ON \(n\)-FLAT MODULES AND \(n\)-VON NEUMANN REGULAR RINGS

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Received 4 May 2006; Revised 20 June 2006; Accepted 21 August 2006

We show that each \(R\)-module is \(n\)-flat (resp., weakly \(n\)-flat) if and only if \(R\) is an \((n, n-1)\)-ring (resp., a weakly \((n, n-1)\)-ring). We also give a new characterization of \(n\)-von Neumann regular rings and a characterization of weak \(n\)-von Neumann regular rings for (CH)-rings and for local rings. Finally, we show that in a class of principal rings and a class of local Gaussian rings, a weak \(n\)-von Neumann regular ring is a (CH)-ring.

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1. Introduction

All rings considered in this paper are commutative with identity elements and all modules are unital. For a nonnegative integer \(n\), an \(R\)-module \(E\) is \(n\)-presented if there is an exact sequence \(F_n \to F_{n-1} \to \cdots \to F_0 \to E \to 0\), in which each \(F_i\) is a finitely generated free \(R\)-module. In particular, “0-presented” means finitely generated and “1-presented” means finitely presented. Also, \(\text{pd}_R E\) will denote the projective dimension of \(E\) as an \(R\)-module. Costa [2] introduced a doubly filtered set of classes of rings throwing a brighter light on the structures of non-Noetherian rings. Namely, for nonnegative integers \(n\) and \(d\), we say that a ring \(R\) is an \((n, d)\)-ring if \(\text{pd}_R (E) \leq d\) for each \(n\)-presented \(R\)-module \(E\) (as usual, \(\text{pd}\) denotes projective dimension); and that \(R\) is a weak \((n, d)\)-ring if \(\text{pd}_R (E) \leq d\) for each \(n\)-presented cyclic \(R\)-module \(E\). The Noetherianness deflates the \((n, d)\)-property to the notion of regular ring. However, outside Noetherian settings, the richness of this classification resides in its ability to unify classic concepts such as von Neumann regular, hereditary/Dedekind, and semi-hereditary/Prüfer rings. For instance, see [2–5, 8–10].

We say that \(R\) is \(n\)-von Neumann regular ring (resp., weak \(n\)-von Neumann regular ring) if it is \((n, 0)\)-ring (resp., weak \((n, 0)\)-ring). Hence, the 1-von Neumann regular rings and the weak 1-von Neumann regular rings are exactly the von Neumann regular ring (see [10, Theorem 2.1] for a characterization of \(n\)-von Neumann regular rings).

According to [1], an \(R\)-module \(E\) is said to be \(n\)-flat if \(\text{Tor}^R_n (E, G) = 0\) for each \(n\)-presented \(R\)-module \(G\). Similarly, an \(R\)-module \(E\) is said to be weakly \(n\)-flat if \(\text{Tor}^R_n (E, G) = 0\) for each \(n\)-presented cyclic \(R\)-module \(G\). Consequently, the 1-flat, weakly 1-flat, and flat
properties are the same. Therefore, each \( R \)-module is 1-flat or weakly 1-flat if and only if \( R \) is a von Neumann regular ring.

In Section 2, we show that each \( R \)-module is \( n \)-flat (resp., weakly \( n \)-flat) if and only if \( R \) is an \( (n,n-1) \)-ring (resp., a weakly \( (n,n-1) \)-ring). Then we give a wide class of non weakly \((n,d)\)-rings for each pair of positive integers \( n \) and \( d \). In Section 3, we give a new characterization of \( n \)-von Neumann regular rings. Also, for (CH)-rings and local rings, a characterization of weak \( n \)-von Neumann regular rings is given. Finally, if \( R \) is a principal ring or a local Gaussian ring, we show that \( R \) is a weak \( n \)-von Neumann regular ring which implies that \( R \) is a (CH)-ring.

2. Rings such that each \( R \)-module is \( n \)-flat

Recall that an \( R \)-module \( E \) is said to be \( n \)-flat (resp., weakly \( n \)-flat) if \( \text{Tor}_n^R(E,G) = 0 \) for each \( n \)-presented \( R \)-module \( G \) (resp., \( n \)-presented cyclic \( R \)-module \( G \)). It is clear but important to see that “all \( R \)-modules are \( n \)-flat” condition is equivalent to “every \( n \)-presented module has flat dimension at most \( n-1 \).”

The following result gives us a characterization of those rings modules are \( n \)-flat (resp., weakly \( n \)-flat).

**Theorem 2.1.** Let \( R \) be a commutative ring and let \( n \geq 1 \) be an integer. Then

1. each \( R \)-module is \( n \)-flat if and only if \( R \) is an \( (n,n-1) \)-ring;
2. each \( R \)-module is weakly \( n \)-flat if and only if \( R \) is a weak \( (n,n-1) \)-ring.

**Proof.** (1) For \( n = 1 \), the result is well known. For \( n \geq 2 \), let \( R \) be an \( (n,n-1) \)-ring and \( N \) be an \( R \)-module. We claim that \( N \) is \( n \)-flat.

Indeed, if \( E \) is an \( n \)-presented \( R \)-module, then \( \text{pd}_R(E) \leq n-1 \) since \( R \) is an \( (n,n-1) \)-ring. Hence, \( f d_R(E) \leq n-1 \) and so \( \text{Tor}_n^R(E,N) = 0 \). Therefore, \( N \) is \( n \)-flat.

Conversely, assume that all \( R \)-modules are \( n \)-flat. Prove that \( R \) is an \( (n,n-1) \)-ring. Let \( E \) be an \( n \)-presented \( R \)-module and consider the exact sequence of \( R \)-modules

\[
0 \rightarrow Q \rightarrow F_{n-2} \rightarrow \cdots \rightarrow F_0 \rightarrow E \rightarrow 0, \tag{2.1}
\]

where \( F_i \) is a finitely generated free \( R \)-module for each \( i \) and \( Q \) an \( R \)-module. It follows that \( Q \) is a finitely presented \( R \)-module since \( E \) is an \( n \)-presented \( R \)-module. On the other hand, \( Q \) is a flat \( R \)-module since \( f d_R(E) \leq n-1 \) (since all \( R \)-modules are \( n \)-flat and \( E \) is \( n \)-presented). Therefore, \( Q \) is a projective \( R \)-module and so \( \text{pd}_R(E) \leq n-1 \) which implies that \( R \) is an \( (n,n-1) \)-ring.

(2) Mimic the proof of (1), when \( E \) is a cyclic \( n \)-presented replace, \( E \) is an \( n \)-presented. \( \square \)

Note that, even if all \( R \)-modules are \( 2 \)-flat, there may exist an \( R \)-module which is not flat. An illustration of this situation is shown in the following example.

**Example 2.2.** Let \( R \) be a Prüfer domain which is not a field. Then all \( R \)-modules are \( 2 \)-flat by [10, Corollary 2.2] since each Prüfer domain is a \( (2,1) \)-domain. But, there exists an \( R \)-module which is not flat since \( R \) is not a von Neumann regular ring (since \( R \) is a domain which is not a field).
Let $A$ be a ring, let $E$ be an $A$-module, and $R = A \propto E$ be the set of pairs $(a,e)$ with pairwise addition and multiplication defined by

$$(a,e)(a',e') = (aa', ae' + a'e).$$

(2.2)

$R$ is called the trivial ring extension of $A$ by $E$. For instance, see [7, 9, 11].

It is clear that every Noetherian nonregular ring is an example of a ring which is not weak $(n,d)$-ring for any $n, d$. Now, we give a wide class of rings which are not a weak $(n,d)$-ring (and so not an $(n,d)$-ring) for each pair of positive integers $n$ and $d$.

**Proposition 2.3.** Let $A$ be a commutative ring and let $R = A \propto A$ be the trivial ring extension of $A$ by $A$. Then, for each pair of positive integers $n$ and $d$, $R$ is not a weak $(n,d)$-ring. In particular, it is not an $(n,d)$-ring.

**Proof.** Let $I := R(0,1)(= 0 \propto A)$. Consider the exact sequence of $R$-modules

$$0 \rightarrow \text{Ker}(u) \rightarrow R \xrightarrow{u} I \rightarrow 0,$$

(2.3)

where $u(a,e) = (a,e)(0,1) = (0,a)$. Clearly, $\text{Ker}(u) = 0 \propto A = R(0,1) = I$. Therefore, $I$ is $m$-presented for each positive integer $m$ by the above exact sequence. It remains to show that $\text{pd}_{R}(I) = \infty$.

We claim that $I$ is not projective. Deny. Then the above exact sequence splits. Hence, $I$ is generated by an idempotent element $(0,a)$, where $a \in A$. Then $(0,a) = (0,a)(0,a) = (0,0)$. So, $a = 0$ and $I = 0$, the desired contradiction (since $I \neq 0$). It follows from the above exact sequence that $\text{pd}_{R}(I) = 1 + \text{pd}_{R}(I)$ since $\text{Ker}(u) = I$. Therefore, $\text{pd}_{R}(I) = \infty$ and then $R$ is not a weak $(n,d)$-ring for each pair of positive integers $n$ and $d$. □

**Remark 2.4.** Let $A$ be a commutative ring and let $R = A \propto A$ be the trivial ring extension of $A$ by $A$. Then, for each positive integer $n$, there exists an $R$-module which is not a weakly $n$-flat, in particular it is not $n$-flat, by Theorem 2.1 and Proposition 2.3.

### 3. Characterization of (weak) $n$-von Neumann regular rings

In [10, Theorem 2.1], the author gives a characterization of $n$-von Neumann regular rings ($(n,0)$-rings). In the sequel, we give a new characterization of $n$-von Neumann regular rings. Recall first that $R$ is a (CH)-ring if each finitely generated proper ideal has a nonzero annihilator.

**Theorem 3.1.** Let $R$ be a commutative ring. Then $R$ is an $n$-von Neumann regular ring if and only if $R$ is a (CH)-ring and all $R$-modules are $n$-flat.

**Proof.** Assume that $R$ is $n$-von Neumann regular. Then $R$ is a (CH)-ring by [10, Theorem 2.1]. On the other hand, $R$ is obviously an $(n,n-1)$-ring since it is an $(n,0)$-ring. So, all $R$-modules are $n$-flat by Theorem 2.1.

Conversely, suppose that $R$ is a (CH)-ring and all $R$-modules are $n$-flat. Then, $R$ is an $(n,n-1)$-ring by Theorem 2.1 and hence $R$ is an $n$-von Neumann regular ring by [10, Corollary 2.3] since $R$ is a (CH)-ring. □
The “(CH)” and “all modules are \( n\)-flat” properties in Theorem 3.1 are not comparable via inclusion as the following two examples show.

**Example 3.2.** Let \( R \) be a Prüfer domain which is not a field. Then

1. all \( R \)-modules are \( n\)-flat for each integer \( n \geq 2 \) by Theorem 2.1 since each Prüfer domain is an \((n, n - 1)\)-domain;
2. \( R \) is not a (CH)-ring since \( R \) is a domain which is not a field.

**Example 3.3.** Let \( A \) be a (CH)-ring and let \( R = A \bowtie A \) be the trivial ring extension of \( A \) by \( A \). Then

1. \( R \) is a (CH)-ring by [11, Lemma 2.6(1)] since \( A \) is a (CH)-ring;
2. \( R \) is not an \((n, d)\)-ring for each pair of positive integers \( n \) and \( d \) by Proposition 2.3.

In particular, \( R \) does not satisfy the property that “all \( R \)-modules are \( n\)-flat” by Theorem 2.1.

Now, we give two characterizations of weak \( n \)-von Neumann regular rings under some hypothesis.

**Theorem 3.4.** Let \( R \) be a commutative ring and let \( n \) be a positive integer.

1. If \( R \) is a (CH)-ring, then \( R \) is a weak \( n \)-von Neumann regular ring if and only if all \( R \)-modules are weakly \( n\)-flat.
2. If \( R \) is a local ring, then \( R \) is a weak \( n \)-von Neumann regular ring if and only if each nonzero proper ideal of \( R \) is not \((n - 1)\)-presented.

**Proof.** (1) Let \( R \) be a (CH)-ring. If \( R \) is a weak \((n, 0)\)-ring, then \( R \) is obviously a weak \((n, n - 1)\)-ring and so each \( R \)-module is a weakly \( n\)-flat by Theorem 2.1(2). Conversely, assume that each \( R \)-module is a weakly \( n\)-flat. Then, \( R \) is a weak \((n, n - 1)\)-ring by Theorem 2.1(2). Our purpose is to show that \( R \) is a weak \((n, 0)\)-ring. Let \( E \) be a cyclic \( n \)-presented \( R \)-module and consider the exact sequence of \( R \)-module

\[
0 \to Q \xrightarrow{i} F_{n-2} \xrightarrow{\phi} F_0 \xrightarrow{\pi} E \to 0, \tag{3.1}
\]

where \( F_i \) is a finitely generated free \( R \)-module for each \( i \) and \( Q \) an \( R \)-module. Hence, \( Q \) is a finitely generated projective \( R \)-module by the same proof of Theorem 2.1(1). Therefore, \( E \) is \( m \)-presented for each positive integer \( m \) and so \( E \) is a projective \( R \)-module by mimicking the end of the proof of [10, Theorem 2.1] since \( R \) is a (CH)-ring.

(2) If each proper ideal of \( R \) is not \((n - 1)\)-presented, then \( R \) is obviously a weak \((n, 0)\)-ring. Conversely, assume that \( R \) is a local weak \((n, 0)\)-ring. We must show that each proper ideal is not \((n - 1)\)-presented. Assume to the contrary that \( I \) is a proper \((n - 1)\)-presented ideal of \( R \). Then, \( R/I \) is an \( n \)-presented cyclic \( R \)-module, so \( R/I \) is a projective \( R \)-module since \( R \) is a weak \((n, 0)\)-ring. Hence, the exact sequence of \( R \)-modules

\[
0 \to I \to R \to R/I \to 0 \tag{3.2}
\]
splits. So, \( I \) is generated by an idempotent, that is, there exists \( e \in R \) such that \( I = Re \) and \( e(e - 1) = 0 \). But \( R \) is a local ring, so \( I \) is a free \( R \)-module (since \( I \) is a finitely generated projective \( R \)-module) and then \( e(e - 1) = 0 \) implies that \( e - 1 = 0 \). So, \( I = Re = R \) and
then $I$ is not a proper ideal, a desired contradiction. Hence, each proper ideal of $R$ is not $(n - 1)$-presented. □

**Remark 3.5.** In Theorem 3.4(2), the condition $R$ local is necessary. In fact, let $R$ be a von Neumann regular ring (i.e., $(1, 0)$-ring) which is not a field. Then, $R$ is a weak $(1, 0)$-ring and there exist many finitely generated proper ideals of $R$.

If $R$ is an $(n, 0)$-ring, then $R$ is a $(CH)$-ring by [10, Theorem 2.1]. The $(1, 0)$-ring is a $(CH)$-ring. So we are led to ask the following question.

**Question 1.** If $R$ is a weak $(n, 0)$-ring for a positive integer $n \geq 2$, does this imply that $R$ is a $(CH)$-ring?

If $R$ is a principal ring (i.e., each finitely generated ideal of $R$ is principal) or a local Gaussian ring, we give an affirmative answer to this question.

For a polynomial $f \in R[X]$, denote by $C(f)$—the content of $f$—the ideal of $R$ generated by the coefficients of $f$. For two polynomials $f$ and $g$ in $R[X]$, $C(fg) \subseteq C(f)C(g)$. A polynomial $f$ is called a Gaussian polynomial if this containment becomes equality for every polynomial $g$ in $R[X]$. A ring $R$ is called a Gaussian ring if every polynomial with coefficients in $R$ is a Gaussian polynomial. For instance, see [6].

**Proposition 3.6.** Let $R$ be a weak $(n, 0)$-ring for a positive integer $n \geq 2$. Then

1. $R$ is a total ring;
2. if $R$ is a principal ring, then $R$ is a $(CH)$-ring;
3. if $R$ is a local Gaussian ring, then $R$ is a $(CH)$-ring.

**Proof.** (1) Let $a(\neq 0)$ be a regular element of $R$. Our aim is to show that $a$ is unit. The ideal $Ra$ is $n$-presented for each positive integer $n$ since $Ra \cong R$ (since $a$ is regular), so $R/Ra$ is $n$-presented for each positive integer $n$ by the exact sequence of $R$-modules $0 \to Ra \to R \to R/Ra \to 0$. Hence, $R/Ra$ is a projective $R$-module (since $R$ is a weak $(n, 0)$-ring) and so the above exact sequence splits. Then $Ra$ is generated by an idempotent, that is, there exists $e \in R$ such that $Ra = Re$ and $e(e - 1) = 0$. But $e$ is regular since so is $a$ (since $Ra = Re$). Hence, $e(e - 1) = 0$ implies that $e - 1 = 0$ and so $Ra = R$, that is, $a$ is unit.

(2) Argue by (1) and since $R$ is principal.

(3) Let $(R, M)$ be a local Gaussian weak $(n, 0)$-ring. By the proof (case 1) of [6, Theorem 3.2], it suffices to show that each $a \in M$ is zero divisor. But $R$ is a total ring by (1). Therefore, each $a \in M$ is a zero divisor and this completes the proof of Proposition 3.6. □

**References**

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