We give a differential geometric characterization for biharmonic curves with null principal normal in Minkowski 3-space.

Copyright © 2006 Hindawi Publishing Corporation. All rights reserved.

1. Introduction

This is a supplement to our previous research note [3]. In [3], we gave a characterization of biharmonic curves in Minkowski 3-space. More precisely, we pointed out that every biharmonic curves with nonnull principal normal in Minkowski 3-space is a helix, whose curvature $\kappa$ and torsion $\tau$ satisfy $\kappa^2 = \tau^2$. In the classification of biharmonic curves in Minkowski 3-space due to Chen and Ishikawa [1], there exist biharmonic spacelike curves with null principal normal. In this supplement, we give a characterization of biharmonic curves with null principal normal.

2. Preliminaries

Let $\mathbb{E}^3_1$ be the Minkowski 3-space with natural Lorentz metric $\langle \cdot, \cdot \rangle = -dx^2 + dy^2 + dz^2$. Let $\gamma = \gamma(s)$ be a spacelike curve parametrized by the arclength parameter; that is, $\gamma$ satisfies $\langle \gamma', \gamma' \rangle = 1$. A spacelike curve $\gamma$ is said to be a Frenet curve if its acceleration vector field $\gamma''$ satisfies the condition $\langle \gamma'', \gamma'' \rangle \neq 0$. Every spacelike Frenet curve admits an orthonormal frame field along it (see [3]). Since biharmonicity for spacelike Frenet curves is studied in [3], hereafter we restrict our attention to spacelike curves with null acceleration vector field. Note that spacelike curves with zero acceleration vector field are lines. There are no timelike curves with null acceleration vector field.

**Lemma 2.1.** Let $\gamma(s)$ be a spacelike curve parametrized by arclength such that $\langle \gamma'', \gamma'' \rangle = 0$. Then there exists a matrix-valued function $F(s) = (f_1(s), f_2(s), f_3(s))$, which satisfies the following ordinary differential equation:
2 Biharmonic curves in Minkowski 3-space. Part II

\[ \nabla \gamma F = F \begin{pmatrix} 0 & 0 & -1 \\ 1 & k & 0 \\ 0 & 0 & -k \end{pmatrix}, \quad f_1 = \gamma'. \] (2.1)

Here \( \nabla \) is the Levi-Civita connection of \( \mathbb{E}^3 \).

Conversely, let \( F(s) = (f_1(s), f_2(s), f_3(s)) \) be a solution to (2.1). Then there exists a spacelike curve \( \gamma(s) \) with arclength parameter \( s \) such that

\[ \gamma' = f_1, \quad \langle \gamma'', \gamma'' \rangle = 0. \] (2.2)

**Proof.** By the assumption, \( f_1' = \gamma'' \) is a null vector field. We set \( f_2 = f_1' \). Since \( f_1 = \gamma' \) is a unit spacelike vector field, there exists a unique null vector field \( f_3 \) along \( \gamma \) such that (cf. [2])

\[ \langle f_2, f_3 \rangle = 1, \quad \langle f_1, f_3 \rangle = 0. \] (2.3)

One can check that \( F = (f_1, f_2, f_3) \) satisfies (2.1). For instance, expand \( f_2 \) as \( f_2 = af_1 + bf_2 + cf_3 \). Then

\[ a = \langle f_2', f_1 \rangle = -\langle f_2, f_1' \rangle = 0, \quad c = \langle f_2', f_2 \rangle = \langle f_2, f_2 \rangle' = 0. \] (2.4)

Hence \( f_2' = bf_2 \). By similar computations, we get

\[ f_3' = -f_1 - bf_3. \] (2.5)

Thus \( F \) satisfies (2.1) with \( k = b \).

Conversely, let \( F \) be a solution to (2.1). Then \( F \) satisfies the following conditions (cf. [2, Section 2]):

\[ \langle f_1, f_1 \rangle = 1, \quad \langle f_2, f_2 \rangle = \langle f_3, f_3 \rangle = 0, \]
\[ \langle f_2, f_3 \rangle = 1, \quad \langle f_1, f_2 \rangle = \langle f_1, f_3 \rangle = 0. \] (2.6)

Integrating \( f_1(s) \) by \( s \), we obtain a spacelike curve \( \gamma(s) \) with null acceleration, since \( \gamma'' = f_1' = f_2 \). \( \square \)

We call the matrix-valued function \( F \), the *null frame* of \( \gamma \). We call \( f_1, f_2, \) and \( f_3 \), the *tangent vector field, principal normal vector field,* and *binormal vector field* of \( \gamma \), respectively. We call the function \( k \) the *curvature function* of \( \gamma \). Note that both principal normal and binormal are null.

**Example 2.2.** Let us consider \( \gamma \) with \( k = 0 \). Since \( f_2' = 0 \), we have

\[ f_1 = sn + u, \] (2.7)

where the constant vectors \( n \) and \( u \) satisfy the relation

\[ \langle n, n \rangle = \langle n, u \rangle = 0, \quad \langle u, u \rangle = 1. \] (2.8)
Thus we obtain

$$\gamma(s) = \frac{s^2}{2} n + su + v,$$

(2.9)

where $v$ is a constant vector. Hence $\gamma$ is congruent to

$$(bs^2, bs^2, s), \quad b \neq 0.$$  

(2.10)

Next, assume that $k$ is a nonzero constant, then $\gamma$ is given by

$$\gamma(s) = \frac{1}{k^2} e^{ks} n + su + v.$$  

(2.11)

Here the constant vectors $n$ and $u$ satisfy (2.8). Hence $\gamma$ is congruent to

$$\left(\frac{a}{k^2} e^{ks}, \frac{a}{k^2} e^{ks}, s\right), \quad a \neq 0.$$  

(2.12)

**Example 2.3.** Let us determine spacelike curves with $1/k = s + c$, where $c$ is a constant. Then $\gamma$ is given by

$$\gamma(s) = \left(\frac{s^3}{3} + \frac{cs^2}{2}\right) n + su + v,$$

(2.13)

where the constant vectors $n$ and $u$ satisfy (2.8). Thus $\gamma$ is congruent to the curve

$$(as^3 + bs^2, as^3 + bs^2, s), \quad a \neq 0.$$  

(2.14)

### 3. Biharmonic curves

We start this section with recalling the notion of biharmonicity.

Let $\gamma$ be a spacelike curve in $\mathbb{E}^3$ parametrized by arclength defined on an open interval $I$. We denote by $\gamma^* T\mathbb{E}^3_1$ the vector bundle over $I$ obtained by pulling back the tangent bundle $T\mathbb{E}^3_1$:

$$\gamma^* T\mathbb{E}^3_1 = \bigcup_{s \in I} T_{\gamma(s)} \mathbb{E}^3_1.$$  

(3.1)

The *Laplace operator* $\Delta$ acting on the space $\Gamma(\gamma^* T\mathbb{E}^3_1)$ of all smooth vector fields along $\gamma$ is given by

$$\Delta = -\nabla \gamma^* \nabla \gamma.$$  

(3.2)

A spacelike curve $\gamma$ is said to be *biharmonic* if $\Delta H = 0$, where $H$ is the mean curvature vector field of $\gamma$.

Chen and Ishikawa obtained the following result.

**Theorem 3.1** [1]. Let $\gamma(s)$ be a spacelike curve parametrized by arclength with null acceleration vector field. Then $\gamma$ is biharmonic if and only if $\gamma$ is congruent to

$$(as^3 + bs^2, as^3 + bs^2, s), \quad a^2 + b^2 \neq 0.$$  

(3.3)
Now we give a geometric characterization of biharmonic spacelike curve with null principal normal. Let $\gamma(s)$ be a spacelike curve parametrized by arclength with null acceleration vector field. Then the mean curvature vector field $H$ is given by

$$H = \nabla_{\gamma'} \gamma' = f_2.$$  \hspace{1cm} (3.4)

Thus we obtain

$$\Delta H = -(k' + k^2)f_2.$$  \hspace{1cm} (3.5)

Hence $\gamma$ is biharmonic if and only if $k' + k^2 = 0$. Hence the curvature function $k$ is given by $k = 0$ or $1/k(s) = s + c$, where $c$ is a constant.

**Proposition 3.2.** A spacelike curve $\gamma(s)$ parametrized by arclength parameter $s$ with null principal normal vector field is biharmonic if and only if its curvature function is given by $k = 0$ or $1/k = s + c$ for some constant $c$. Hence such curves are congruent to the curve (3.3). The former case ($k = 0$) corresponds to the case $a = 0$ (2.10) and the latter case ($1/k = s + c$) to $a \neq 0$ (2.14), respectively.

**References**


Jun-Ichi Inoguchi: Department of Mathematics Education, Faculty of Education, Utsunomiya University, Utsunomiya, 321-8505, Japan

_E-mail address: inoguchi@cc.ustunomiya-u.ac.jp_
Submit your manuscripts at 
http://www.hindawi.com