FILTERS OF R_0 -ALGEBRAS

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The notion of fated filters in R_0 -algebras is introduced. Characterizations of (fated) filters are given. A filter generated by a set is established. By introducing the notion of finite $\odot$ -property, we show that if F is a nonempty subset of an R_0 -algebra L that has the finite $\odot$ -property, then there exists a maximal filter of L containing F .

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1. Introduction

In order to research the logical system whose propositional value is given in a lattice from the semantic viewpoint, Xu [7] proposed the concept of lattice implication algebras, and discussed some of their properties. Xu and Qin [8] introduced the notion of implicative filters in a lattice implication algebra, and investigated some of their properties. Turunen [5] introduced the notion of Boolean deductive system, or equivalently, Boolean filter in BL -algebras which rise as Lindenbaum algebras from many valued logic introduced by Hájek [2]. Boolean filters are important because the quotient algebras induced by Boolean filters are Boolean algebras, and a BL -algebra is bipartite if and only if it has proper Boolean filter. In [6], Wang introduced the notion of R_0 -algebras in order to provide an algebraic proof of the completeness theorem of a formal deductive system. We note that R_0 -algebras are different from BL -algebras because the identity $x \wedge y = x \odot (x \rightarrow y)$ holds in BL -algebras, but does not hold in R_0 -algebras. R_0 -algebras are also different from lattice implication algebras because the identity $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$ holds in lattice implication algebras, but does not hold in R_0 -algebras. Although they are different in essence, they have some similarities, that is, they all have the implication operator $\rightarrow$. Therefore, it is meaningful to generalize some aspects of lattice implication algebras and BL -algebras to R_0 -algebras. In [1], Esteva and Godo introduced the MTL -algebra; the MTL -algebra is an extension of a BL -algebra, which is obtained by eliminating the condition $x \wedge y = x \odot (x \rightarrow y)$ in a BL -algebra. In fact, the MTL -algebra is an algebra induced by a left continuous t -norm and its corresponding residuum, but the BL -algebra is an algebra induced by a continuous t -norm and its corresponding residuum. It is proved that an R_0 -algebra is a particular MTL -algebra and

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its t -norm $\odot$ is a nilpotent minimum t -norm [1], which is obtained by taking a negation operator as $1 - x$. Hence, the theory of R_0 -algebras becomes one of the guides to the development of the theory of MTL -algebras. Lianzhen and Kaitai [3] extended the notions of implicative filters and Boolean filters to R_0 -algebras, considered the fuzzification of such notions, and gave characterizations of fuzzy implicative filters. They also proved that fuzzy implicative filters and fuzzy Boolean filters coincide in R_0 -algebras.

In this paper, we introduce the notion of fated filters, and give characterizations of fated filters. We study how to generate a filter by a set. By introducing the notion of finite $\odot$ -property, we show that if F is a nonempty subset of an R_0 -algebra L that has the finite $\odot$ -property, then there exists a maximal filter of L containing F .

2. Preliminaries

Definition 2.1 [6]. Let L be a bounded distributive lattice with order-reversing involution $\neg$ and a binary operation $\rightarrow$. Then $(L, \wedge, \vee, \neg, \rightarrow)$ is called an R_0 -algebra if it satisfies the following axioms:

- (R1) $x \rightarrow y = \neg y \rightarrow \neg x$,
- (R2) $1 \rightarrow x = x$,
- (R3) $(y \rightarrow z) \wedge ((x \rightarrow y) \rightarrow (x \rightarrow z)) = y \rightarrow z$,
- (R4) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
- (R5) $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$,
- (R6) $(x \rightarrow y) \vee ((x \rightarrow y) \rightarrow (\neg x \vee y)) = 1$.

Let L be an R_0 -algebra. For any $x, y \in L$, we define $x \odot y = \neg(x \rightarrow \neg y)$ and $x \oplus y = \neg x \rightarrow y$. It is proved that $\odot$ and $\oplus$ are commutative, associative, and $x \oplus y = \neg(\neg x \odot \neg y)$, and $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice.

Example 2.2 [3]. Let $L = [0, 1]$. For any $x, y \in L$, we define $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$, $\neg x = 1 - x$, and

$$x \rightarrow y := \begin{cases} 1 & \text{if } x \leq y, \\ \neg x \vee y & \text{if } x > y. \end{cases} \quad (2.1)$$

Then $(L, \wedge, \vee, \neg, \rightarrow)$ is an R_0 -algebra which is neither a BL -algebra nor a lattice implication algebra.

An R_0 -algebra has the following useful properties.

PROPOSITION 2.3 [4]. *For any elements x, y , and z of an R_0 -algebra L , there exist the following properties:*

- (a1) $x \leq y$ if and only if $x \rightarrow y = 1$,
- (a2) $x \leq y \rightarrow x$,
- (a3) $\neg x = x \rightarrow 0$,
- (a4) $(x \rightarrow y) \vee (y \rightarrow x) = 1$,
- (a5) $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$,
- (a6) $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$,
- (a7) $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$,

- (a8) $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$,
- (a9) $x \odot \neg x = 0$ and $x \oplus \neg x = 1$,
- (a10) $x \odot y \leq x \wedge y$ and $x \odot (x \rightarrow y) \leq x \wedge y$,
- (a11) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$,
- (a12) $x \leq y \rightarrow (x \odot y)$,
- (a13) $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$,
- (a14) $x \leq y$ implies $x \odot z \leq y \odot z$,
- (a15) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,
- (a16) $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$.

3. Fated filters

Definition 3.1 [4]. A nonempty subset F of L is called a *filter* of L if it satisfies

- (i) $1 \in F$,
- (ii) (for all $x \in F$) (for all $y \in L$) $(x \rightarrow y \in F \Rightarrow y \in F)$.

Definition 3.2 [3]. A nonempty subset F of L is called an *implicative filter* of L if it satisfies

- (i) $1 \in F$,
- (ii) (for all $x, y, z \in L$) $(x \rightarrow (y \rightarrow z) \in F, x \rightarrow y \in F \Rightarrow x \rightarrow z \in F)$.

Note that every implicative filter is a filter. The following is a characterization of filters.

LEMMA 3.3 [4]. Let F be a nonempty subset of L . Then F is a filter of L if and only if it satisfies

- (i) (for all $x \in F$) (for all $y \in L$) $(x \leq y \Rightarrow y \in F)$,
- (ii) (for all $x, y \in F$) $(x \odot y \in F)$.

PROPOSITION 3.4. Let F be a nonempty subset of L . Then F is a filter of L if and only if it satisfies

$$(\forall x, y \in L) \quad (\forall a \in F)((x \rightarrow a) \rightarrow y \in F \Rightarrow x \rightarrow y \in F). \quad (3.1)$$

Proof. Suppose that F is a filter of L . Let $x, y \in L$ and $a \in F$ be such that $(x \rightarrow a) \rightarrow y \in F$. Since $a \leq x \rightarrow a$ by (a2), we have $x \rightarrow a \in F$ by Lemma 3.3(i) and so $y \in F$ by Definition 3.1(ii). Using (a2) and Lemma 3.3, we conclude that $x \rightarrow y \in F$. Conversely, let F be a nonempty subset of L for which (3.1) is valid. Then there exists $a \in F$. Since $(0 \rightarrow a) \rightarrow a = 1 \rightarrow a = a \in F$, it follows from (3.1) that $1 = 0 \rightarrow a \in F$. Let $x \in F$ and $y \in L$ be such that $x \rightarrow y \in F$. Then $(1 \rightarrow x) \rightarrow y = x \rightarrow y \in F$, and thus $y = 1 \rightarrow y \in F$ by (3.1). Therefore, F is a filter of L . $\square$

PROPOSITION 3.5. Every filter F of L satisfies the following implication:

$$(\forall x, y, z \in L) \quad ((x \rightarrow y) \rightarrow z \in F \Rightarrow x \rightarrow (y \rightarrow z) \in F). \quad (3.2)$$

Proof. Let $x, y, z \in L$ be such that $(x \rightarrow y) \rightarrow z \in F$. Then

$$1 = y \rightarrow (x \rightarrow y) \leq ((x \rightarrow y) \rightarrow z) \rightarrow (y \rightarrow z), \quad (3.3)$$

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and so $((x \rightarrow y) \rightarrow z) \rightarrow (y \rightarrow z) \in F$ by Lemma 3.3(i). Using Definition 3.1(ii), we have $y \rightarrow z \in F$. Since $y \rightarrow z \leq x \rightarrow (y \rightarrow z)$ by (a2), it follows from Lemma 3.3(i) that $x \rightarrow (y \rightarrow z) \in F$. This completes the proof. $\square$

The following example shows that the converse of Proposition 3.5 may not be true in general.

Example 3.6. Let $L = \{0, a, b, c, d, 1\}$ be a set with the order $0 \leq a \leq b \leq c \leq d \leq 1$, and the following Cayley tables:

x	$\neg x$	$\rightarrow$	0	a	b	c	d	1
0	1	0	1	1	1	1	1	1
a	d	a	d	1	1	1	1	1
b	c	b	c	c	1	1	1	1
c	b	c	b	b	b	1	1	1
d	a	d	a	a	b	c	1	1
1	0	1	0	a	b	c	d	1

Then $(L, \wedge, \vee, \neg, \rightarrow)$ is an R_0 -algebra (see [3]), where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Then the set $D := \{a, b, c\}$ satisfies the condition (3.2), but D is not a filter of L .

Let F be a nonempty subset of L . Then the least filter containing F is called the *filter generated by F* and denoted by $\langle F \rangle$.

The next statement gives a description of elements of $\langle F \rangle$.

THEOREM 3.7. *If F is a nonempty subset of L , then*

$$\langle F \rangle = \left\{ x \in L \mid \begin{array}{l} a_1 \rightarrow (a_2 \rightarrow (\cdots \rightarrow (a_n \rightarrow x) \cdots)) = 1 \\ \text{for some } a_1, a_2, \dots, a_n \in F \end{array} \right\}. \quad (3.4)$$

Proof. Denote

$$G := \left\{ x \in L \mid \begin{array}{l} a_1 \rightarrow (a_2 \rightarrow (\cdots \rightarrow (a_n \rightarrow x) \cdots)) = 1 \\ \text{for some } a_1, a_2, \dots, a_n \in F \end{array} \right\}. \quad (3.5)$$

Obviously, $G \neq \emptyset$ and $F \subseteq G$. Let $x, y \in L$ and $z \in G$ be such that $(x \rightarrow z) \rightarrow y \in G$. Then there exist $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m \in F$ such that

$$a_1 \rightarrow (a_2 \rightarrow (\cdots \rightarrow (a_n \rightarrow ((x \rightarrow z) \rightarrow y)) \cdots)) = 1, \quad (3.6)$$

$$b_1 \rightarrow (b_2 \rightarrow (\cdots \rightarrow (b_m \rightarrow z) \cdots)) = 1. \quad (3.7)$$

Using (R4) repeatedly, (3.7) implies that

$$\begin{aligned} & b_1 \rightarrow (b_2 \rightarrow (\cdots \rightarrow (b_m \rightarrow (x \rightarrow z)) \cdots)) \\ &= x \rightarrow (b_1 \rightarrow (b_2 \rightarrow (\cdots \rightarrow (b_m \rightarrow z) \cdots))) \\ &= x \rightarrow 1 = 1. \end{aligned} \quad (3.8)$$

Since $x \rightarrow z \leq ((x \rightarrow z) \rightarrow y) \rightarrow y$, we have

$$b_1 \rightarrow (b_2 \rightarrow (\cdots \rightarrow (b_m \rightarrow (((x \rightarrow z) \rightarrow y) \rightarrow y)) \cdots)) = 1, \quad (3.9)$$

that is, $((x \rightarrow z) \rightarrow y) \rightarrow (b_1 \rightarrow (b_2 \rightarrow (\cdots \rightarrow (b_m \rightarrow y) \cdots))) = 1$. It follows from (3.6) and (a6) that

$$\begin{aligned} 1 &= a_1 \rightarrow (a_2 \rightarrow (\cdots \rightarrow (a_n \rightarrow ((x \rightarrow z) \rightarrow y)) \cdots)) \\ &\leq a_1 \rightarrow (a_2 \rightarrow (\cdots \rightarrow (a_n \rightarrow (b_1 \rightarrow (b_2 \rightarrow (\cdots \rightarrow (b_m \rightarrow y) \cdots)))) \cdots)), \end{aligned} \quad (3.10)$$

so that

$$a_1 \rightarrow (a_2 \rightarrow (\cdots \rightarrow (a_n \rightarrow (b_1 \rightarrow (b_2 \rightarrow (\cdots \rightarrow (b_m \rightarrow y) \cdots)))) \cdots)) = 1. \quad (3.11)$$

Consequently,

$$\begin{aligned} 1 &= x \rightarrow 1 \\ &= x \rightarrow (a_1 \rightarrow (a_2 \rightarrow (\cdots \rightarrow (a_n \rightarrow (b_1 \rightarrow (b_2 \rightarrow (\cdots \rightarrow (b_m \rightarrow y) \cdots)))) \cdots)) \\ &= a_1 \rightarrow (a_2 \rightarrow (\cdots \rightarrow (a_n \rightarrow (b_1 \rightarrow (b_2 \rightarrow (\cdots \rightarrow (b_m \rightarrow (x \rightarrow y)) \cdots)))) \cdots)), \end{aligned} \quad (3.12)$$

which shows that $x \rightarrow y \in G$. Using Proposition 3.4, we know that G is a filter of L . Now let H be a filter of L containing F and let $x \in G$. Then

$$a_1 \rightarrow (a_2 \rightarrow (\cdots \rightarrow (a_n \rightarrow x) \cdots)) = 1 \quad (3.13)$$

for some $a_1, a_2, \dots, a_n \in F \subseteq H$. Since H is a filter of L , it follows that $x \in H$ so that $G \subseteq H$. Therefore, $G = \langle F \rangle$, completing the proof. $\square$

PROPOSITION 3.8. For any $x, a_1, a_2, \dots, a_n \in L$,

$$(a_1 \odot a_2 \odot \cdots \odot a_n) \rightarrow x = 1 \iff a_1 \rightarrow (a_2 \rightarrow (\cdots \rightarrow (a_n \rightarrow x) \cdots)) = 1. \quad (3.14)$$

Proof. We will prove (3.14) by induction on n . If $n = 1$, then it is clear. If $n = 2$, then we have

$$\begin{aligned} (a_1 \odot a_2) \rightarrow x = 1 &\iff \neg(a_1 \rightarrow \neg a_2) \rightarrow x = 1 \\ &\iff \neg x \rightarrow (a_1 \rightarrow \neg a_2) = 1 \iff a_1 \rightarrow (\neg x \rightarrow \neg a_2) = 1 \\ &\iff a_1 \rightarrow (a_2 \rightarrow x) = 1. \end{aligned} \quad (3.15)$$

Assume that (3.14) holds for $n = k$, that is,

$$(a_1 \odot a_2 \odot \cdots \odot a_k) \rightarrow x = 1 \iff a_1 \rightarrow (a_2 \rightarrow (\cdots \rightarrow (a_k \rightarrow x) \cdots)) = 1. \quad (3.16)$$

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Then

$$\begin{aligned}
 (a_1 \odot a_2 \odot \cdots \odot a_k \odot a_{k+1}) \longrightarrow x = 1 \\
 \iff \neg((a_1 \odot a_2 \odot \cdots \odot a_k) \longrightarrow \neg a_{k+1}) \longrightarrow x = 1 \\
 \iff \neg x \longrightarrow ((a_1 \odot a_2 \odot \cdots \odot a_k) \longrightarrow \neg a_{k+1}) = 1 \\
 \iff (a_1 \odot a_2 \odot \cdots \odot a_k) \longrightarrow (\neg x \longrightarrow \neg a_{k+1}) = 1 \\
 \iff (a_1 \odot a_2 \odot \cdots \odot a_k) \longrightarrow (a_{k+1} \longrightarrow x) = 1 \\
 \iff a_1 \longrightarrow (a_2 \longrightarrow (\cdots \longrightarrow (a_k \longrightarrow (a_{k+1} \longrightarrow x)) \cdots)) = 1,
 \end{aligned} \tag{3.17}$$

which shows that (3.14) holds for $n = k + 1$. This completes the proof. $\square$

Combining Theorem 3.7 and Proposition 3.8, the following corollary is straightforward.

COROLLARY 3.9 [4]. *If F is a nonempty subset of L , then*

$$\langle F \rangle = \left\{ x \in L \mid \begin{array}{l} (a_1 \odot a_2 \odot \cdots \odot a_n) \longrightarrow x = 1 \\ \text{for some } a_1, a_2, \dots, a_n \in F \end{array} \right\}. \tag{3.18}$$

A subset F of L is said to have the *finite $\odot$ -property* if $a_1 \odot a_2 \odot \cdots \odot a_n \neq 0$ for any finite members $a_1, a_2, \dots, a_n$ of F .

Example 3.10. Let $L = \{0, a, b, c, d, 1\}$ be an R_0 -algebra in Example 3.6. The set $\{c, d\}$ satisfies the finite $\odot$ -property, but the set $\{a, b, c\}$ does not satisfy the finite $\odot$ -property since $a \odot b = \neg(a \rightarrow \neg b) = \neg(a \rightarrow c) = \neg 1 = 0$.

The following corollary is an immediate consequence of Corollary 3.9.

COROLLARY 3.11. *If F is a nonempty subset of L , then $\langle F \rangle$ is a proper filter of L if and only if F has the finite $\odot$ -property.*

THEOREM 3.12. *Let F be a filter of L that satisfies*

$$(\forall x \in L) \quad (x \in F \iff \neg x \in L \setminus F). \tag{3.19}$$

Then F is a maximal filter of L .

Proof. Let F be a filter that satisfies the condition (3.19). Since $1 \in F$, we get $0 = \neg 1 \in L \setminus F$ and so F is proper. If G is a filter of L and $F \subsetneq G$, then there exists $a \in G \setminus F \subseteq L \setminus F$. Thus $\neg a \in F \subsetneq G$. By (a9) and Lemma 3.3, we have $0 = a \odot \neg a \in G$ and thus $G = L$. Therefore, F is a maximal filter of L . $\square$

PROPOSITION 3.13. *Let F be a proper filter of L . Then F satisfies condition (3.19) if and only if it satisfies*

$$(\forall x, y \in L) \quad (x \oplus y \in F \implies x \in F \text{ or } y \in F). \tag{3.20}$$

Proof. Assume that F satisfies the condition (3.20). Let $x \in L$ be such that $\neg x \in L \setminus F$. Then $x \oplus \neg x = \neg x \rightarrow \neg x = 1 \in F$, and so $x \in F$ by (3.20). Suppose that $x \in F$. If $\neg x \in F$, then $0 = x \odot \neg x \in F$ since F is closed under $\odot$. This is a contradiction because F is proper. Hence, $\neg x \in L \setminus F$. Conversely, suppose that F satisfies (3.19). Let $x, y \in L$ be such that $x \oplus y \in F$. If $x \notin F$, then $\neg x \in F$ by (3.19). Since F is a filter, it follows from $\neg x \rightarrow y = x \oplus y \in F$ that $y \in F$. Hence, (3.20) is valid. $\square$

COROLLARY 3.14. *Every proper filter F of L satisfying the condition (3.20) is maximal.*

The following example shows that there exists a maximal filter F which does not satisfy the condition (3.19).

Example 3.15. Let $L = \{0, a, b, c, 1\}$ be a set with the order $0 \leq a \leq b \leq c \leq 1$, and the following Cayley tables:

x	$\neg x$	$\rightarrow$	0	a	b	c	1
0	1	0	1	1	1	1	1
a	c	a	c	1	1	1	1
b	b	b	b	b	1	1	1
c	a	c	a	a	b	1	1
1	0	1	0	a	b	c	1

Then $(L, \wedge, \vee, \neg, \rightarrow)$ is an R_0 -algebra (see [3]), where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. This is neither a BL -algebra nor a lattice implication algebra because $c \wedge a \neq c \odot (c \rightarrow a)$ and $(b \rightarrow c) \rightarrow c \neq (c \rightarrow b) \rightarrow b$, respectively. It is not difficult to verify that the set $F = \{1, c\}$ is a maximal filter of L and $\neg b \in L \setminus F$ but $b \notin F$.

THEOREM 3.16. *Let F be a nonempty subset of L . If F has the finite $\odot$ -property, then there exists a maximal filter of L containing F .*

Proof. Let $\Omega := \{G \mid G \text{ is a proper filter of } L \text{ containing } F\}$. Then $\Omega \neq \emptyset$ since $\langle F \rangle \in \Omega$. Suppose that $G_1 \subseteq G_2 \subseteq \dots$ is a chain of elements of Ω and let $H = \bigcup_i G_i$. Then (i) $F \subseteq H$, (ii) $0 \notin H$ (because $0 \notin G_i$ for all i), (iii) $1 \in H$, and (iv) if $x \in H$ and $x \rightarrow y \in H$, then there exists i such that $x \in G_i$ and $x \rightarrow y \in G_i$. It follows from Definition 3.1(ii) that $y \in G_i \subseteq H$. This shows that H is a proper filter of L containing F , and so $H \in \Omega$. Using Zorn's lemma, Ω has a maximal element. This completes the proof. $\square$

Example 3.17. Let $L = \{0, a, b, c, d, 1\}$ be an R_0 -algebra in Example 3.6. Then $\{c, d\}$ satisfies the finite $\odot$ -property, and we know that $F = \{1, c, d\}$ is a maximal proper filter of L containing $\{c, d\}$.

Definition 3.18. A nonempty subset F of L is called a *fated filter* of L if it satisfies

- (i) $1 \in F$,
- (ii) (for all $x, y \in L$) (for all $a \in F$) $(a \rightarrow ((x \rightarrow y) \rightarrow x) \in F \Rightarrow x \in F)$.

Example 3.19. Let $L = \{0, a, b, c, d, 1\}$ be an R_0 -algebra in Example 3.6. It is not difficult to verify that the set $F := \{1, c, d\}$ is a fated filter of L .

THEOREM 3.20. *Every fated filter is a filter.*

Proof. Let F be a fated filter of L and let $x, y \in L$ be such that $x \in F$ and $x \rightarrow y \in F$. Replacing a and x by x and y , respectively, in Definition 3.18(ii), we have

$$x \rightarrow ((y \rightarrow y) \rightarrow y) = x \rightarrow (1 \rightarrow y) = x \rightarrow y \in F, \quad (3.21)$$

and so $y \in F$ by Definition 3.18(ii). Hence, F is a filter of L . $\square$

The following example shows that the converse of Theorem 3.20 is not true in general.

Example 3.21. Let $L = \{0, a, b, c, 1\}$ be a R_0 -algebra in Example 3.15. Then $F = \{1, c\}$ is a filter of L but not a fated filter of L since $b \notin F$ and $c \rightarrow ((b \rightarrow a) \rightarrow b) = 1 \in F$.

THEOREM 3.22. *A filter F of L is fated if and only if it satisfies*

$$(\forall x, y \in L) \quad ((x \rightarrow y) \rightarrow x \in F \implies x \in F). \quad (3.22)$$

Proof. Assume that F is a fated filter of L and let $x, y \in L$ be such that $(x \rightarrow y) \rightarrow x \in F$. Since $1 \rightarrow ((x \rightarrow y) \rightarrow x) = (x \rightarrow y) \rightarrow x \in F$ and $1 \in F$, it follows from Definition 3.18(ii) that $x \in F$. Thus (3.22) is valid. Conversely, let F be a filter of L that satisfies the condition (3.22). Let $a \in F$ and $x, y \in L$ be such that $a \rightarrow ((x \rightarrow y) \rightarrow x) \in F$. Then $(x \rightarrow y) \rightarrow x \in F$ by Definition 3.1(ii), and so $x \in F$ by (3.22). Therefore, F is a fated filter of L . $\square$

THEOREM 3.23. *Let F be a filter of L . Then F is a fated filter if and only if it is an implicative filter.*

Proof. Let F be a fated filter of L and assume that $x \rightarrow (y \rightarrow z) \in F$ and $x \rightarrow y \in F$ for all $x, y, z \in L$. Using (R4) and (a15), we have

$$x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow (x \rightarrow z)), \quad (3.23)$$

and so $(x \rightarrow y) \rightarrow (x \rightarrow (x \rightarrow z)) \in F$ by Lemma 3.3(i). Since $x \rightarrow y \in F$, it follows from Definition 3.1(ii) that $x \rightarrow (x \rightarrow z) \in F$. Since

$$((x \rightarrow z) \rightarrow z) \rightarrow (x \rightarrow z) = x \rightarrow (((x \rightarrow z) \rightarrow z) \rightarrow z) = x \rightarrow (x \rightarrow z) \in F, \quad (3.24)$$

it follows from Theorem 3.22 that $x \rightarrow z \in F$. Therefore, F is an implicative filter of L . Conversely, suppose that F is an implicative filter of L . Let $x, y \in L$ be such that $(x \rightarrow y) \rightarrow x \in F$. Since $\neg x \leq x \rightarrow y$ and $\neg$ is involution, we have

$$(x \rightarrow y) \rightarrow x \leq \neg x \rightarrow x = \neg x \rightarrow (\neg x \rightarrow 0) \quad (3.25)$$

by (a5). Since F is a filter, we get $\neg x \rightarrow (\neg x \rightarrow 0) \in F$ by Lemma 3.3(i). Since F is an implicative filter and $\neg x \rightarrow \neg x = 1 \in F$, it follows from Definition 3.2(ii) that $x = \neg x \rightarrow 0 \in F$. Hence, by Theorem 3.22, we conclude that F is a fated filter of L . $\square$

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