathbb{R} \).

(ii) \( \mathcal{F}_a(\mathbb{R}) \) is the subspace of \( \mathcal{E}_a(\mathbb{R}) \), consisting of functions rapidly decreasing together with all their derivatives.

(iii) \( \mathcal{F}_a^0(\mathbb{R}) \) is the space of even tempered distributions on \( \mathbb{R} \).
(iv) $\mathcal{S}_a'(\mathbb{R}^2)$ is the space of tempered distributions on $\mathbb{R}^2$, even with respect to the first variable.

Each of these spaces is equipped with its usual topology.

(i) For $a \in \mathbb{R}$, $a \geq -1/2$, $d\omega_a(r)$ is the measure defined on $[0, +\infty[\text{ by}$

$$
d\omega_a(r) = \frac{1}{2^{a} \Gamma(a+1)} r^{2a+1} dr. \quad (4.30)
$$

(ii) $\ell_a$ is the Bessel operator defined on $]0, +\infty[$\text{ by}$

$$
\ell_a = \frac{d^2}{dr^2} + \frac{2a+1}{r} \frac{d}{dr}, \quad a \geq -\frac{1}{2}.
$$

(iii) For an even measurable function $f$ on $\mathbb{R}$, $T^\omega_{a} f$ is the element of $\mathcal{S}_a'(\mathbb{R})$, defined by

$$
\langle T^\omega_{a} f, \varphi \rangle = \int_{0}^{+\infty} f(r) \varphi(r) d\omega_a(r), \quad \varphi \in \mathcal{S}_a(\mathbb{R}). \quad (4.32)
$$

(iv) For a measurable function $g$ on $\mathbb{R}^2$, even with respect to the first variable, $T^\nu_{g}$ (resp., $T^m_{g}$) is the element of $\mathcal{S}_a'(\mathbb{R}^2)$, defined by

$$
\langle T^\nu_{g}, \varphi \rangle = \int_{\mathbb{R}} \int_{0}^{+\infty} g(r, x) \varphi(r, x) d\nu(r, x),
$$

\text{resp.,} \quad \langle T^m_{g}, \varphi \rangle = \int_{\mathbb{R}} \int_{0}^{+\infty} g(r, x) \varphi(r, x) dm(r, x), \quad \varphi \in \mathcal{S}_a(\mathbb{R}^2), \quad (4.33)

where $d\nu$ and $dm$ are the measures defined by the relations (3.1) and (3.26).

**Definition 4.7.** (i) The translation operator $\tau_a^r$, $r \in \mathbb{R}$, associated with Bessel operator $\ell_a$ is defined on $\mathcal{S}_a(\mathbb{R})$ by the following. For all $s \in \mathbb{R}$,

$$
\tau_a^r f(s) = \begin{cases} 
\frac{\Gamma(a+1)}{\sqrt{\pi} \Gamma(a+1/2)} \int_{0}^{\pi} f\left(\sqrt{r^2 + s^2 + 2rs \cos \theta}\right) \sin^{2a} \theta \ d\theta & \text{if } a > -\frac{1}{2}, \\
\frac{f(r+s) + f(|r-s|)}{2} & \text{if } a = -\frac{1}{2}.
\end{cases} \quad (4.34)
$$

(ii) The convolution product of $f \in \mathcal{S}_a(\mathbb{R})$ and $T \in \mathcal{S}_a'(\mathbb{R})$ is defined by

$$
\forall r \in \mathbb{R}, \quad T \ast_a f(r) = \langle T, \tau_a^r f \rangle. \quad (4.35)
$$

(iii) The Fourier Bessel transform is defined on $\mathcal{S}_a(\mathbb{R})$ by

$$
\forall \mu \in \mathbb{R}, \quad F_a(f)(\mu) = \int_{0}^{+\infty} f(r) j_a(\mu r) d\omega_a(r), \quad (4.36)
$$

and on $\mathcal{S}_a'(\mathbb{R})$ by

$$
\forall \varphi \in \mathcal{S}_a(\mathbb{R}), \quad \langle F_a(T), \varphi \rangle = \langle T, F_a(\varphi) \rangle. \quad (4.37)
$$
We have the following properties (we refer to [19]).

(i) $F_\alpha$ is an isomorphism from $\mathcal{S}_*(\mathbb{R})$ (resp., $\mathcal{S}'_*(\mathbb{R})$) onto itself, and we have

$$F_\alpha^{-1} = F_\alpha. \quad (4.38)$$

(ii) For $f \in \mathcal{S}_*(\mathbb{R})$, and $r \in \mathbb{R}$, $\tau^r f$ belongs to $\mathcal{S}_*(\mathbb{R})$, and we have

$$F_\alpha(\tau^r f)(\mu) = j_a(r \mu)F_\alpha(f)(\mu). \quad (4.39)$$

(iii) For $f \in \mathcal{S}_*(\mathbb{R})$ and $T \in \mathcal{S}'_*(\mathbb{R})$, the function $T *_a f$ belongs to $\mathcal{E}_*(\mathbb{R})$, and is slowly increasing, moreover

$$F_\alpha(T^{\omega a}_{*_a f}) = F_\alpha(f)F_\alpha(T). \quad (4.40)$$

In the following, we will define the fractional powers of Bessel operator and the Laplacian operator defined on $\mathbb{R}^2$ by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial x^2} \quad (4.41)$$

that we use to give simple expressions of $K^1_\alpha$ and $K^2_\alpha$.

In [16], the author has proved that the mappings

$$z \mapsto T^{\omega a}_{|r|^z}, \quad z \mapsto T^{\omega a}_{(2^{2+a+1}\Gamma(z/2+a+1)/\Gamma(-z/2))|r|^{-2z-2}}, \quad (4.42)$$

defined initially for $-2(a+1) < \Re e(z) < 0$, can be extended to a valued functions on $\mathcal{S}'_*(\mathbb{R})$, analytic on $\mathbb{C}\setminus\{-(k+a), k \in \mathbb{N}^*\}$, and we have

$$T^{\omega a}_{|r|^z} = F_\alpha(T^{\omega a}_{(2^{2+a+1}\Gamma(z/2+a+1)/\Gamma(-z/2))|r|^{-2z-2+1}}), \quad (4.43)$$

**Definition 4.8.** For $z \in \mathbb{C}\setminus\{-(k+a), k \in \mathbb{N}^*\}$, the fractional power of Bessel operator $\ell_a$ is defined on $\mathcal{S}_*(\mathbb{R})$ by

$$(-\ell_a)^z f(r) = \left( T^{\omega a}_{(2^{2+a+1}\Gamma(z+a+1)/\Gamma(-z))|r|^{-2z-2}} \right) *_a f(r). \quad (4.44)$$

From the relations (4.40) and (4.43), we deduce that for $f \in \mathcal{S}_*(\mathbb{R})$ and $z \in \mathbb{C}\setminus\{-(k+a), k \in \mathbb{N}^*\}$, we have

$$F_\alpha(T^{\omega a}_{(-\ell_a)^z f}) = F_\alpha(f)T^{\omega a}_{|r|^z}. \quad (4.45)$$

On the other hand, from [8, 10], we deduce that the mappings

$$z \mapsto T^{m}_{(r^2+x^2)^{z}}, \quad \Gamma\left(\frac{m}{2}\right)T^{m}_{(r^2+x^2)^{z}}(2x^{2a+1}\Gamma(z+1)\Gamma(a+1)/\Gamma(-z)(r^2+x^2)^{z+1}, \quad (4.46)$$

defined initially for $-1 < \Re e(z) < 0$, can be extended to a valued functions in $\mathcal{S}'_*(\mathbb{R}^2)$, analytic on $\mathbb{C}\setminus\{-k, k \in \mathbb{N}^*\}$, and we have

$$T^{m}_{(r^2+x^2)^{z}} = \Lambda_a\left( T^{m}_{\gamma(2^{2+a+1}\Gamma(z+1)\Gamma(a+1)/\Gamma(-z))(r^2+x^2)^{z+1}} \right), \quad (4.47)$$
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where $\Lambda_\alpha$ is defined on $\mathcal{F}_*(\mathbb{R}^2)$ by

$$
\langle \Lambda_\alpha(T), \varphi \rangle = \langle T, \Lambda_\alpha(\varphi) \rangle, \quad \varphi \in \mathcal{F}_*(\mathbb{R}^2),
$$

and $\Lambda_\alpha(\varphi)$ is given in Lemma 3.6.

**Definition 4.9.** For $z \in \mathbb{C} \setminus \{-k, k \in \mathbb{N}^*\}$, the fractional power of the Laplacian operator $\Delta$ is defined on $\mathcal{F}_*(\mathbb{R}^2)$ by

$$
(-\Delta)^z f(r,x) = \left( T_{-\ell + 1/2}^{m} \frac{2 \pi^{z+1} \Gamma(z+1)}{\Gamma(-z)} (s^2+y^2)^{-z-1} \right) f(r,x),
$$

where

(i) $\ast$ is the usual convolution product defined by

$$
T \ast f(r,x) = \langle T, \sigma(r,x) \hat{f} \rangle, \quad T \in \mathcal{F}'_*(\mathbb{R}^2), \quad f \in \mathcal{F}_*(\mathbb{R}^2);
$$

(ii)

$$
\sigma(r,x) f(s,y) = \frac{1}{2} \left[ f(r+s,y-x) + f(r-s,y-x) \right], \quad f \in \mathcal{F}_*(\mathbb{R}^2).
$$

It is well known that for $f \in \mathcal{F}_*(\mathbb{R}^2)$ and $T \in \mathcal{F}'_*(\mathbb{R}^2)$, the function $T \ast f$ belongs to $\mathcal{C}_*(\mathbb{R}^2)$ and is slowly increasing, and we have

$$
\Lambda_\alpha \left( T_{-\ell + 1/2}^{m} \right) = \Lambda_\alpha(f) \Lambda_\alpha(T),
$$

thus from the relations (4.47) and (4.52), we deduce that for $f \in \mathcal{F}_*(\mathbb{R}^2)$ and $z \in \mathbb{C} \setminus \{-k, k \in \mathbb{N}^*\}$,

$$
\Lambda_\alpha \left( T_{\sqrt{2\pi^{z+1} \Gamma(z+1)}}^{m} (-\Delta)^z f \right) = \Lambda_\alpha(f) T_{\sqrt{2\pi^{z+1} \Gamma(z+1)}}^{m} (-\Delta)^z f.
$$

**Theorem 4.10.** The operator $K_\alpha^1$ defined in Theorem 4.5 can be written as

$$
K_\alpha^1(f) = \frac{\pi}{2 \alpha + 1} \frac{\sqrt{\Gamma(\alpha + 1)}}{\sqrt{\Gamma(\alpha + 1)}} \int_0^\infty \int_0^\infty \left( -\frac{\partial^2}{\partial r^2} \right)^{1/2} \left( -\ell \right) \frac{f(r,x)}{\sqrt{2\pi^{z+1} \Gamma(z+1)}} \exp(-i \ell y) dx dy
$$

where

$$
\left( -\frac{\partial^2}{\partial r^2} \right)^{1/2} f(r,x) = \left( -\ell \right) \frac{f(r',x)}{\sqrt{2\pi^{z+1} \Gamma(z+1)}} \exp(-i \ell y).
$$

**Proof.** Let $f \in \mathcal{F}_*,0(\mathbb{R}^2)$. Using Fubini’s theorem, we get for every $\varphi \in \mathcal{F}_*(\mathbb{R}^2)$ the following:

$$
\langle \Lambda_\alpha \left( T_{\sqrt{2\pi^{z+1} \Gamma(z+1)}}^{m} (-\Delta)^z f \right), \varphi \rangle
$$

$$
= \frac{1}{2 \alpha + 1} \int_0^\infty \int_0^\infty \left( T_{\sqrt{2\pi^{z+1} \Gamma(z+1)}}^{m} \right) \frac{f(r',x)}{\sqrt{2\pi^{z+1} \Gamma(z+1)}} \exp(-i \ell y) dx dy
$$

(4.56)
and by the relation (4.45), we obtain
\[
\left\langle \Lambda_\alpha \left( T^m_{(-\partial^2/\partial r^2)^{1/2}f} \right), \varphi \right\rangle = \frac{1}{2^{2a+2} \Gamma^2(a+1)} \int_\mathbb{R} \int_0^{+\infty} F_{-1/2} \left( f(\cdot, x) \right) T^m_{-\partial^2/\partial r^2} \varphi(\cdot, y) \right \rangle \times \exp(-i xy) \, dx \, dy,
\]
which involves that
\[
\left\langle \Lambda_\alpha \left( T^m_{(-\partial^2/\partial r^2)^{1/2}f} \right), \varphi \right\rangle = \frac{1}{2} \int_\mathbb{R} \int_0^{+\infty} r \Lambda_\alpha (f)(r, y) \varphi(r, y) \, dm(r, y),
\]
this shows that
\[
\Lambda_\alpha \left( T^m_{(-\partial^2/\partial r^2)^{1/2}f} \right) = T^m_{|r| \Lambda_\alpha f}.
\]
Now, from Lemma 4.1, we deduce that the function
\[
(\mu, \lambda) \mapsto |\mu| \Lambda_\alpha (f)(\mu, \lambda)
\]
belongs to the subspace \( \mathcal{N} \). Then, from the relation (4.59), it follows that the function \((-\partial^2/\partial r^2)^{1/2}f\) belongs to the subspace \( \mathcal{F}_{*,0}(\mathbb{R}^2) \), and we have
\[
\forall (\mu, \lambda) \in \mathbb{R}^2, \quad \Lambda_\alpha \left( \left( -\frac{\partial^2}{\partial r^2} \right)^{1/2} f \right)(\mu, \lambda) = |\mu| \Lambda_\alpha (\mu, \lambda).
\]
By the same way, and using the relation (4.53), we deduce that for every \( f \in \mathcal{F}_{*,0}(\mathbb{R}^2) \), the function \((-\Delta)^a f\) belongs to the subspace \( \mathcal{F}_{*,0}(\mathbb{R}^2) \), and we have that for all \((\mu, \lambda) \in \mathbb{R}^2\),
\[
\Lambda_\alpha (\sqrt{2\pi}^2 \Gamma(a+1)(-\Delta)^a f)(\mu, \lambda) = (\mu^2 + \lambda^2)^a \Lambda_\alpha (f)(\mu, \lambda).
\]
Hence, the theorem follows from the relations (4.61) and (4.62).

**Definition 4.11.** Let \( a, b \in \mathbb{R}, b \geq a \geq -1/2 \).

(i) The Sonine transform is the mapping defined on \( \mathcal{C}_*(\mathbb{R}) \) by the following. For all \( r \in \mathbb{R} \),
\[
S_{b,a}(f)(r) = \begin{cases} 
\frac{2 \Gamma(b+1)}{\Gamma(b-a) \Gamma(a+1)} \int_0^1 (1-t^2)^{b-a-1} f(rt) t^{2a+1} \, dt & \text{if } b > a, \\
\frac{2 \Gamma(b+1)}{\Gamma(b-a) \Gamma(a+1)} \int_0^1 (1-t^2)^{b-a-1} f(rt) t^{2a+1} \, dt & \text{if } b = a.
\end{cases}
\]

(ii) The dual transform \( S_{b,a}^* \) is the mapping defined on \( \mathcal{F}_*(\mathbb{R}) \) by the following. For all \( r \in \mathbb{R} \),
\[
S_{b,a}^*(f)(r) = \begin{cases} 
\frac{2 \Gamma(b+1)}{\Gamma(b-a) \Gamma(a+1)} \int_r^{+\infty} (t^2-r^2)^{b-a-1} f(t) t \, dt & \text{if } b > a, \\
\frac{2 \Gamma(b+1)}{\Gamma(b-a) \Gamma(a+1)} \int_r^{+\infty} (t^2-r^2)^{b-a-1} f(t) t \, dt & \text{if } b = a.
\end{cases}
\]
Then, we have the following.
(i) The Sonine transform is an isomorphism from $\mathcal{S}_a(\mathbb{R})$ onto itself.
(ii) The dual Sonine transform is an isomorphism from $\mathcal{S}_a(\mathbb{R})$ onto itself.
(iii) For $f \in \mathcal{S}_a(\mathbb{R})$, $f$ bounded, and $g \in \mathcal{S}_a(\mathbb{R})$, we have
\[
\int_0^{+\infty} S_{b,a}(f)(r)g(r)r^{2a+1} dr = \int_0^{+\infty} f(r)S_{b,a}(g)(r)r^{2a+1} dr.
\] (4.65)
(iv) $j_b = S_{b,a}(j_a)$.
(v)
\[
F_b = \frac{\Gamma(a+1)}{2^{b-a}\Gamma(b+1)} F_a \circ tS_{b,a}.
\] (4.66)

For more details, we refer to [18, 20, 21].

We denote the following.
(i) For $T \in \mathcal{S}'_a(\mathbb{R}^2)$, $\varphi \in \mathcal{S}_a(\mathbb{R}^2)$,
\[
\langle S_{a,0}(T), \varphi \rangle = \langle T, \psi \rangle,
\] (4.67)
with $\psi(r,x) = tS_{a,0}(\varphi(\cdot,x))(r)$.
(ii) For all $(r,x) \in \mathbb{R}^2$,
\[
T#\varphi(r,x) = \langle T, \mathcal{T}(r,-x)\tilde{\varphi} \rangle,
\] (4.68)
where $\mathcal{T}(r,x)$ is the translation operator given by Definition 3.1.
(iii) $\tilde{\mathcal{S}}_a$ is the mapping defined on $\mathcal{S}'_a(\mathbb{R}^2)$ by
\[
\forall \varphi \in \mathcal{S}_a(\mathbb{R}^2), \quad \langle \tilde{\mathcal{S}}_a(T), \varphi \rangle = \langle T, \tilde{\mathcal{S}}_a(\varphi) \rangle.
\] (4.69)
(iv) $L_\alpha$ is the operator defined on $\mathcal{S}_a(\mathbb{R}^2)$ by
\[
L_\alpha f(r,x) = (-\ell_\alpha)^{2a}(f(\cdot,x))(r),
\] (4.70)
where $(-\ell_\alpha)^{2}$ is the fractional power of Bessel given by Definition 4.8.

**Theorem 4.12.** The operator $K_{a,\alpha}^2$, defined in Theorem 4.5, is given by
\[
K_{a,\alpha}^2 f(r,x) = \frac{\pi}{2^{2a+1}\Gamma(a+1)} S_{a,0}(T)\# (-\Delta_2)L_\alpha(\hat{f})(r,-x), \quad f \in \mathcal{S}'_a(\mathbb{R}^2),
\] (4.71)
where
(i) $T$ is the distribution defined by
\[
\langle T, \varphi \rangle = \int_{\mathbb{R}} \varphi(y,y) dy;
\] (4.72)
(ii) $\Delta_2$ is the operator defined in Section 2.
Proof. By the definition of $K_α^2$, and the relation (3.12), we have that for $f \in \mathcal{S}_α^0(\mathbb{R}^2)$, 

$$K_α^2(f)(r,x) = \frac{\sqrt{\pi/2}}{2^{3α+1} Γ^3(α+1)} \int_0^{+∞} \int_0^r \mu^2(μ^2 + λ^2)^{2α} \tilde{S}_α(f)(\sqrt{μ^2 + λ^2}) j_α(μ \sqrt{μ^2 + λ^2}) \exp(iλx) dμdλ. \quad (4.73)$$

By a change of variables, and using Fubini’s theorem, we get

$$K_α^2(f)(r,x) = \frac{\sqrt{π/2}}{2^{3α+1} Γ^3(α+1)} \int_0^{+∞} \int_{-y}^y ν^4(ν^2 - λ^2) \tilde{S}_α(f)(ν,λ) \frac{\exp(iλx)}{√ν^2 - λ^2} j_α(νy) ν dνdλ. \quad (4.74)$$

On the other hand, for $f \in \mathcal{S}_α^0(\mathbb{R}^2)$, the function $L_α f$ belongs to $\mathcal{C}_α(\mathbb{R}^2)$, and is slowly increasing. Moreover, we have

$$\tilde{S}_α(T^r_{L_α f}) = T^{v}_r |μ|^{4α} \tilde{S}_α(f), \quad (4.75)$$

But, for $f \in \mathcal{S}_α^0(\mathbb{R}^2)$, the function $\tilde{S}_α(f)$ belongs to the subspace $\mathcal{N}$; according to Theorem 4.2, we deduce that the function $L_α f$ belongs to $\mathcal{S}_α(\mathbb{R}^2)$, and we have

$$∀ (μ,λ) ∈ \mathbb{R}^2, \quad \tilde{S}_α(L_α f)(μ,λ) = |μ|^{4α} \tilde{S}_α(f)(μ,λ). \quad (4.76)$$

This involves that

$$K_α^2(f)(r,x) = \frac{\sqrt{π/2}}{2^{3α+1} Γ^3(α+1)} \int_0^{+∞} \int_{-y}^y (√ν^2 - λ^2) \tilde{S}_α(L_α f)(ν,λ) \frac{\exp(iλx)}{√ν^2 - λ^2} j_α(νy) ν dνdλ$$

$$= \frac{\sqrt{π/2}}{2^{3α+1} Γ^3(α+1)} \int_0^{+∞} \int_{-y}^y \tilde{S}_α((- Δ_2)L_α f)(ν,λ) \frac{\exp(iλx)}{√ν^2 - λ^2} j_α(νy) ν dνdλ. \quad (4.77)$$

Since for every $f ∈ \mathcal{S}_α^0(\mathbb{R}^2)$, we have that

$$∀ (r,x), (μ,λ) ∈ \mathbb{R}^2, \quad \tilde{S}_α(\mathcal{T}_{(r,x)} f)(ν,λ) = j_α(νy) \exp(iλx) \tilde{S}_α(f)(ν,λ), \quad (4.78)$$

we get

$$K_α^2(f)(r,x) = \frac{\sqrt{π/2}}{2^{3α+1} Γ^3(α+1)} \int_0^{+∞} \int_{-y}^y \tilde{S}_α(\mathcal{T}_{(r,x)}((- Δ_2)L_α f))(ν,λ) \frac{ν dνdλ}{√ν^2 - λ^2}. \quad (4.79)$$

Using the expression of $\tilde{S}_α$, we obtain

$$K_α^2(f)(r,x) = \frac{1}{2^{2α+2} Γ^3(α+1)} \int_0^{+∞} \int_{-y}^y [\int_0^∞ (\mathcal{T}_{(r,x)}((- Δ_2)L_α f))(s,y) \times j_α(sy) \exp(-iλy)s^{2α+1} ds dy] \frac{dλ}{√ν^2 - λ^2} ν dν. \quad (4.80)$$
From the fact that
\[
\int_{-\nu}^{\nu} \frac{\exp(-i\lambda y)}{\sqrt{\nu^2 - \lambda^2}} d\lambda = \pi j_0(\nu y),
\] (4.81)
and using Fubini’s theorem, we deduce that
\[
K_2^2(\alpha)(f)(r,x) = \frac{\pi}{2^{4\alpha+2}\Gamma^4(\alpha + 1)} \int_{\mathbb{R}} \left\{ \int_{-\infty}^{+\infty} \mathcal{T}_{(r,x)}(\Delta_2) L_\alpha f)(sv) j_\alpha(sv) \times j_0(\nu y)s^{2\alpha+1} dsv dy \right\} dy
\]
\[
= \frac{\pi}{2^{3\alpha+2}\Gamma^3(\alpha + 1)} \int_{\mathbb{R}} \left\{ \int_{0}^{+\infty} F_\alpha((\mathcal{T}_{(r,x)}(\Delta_2) L_\alpha f)(\cdot, y))(s) j_0(\nu y) s^2 dy \right\} dy,
\] (4.82)
and from the relation (4.66), we have
\[
K_2^2(\alpha)(f)(r,x) = \frac{\pi}{2^{4\alpha+2}\Gamma^4(\alpha + 1)} \int_{\mathbb{R}} \left\{ \int_{0}^{+\infty} F_0 \circ S_{a,0}((\mathcal{T}_{(r,x)}(\Delta_2) L_\alpha f)(\cdot, y))(\cdot)(s) j_0(\nu y) s^2 dy \right\} dy,
\] (4.83)
and the relation (4.38) implies that
\[
K_2^2(\alpha)(f)(r,x) = \frac{\pi}{2^{4\alpha+2}\Gamma^4(\alpha + 1)} \int_{\mathbb{R}} S_{a,0}((\mathcal{T}_{(r,x)}(\Delta_2) L_\alpha f)(\cdot, y))(y)dy.
\] (4.84)

4.3. Plancherel theorem for $^1\mathcal{R}_a$

**Proposition 4.13.** The operator $K_3^2$ defined by
\[
K_3^2(\alpha)(f) = \frac{\pi}{2^{4\alpha+2}\Gamma^4(\alpha + 1)} \int_{\mathbb{R}} \left( - \frac{\partial^2}{\partial r^2} \right)^{1/4} (-\Delta)^{\alpha/2} f
\] (4.85)
is an isomorphism from $\mathcal{F}_{*0}(\mathbb{R}^2)$ onto itself, where
\[
\left( - \frac{\partial^2}{\partial r^2} \right)^{1/4} f(r,x) = (-\mathcal{L}_{-1/2})^{1/4}(f(\cdot, x))(r).
\] (4.86)

**Proof.** Let $f \in \mathcal{F}_{*0}(\mathbb{R}^2)$. From the relations (4.45) and (4.53), we deduce that for all $(\mu, \lambda) \in \mathbb{R}^2$,
\[
\sqrt{|\mu|}(\mu^2 + \lambda^2)^{\alpha/2} \Lambda_\alpha(f)(\mu, \lambda) = \Lambda_\alpha\left(2\pi 2^\alpha \Gamma(\alpha + 1) \left( - \frac{\partial^2}{\partial r^2} \right)^{1/4} (-\Delta)^{\alpha/2} f \right)(\mu, \lambda),
\] (4.87)
which implies that for all \((\mu, \lambda) \in \mathbb{R}^2,\)

\[
\Lambda_\alpha(K_3^3(f))(\mu, \lambda) = \sqrt{\pi} \frac{1}{2^a \Gamma(a + 1)} \sqrt{|\mu|^a (\mu^2 + \lambda^2)^{a/2}} \Lambda_\alpha(f)(\mu, \lambda).
\] (4.88)

Then, the result follows from Lemma 4.1 and Theorem 4.2. \(\Box\)

**Proposition 4.14.** For \(g \in \mathcal{S}_a^0(\mathbb{R}^2),\) there exists the Plancherel formula

\[
\int_{\mathbb{R}} \int_{0}^{+\infty} |g(r, x)|^2 d\nu(r, x) = \int_{\mathbb{R}} \int_{0}^{+\infty} |K_3^3(\mathcal{F}_\alpha(g))(r, x)|^2 dm(r, x).
\] (4.89)

**Proof.** Let \(g \in \mathcal{S}_a^0(\mathbb{R}^2),\) from Theorem 3.5 (Plancherel formula), we have

\[
\int_{\mathbb{R}} \int_{0}^{+\infty} |g(r, x)|^2 d\nu(r, x) = \int_{\mathbb{R}} \int_{0}^{+\infty} |\mathcal{F}_\alpha(g)(\mu, \lambda)|^2 d\gamma(\mu, \lambda).
\] (4.90)

From the relation (3.12), Lemma 3.6, and the fact that

\[
\text{supp } \hat{\mathcal{F}}_\alpha(g) \subset \{(\mu, \lambda) \in \mathbb{R}^2/|\mu| \geq |\lambda|\},
\] (4.91)

we get

\[
\int_{\mathbb{R}} \int_{0}^{+\infty} |g(r, x)|^2 d\nu(r, x) = \int_{\mathbb{R}} \int_{0}^{+\infty} |\sqrt{\mu}(\mu^2 + \lambda^2)^{\alpha/2} \Lambda_\alpha(\mathcal{F}_\alpha(g)(\mu, \lambda)|^2 dm(r, x).
\] (4.92)

We complete the proof by using the formula (4.88), and the fact that for every \(f \in \mathcal{S}_a(\mathbb{R}^2),\)

\[
\int_{\mathbb{R}} \int_{0}^{+\infty} |\Lambda_\alpha(f)(\mu, \lambda)|^2 dm(\mu, \lambda) = \frac{\pi}{2^{2a+1} \Gamma^2(a+1)} \int_{\mathbb{R}} \int_{0}^{+\infty} |f(\mu, \lambda)|^2 dm(\mu, \lambda).
\] (4.93)

We denote by

(i) \(L_0^2(d\nu)\) the subspace of \(L^2(d\nu)\) consisting of functions \(g\) such that

\[
\text{supp } \hat{\mathcal{F}}_\alpha(g) \subset \{(\mu, \lambda) \in \mathbb{R}^2/|\mu| \geq |\lambda|\};
\] (4.94)

(ii) \(L^2(dm)\) the space of square integrable functions on \([0, +\infty[ \times \mathbb{R}\) with respect to the measure \(dm(r, x).\)

**Theorem 4.15.** The operator \(K_3^3 \circ \mathcal{F}_\alpha\) can be extended to an isometric isomorphism from \(L_0^2(d\nu)\) onto \(L^2(dm).\)
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Proof. The theorem follows from Propositions 4.13, 4.14, and the density of $\mathcal{S}_{*,0}(\mathbb{R}^2)$ (resp., $\mathcal{S}^0_{*}(\mathbb{R}^2)$) in $L^2(dm)$ (resp., $L^2_0(d\nu)$).

5. Transmutation operators

Proposition 5.1. The Riemann-Liouville transform and its dual satisfy the following permutation properties.

(i) For all $f \in \mathcal{S}_{*}(\mathbb{R}^2)$,

$$\mathcal{i} R_a(\Delta_2 f) = \frac{\partial^2}{\partial r^2} \mathcal{i} R_a(f), \quad \mathcal{i} R_a(\Delta_1 f) = \Delta_1 \mathcal{i} R_a(f). \tag{5.1}$$

(ii) For all $f \in \mathcal{E}_{*}(\mathbb{R}^2)$,

$$\Delta_2 R_a(f) = R_a\left(\frac{\partial^2 f}{\partial r^2}\right), \quad \Delta_1 R_a(f) = R_a(\Delta_1 f). \tag{5.2}$$

Proof. (i) We know that the operators $\Delta_1, \Delta_2, \partial^2/\partial r^2,$ and $\mathcal{i} R_a$ are continuous mappings from $\mathcal{S}_{*}(\mathbb{R}^2)$ into itself. Then, by applying the usual Fourier transform $\Lambda_a$, we have

$$\Lambda_a(\mathcal{i} R_a(\Delta_2 f))(\mu, \lambda) = -\mu^2 \Lambda_a \circ \mathcal{i} R_a(f)(\mu, \lambda) = \Lambda_a\left(\frac{\partial^2}{\partial r^2} \mathcal{i} R_a(f)\right)(\mu, \lambda),$$

$$\Lambda_a(\Delta_1 \mathcal{i} R_a f)(\mu, \lambda) = i\lambda \Lambda_a(\mathcal{i} R_a(f))\mu, \lambda) = \Lambda_a(\mathcal{i} R_a(\Delta_1 f))(\mu, \lambda). \tag{5.3}$$

Consequently, (i) follows from the fact that $\Lambda_a$ is an isomorphism from $\mathcal{S}_{*}(\mathbb{R}^2)$ onto itself.

(ii) We obtain the result from (i), Lemma 2.4, and the fact that for $f \in \mathcal{E}_{*}(\mathbb{R}^2)$, and $g \in \mathbb{D}_{*}(\mathbb{R}^2)$,

$$\int_{\mathbb{R}} \int_0^{+\infty} \Delta_2 f(r,x)g(r,x)d\nu(r,x) = \int_{\mathbb{R}} \int_0^{+\infty} f(r,x)\Delta_2 g(r,x)d\nu(r,x). \tag{5.4}$$

Theorem 5.2. (i) The Riemann-Liouville transform $R_a$ is a transmutation operator of

$$\frac{\partial^2}{\partial r^2}, \Delta_1 \text{ into } \Delta_2, \Delta_1 \tag{5.5}$$

from

$$\mathcal{S}_{*,0}(\mathbb{R}^2) \text{ onto } \mathcal{S}^0_{*}(\mathbb{R}^2). \tag{5.6}$$
(ii) The dual transform $\mathcal{R}_\alpha$ is a transmutation operator of

$$
\Delta_2, \Delta_1 \text{ into } \frac{\partial^2}{\partial r^2}, \Delta_1
$$

from

$$\mathcal{S}_{*,0}^0(\mathbb{R}^2) \text{ onto } \mathcal{S}_{*,0}(\mathbb{R}^2).$$

This theorem follows from Proposition 5.1 and the fact that $\mathcal{R}_\alpha$ is an isomorphism from $\mathcal{S}_{*,0}(\mathbb{R}^2)$ onto $\mathcal{S}_{*,0}(\mathbb{R}^2)$ and $\mathcal{R}_\alpha^\dagger$ is an isomorphism from $\mathcal{S}_{*,0}^0(\mathbb{R}^2)$ onto $\mathcal{S}_{*,0}(\mathbb{R}^2)$.

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