A NOTE ON COMPREHENSIVE BACKWARD BIORTHOGONALIZATION

DAVID K. RUCH

Received 16 February 2006; Revised 21 July 2006; Accepted 25 July 2006

We present a backward biorthogonalization technique for giving an orthogonal projection of a biorthogonal expansion onto a smaller subspace, reducing the dimension of the initial space by dropping d basis functions. We also determine which basis functions should be dropped to minimize the L^2 distance between a given function and its projection. This generalizes some recent results of Rebollo-Neira.

Copyright © 2006 Hindawi Publishing Corporation. All rights reserved.

In [3], Rebollo-Neira gives a backward biorthogonalization technique for projecting a biorthogonal expansion onto a subspace, reducing the dimension N of the initial space by dropping $d = 1$ basis function. In this note, we generalize this method to reduce the space by an arbitrary number d of basis functions, $d < N$. Proposition 3.4 in [3] indicates which single basis function is to be removed in order to minimize the L^2 distance between a function f and its orthogonal projection into the reduced space. We will also generalize this result in Proposition 7. If more than one basis function is to be dropped, Rebollo-Neira recommends iterating the $d = 1$ process. We show via Example 8 that in some circumstances iterating the $d = 1$ process k times leads to results inferior to using Proposition 7 and dropping $k = d$ basis functions *simultaneously*.

We begin with a Hilbert space H and an N -dimensional subspace V . Assume biorthogonal bases of V given by $\{x'_i\}_{i=1}^N$ and $\{x_i\}_{i=1}^N$ such that $\langle x'_i, x_j \rangle = \delta_{ij}$. Now drop d basis elements from each set, without loss of generality the first d elements for notational purposes, and form the reduced subspaces $\tilde{V} = \text{span}\{x_i\}_{i=d+1}^N$ and $\tilde{V}' = \text{span}\{x'_i\}_{i=1}^d$. We wish to modify the x'_i so that the projection from V to $\tilde{V}$ is orthogonal. We next recursively construct the sequence $\{v'_i\}_{i=1}^d \subset \tilde{V}'$ by

$$v'_1 = x'_1, \quad v'_i = x'_i - \sum_{\ell=1}^{i-1} \frac{\langle x'_i, v'_\ell \rangle}{\langle v'_\ell, v'_\ell \rangle} v'_\ell, \quad i \leq d. \quad (1)$$

2 A note on comprehensive backward biorthogonalization

We observe that the set $\{v'_i\}_{i=1}^d$ forms an orthogonal basis of $\tilde{V}'$ by construction. We then construct the sequence $\{\tilde{x}'_i\}_{i=d+1}^N$ by

$$\tilde{x}'_i = x'_i - \sum_{\ell=1}^d \frac{\langle x'_i, v'_\ell \rangle}{\langle v'_\ell, v'_\ell \rangle} v'_\ell \quad (2)$$

and set $U = \text{span}\{\tilde{x}'_i\}_{i=d+1}^N$. We will see that this formula generalizes the dual modification of [3, Theorem 3.1] for $d \geq 1$. Note that each $\tilde{x}'_i$ is created to be orthogonal to $\tilde{V}'$ by subtracting from x'_i its projection onto $\tilde{V}'$.

PROPOSITION 1. *The spaces U and $\tilde{V}'$ are orthogonal complements in V , $V = \tilde{V} \oplus \tilde{V}'$.*

Proof. Choose i, j such that $j \leq d < i$ and use the definition of $\tilde{x}'_i$ and the orthogonality of $\{v'_i\}$,

$$\langle \tilde{x}'_i, v'_j \rangle = \langle x'_i, v'_j \rangle - \sum_{\ell=1}^d \frac{\langle x'_i, v'_\ell \rangle}{\langle v'_\ell, v'_\ell \rangle} \langle v'_\ell, v'_j \rangle = \langle x'_i, v'_j \rangle - \langle x'_i, v'_j \rangle = 0. \quad (3)$$

Thus U and $\tilde{V}'$ are orthogonal subspaces of V , and their dimensions add to N . □

We next verify that U and $\tilde{V}$ are actually the same space.

LEMMA 2. *The spaces U and $\tilde{V}'$ are orthogonal complements in V , and $U = \tilde{V}$.*

Proof. By (1), we can write $v'_j = \sum_{n=1}^j a_n x'_n$ for some constants a_n , so the original biorthogonality condition $\langle x'_i, x_j \rangle = \delta_{ij}$ says that, for $j < i$, $\langle v'_j, x_i \rangle = \sum_{n=1}^j a_n \langle x'_n, x_i \rangle = 0$. Thus $\tilde{V}$ and $\tilde{V}'$ are orthogonal subspaces of V , and their dimensions add to N . By the previous proposition, $U = \tilde{V}$. □

Next we give the desired biorthogonal bases of the reduced subspace $\tilde{V}$.

PROPOSITION 3. *The reduced spaces U and $\tilde{V}$ are identical and have biorthogonal bases $\{\tilde{x}'_i\}_{i=d+1}^N$ and $\{x_j\}_{j=d+1}^N$.*

Proof. Using Lemma 2 and (2), we have for $i, j > d \geq \ell$,

$$\langle \tilde{x}'_i, x_j \rangle = \langle x'_i, x_j \rangle - \sum_{\ell=1}^d \frac{\langle x'_i, v'_\ell \rangle}{\langle v'_\ell, v'_\ell \rangle} \langle v'_\ell, x_j \rangle = \delta_{ij} - \sum_{\ell=1}^d \frac{\langle x'_i, v'_\ell \rangle}{\langle v'_\ell, v'_\ell \rangle} \cdot 0 = \delta_{ij}. \quad (4)$$
□

In order to give an explicit method for determining which basis functions to drop to minimize the residual, we give a formula for the projection operator.

PROPOSITION 4. *The orthogonal projection of V onto $\tilde{V}$ is $P(\cdot) = \sum_{i=d+1}^N \tilde{x}'_i(\cdot) x_i$.*

Proof. By Proposition 3, $P(w) = w$ for all $w \in \tilde{V}$ and $\text{Range}(P) = \tilde{V}$. From Propositions 1 and 3, $\tilde{V}'$ is the null space of P , and $\text{Range}(P)$ and $\tilde{V}' = \text{Null}(P)$ are orthogonal, so P is an orthogonal projection. □

The following generalizes [3, Corollary 3.2] to give the coefficients of $P(f)$ for the case $d \geq 1$.

THEOREM 5. *If $f = \sum_{i=1}^N c_i x_i$, where $c_i = \langle x'_i, f \rangle$, then*

$$P(f) = \sum_{i=d+1}^N c'_i x_i, \quad \text{where } c'_i = c_i - \sum_{\ell=1}^d \frac{\langle x'_i, v'_\ell \rangle}{\langle v'_\ell, v'_\ell \rangle} \langle v'_\ell, f \rangle. \quad (5)$$

Proof. We calculate, using (2),

$$P(f) = \sum_{i=d+1}^N \tilde{x}'_i(f) x_i = \sum_{i=d+1}^N \left(\langle x'_i, f \rangle - \sum_{\ell=1}^d \frac{\langle x'_i, v'_\ell \rangle}{\langle v'_\ell, v'_\ell \rangle} \langle v'_\ell, f \rangle \right) x_i. \quad (6)$$

so $P(f) = \sum_{i=d+1}^N c'_i x_i$, where

$$c'_i = c_i - \sum_{\ell=1}^d \frac{\langle x'_i, v'_\ell \rangle}{\langle v'_\ell, v'_\ell \rangle} \langle v'_\ell, f \rangle. \quad (7) \quad \square$$

The following generalizes [3, Corollary 3.3] for the case $d \geq 1$.

COROLLARY 6. *If $f = \sum_{i=1}^N c_i x_i$, where $c_i = \langle x'_i, f \rangle$, then*

$$\|f\|^2 = \|P(f)\|^2 + \sum_{i=1}^d \frac{1}{\|v'_i\|^2} \left| \sum_{k=1}^i c_k \langle v'_i, x_k \rangle \right|^2. \quad (8)$$

Proof. Since $V = \tilde{V} \oplus \tilde{V}'$, we can write $f = P(f) \oplus \text{proj}_{\tilde{V}'}(f)$, where $\text{proj}_{\tilde{V}'}(f) = \sum_{i=1}^d \langle v'_i / \|v'_i\|, f \rangle (v'_i / \|v'_i\|)$ is the projection of f onto $\tilde{V}'$ using the orthogonal basis $\{v'_i\}$. Thus by Parseval and then Lemma 2, we have

$$\begin{aligned} \|f\|^2 &= \|P(f)\|^2 + \left\| \sum_{i=1}^d \left\langle \frac{v'_i}{\|v'_i\|}, f \right\rangle \frac{v'_i}{\|v'_i\|} \right\|^2 = \|P(f)\|^2 + \sum_{i=1}^d \frac{1}{\|v'_i\|^2} |\langle v'_i, f \rangle|^2 \\ &= \|P(f)\|^2 + \sum_{i=1}^d \frac{1}{\|v'_i\|^2} \left| \sum_{k=1}^i c_k \langle v'_i, x_k \rangle \right|^2. \end{aligned} \quad (9) \quad \square$$

Next we generalize [3, Proposition 3.4] for the case $d \geq 1$.

PROPOSITION 7. *By reindexing the original x_i and x'_i to examine all possible $\binom{N}{d}$ combinations of d components dropped from the original basis of V and to minimize the L^2 distance between f and $P(f)$, choose the set of d basis elements x_i that minimizes*

$$\sum_{i=1}^d \frac{1}{\|v'_i\|^2} \left| \sum_{k=1}^i c_k \langle v'_i, x_k \rangle \right|^2. \quad (10)$$

We now give an example demonstrating that iterating the process k times with $d = 1$ may give a projection considerably farther from the original f than reducing by $k = d$ basis functions *simultaneously*.

Figure 1. Drop two basis functions: iteratively (a), and simultaneously (b), for Example 8.

Example 8. For simplicity, we consider a function $f(t)$ in the four-dimensional subspace V with basis functions generated from cardinal spline wavelets. Let $B_3(x)$ be the standard quadratic cardinal spline supported on $[-1, 2]$ and let $w(t)$ be the standard associated wavelet for the Riesz basis of $L^2(\mathbb{R})$ generated by $B_3(x)$ as mentioned in [1] or [2]. Let $V = \text{span}\{x_1, x_2, x_3, x_4\}$, where $x_1(t) = B_3(t+2)/\|B_3\|$, $x_2(t) = B_3(t-2)/\|B_3\|$, $x_3(t) = (B_3(t-2) + B_3(t+2) + 0.2B_3(t))/\|B_3\|$, $x_4 = w(t)$. The function f can be expressed as $f(t) = 0.7x_1(t) + 0.5x_2(t) + 0.4x_3(t) + x_4(t)$. We wish to drop $d = 2$ basis elements and obtain the best two-dimensional approximation to f . If we iteratively drop one basis element at a time using Proposition 7 with $d = 1$, then we remove x_3 and then x_2 leaving projection $P(f) = 0.9x_1 + x_4$ as shown in Figure 1(a) with residual error $\|f - P(f)\|^2 = 0.82$. However, if we simultaneously drop two elements with $d = 2$, then we instead drop x_1 and x_2 leaving projection $P(f) = 1.1x_3 + x_4$ as shown in Figure 1(b) with residual error $\|f - P(f)\|^2 = 0.03$. As can be seen from these errors and the plots in Figure 1, there is a considerable advantage for $t \geq 1.5$ in removing two basis elements together, rather than dropping them iteratively.

When the value of $\binom{N}{d}$ is large, the computational expense of choosing the optimal set of basis elements to be dropped can be quite large. Investigation of this issue merits further study.

References

- [1] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 61, SIAM, Pennsylvania, 1992.
- [2] P. R. Massopust, D. K. Ruch, and P. J. Van Fleet, *On the support properties of scaling vectors*, Applied and Computational Harmonic Analysis **3** (1996), no. 3, 229–238.
- [3] L. Rebollo-Neira, *Backward adaptive biorthogonalization*, International Journal of Mathematics and Mathematical Sciences **2004** (2004), no. 35, 1843–1853.

David K. Ruch: Department of Mathematical and Computer Sciences, Metropolitan State College of Denver, Denver, CO 80217-3362, USA
E-mail address: ruch@mscd.edu