SUBORDINATION PROPERTIES OF p -VALENT FUNCTIONS DEFINED BY INTEGRAL OPERATORS

SAEID SHAMS, S. R. KULKARNI, AND JAY M. JAHANGIRI

Received 20 June 2005; Revised 14 November 2005; Accepted 28 November 2005

By applying certain integral operators to p -valent functions we define a comprehensive family of analytic functions. The subordination properties of this family are studied, which in certain special cases yield some of the previously obtained results.

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1. Introduction

For the natural numbers p let $A(p)$ denote the class of functions of the form $f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots$, which are analytic in the open unit disk $U = \{z: |z| < 1\}$. For $f(z) \in A(p)$ we define

$$\begin{aligned} I^\sigma f(z) &= \frac{(p+1)^\sigma}{z\Gamma(\sigma)} \int_0^z \left(\log \frac{z}{t}\right)^{\sigma-1} f(t) dt \\ &= z^p + \sum_{n=p+1}^{\infty} \left(\frac{p+1}{n+1}\right)^\sigma a_n z^n, \quad \sigma > 0. \end{aligned} \quad (1.1)$$

Also, for $-1 \leq B < A \leq 1$ and $\lambda \geq 0$, let $\Omega_p^\sigma(A, B, \lambda)$ be the class of functions $f \in A(p)$ so that

$$\frac{\lambda}{p} \frac{I^{\sigma-1} f(z)}{z^p} + \frac{p-\lambda}{p} \frac{I^\sigma f(z)}{z^p} \prec \frac{1+Az}{1+Bz}, \quad \lambda \geq 0, \quad (1.2)$$

where “ $\prec$ ” denotes the usual subordination. See [2].

The family $\Omega_p^\sigma(A, B, \lambda)$ is a comprehensive family containing various well-known as well as new classes of analytic functions. For example, for $\sigma = 0$ and $\lambda = p+1$ we obtain the class $\Omega_p^0(A, B, p+1)$ studied by Patel and Mohanty [3] or for nonzero σ see Liu [1].

2. Main results

Our first theorem examines the containment properties of the family $\Omega_p^\sigma(A, B, \lambda)$.

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THEOREM 2.1. For $f \in A(p)$ suppose that $f \in \Omega_p^\sigma(A, B, \lambda)$ and $0 \leq \lambda \leq p(p+1)$. Then $f \in \Omega_p^\sigma(A, B, 0)$.

To prove our theorem we will need the following lemma which is due to Miller and Mocanu [2].

LEMMA 2.2. Let $g(z)$ be analytic and convex univalent in U and $g(0) = 1$. Also let $p(z)$ be analytic in U with $p(0) = 1$. If $p(z) + (zp'(z))/\gamma \prec g(z)$, where $\gamma \neq 0$ and $\operatorname{Re} \gamma \geq 0$, then $p(z) \prec \gamma z^{-\gamma} \int_0^z t^{\gamma-1} g(t) dt$.

Proof of Theorem 2.1. First, we note that

$$z(I^\sigma f(z))' = (p+1)I^{\sigma-1}f(z) - I^\sigma f(z). \quad (2.1)$$

Setting $p(z) = (I^\sigma f(z))/z^p$ we also observe that

$$\begin{aligned} \frac{(I^\sigma f(z))'}{pz^{p-1}} &= p(z) + \frac{zp'(z)}{p}, \\ \frac{I^{\sigma-1}f(z)}{z^p} &= p(z) + \frac{zp'(z)}{p+1}. \end{aligned} \quad (2.2)$$

Therefore, for $f \in \Omega_p^\sigma(A, B, \lambda)$, we conclude that

$$p(z) + \frac{\lambda}{p(p+1)}zp'(z) \prec \frac{1+Az}{1+Bz}. \quad (2.3)$$

Now from Lemma 2.2 for $\gamma = p(p+1)/\lambda$ it follows that

$$\frac{I^\sigma f(z)}{z^p} \prec \frac{p(p+1)}{\lambda} z^{-p(p+1)/\lambda} \int_0^z t^{p(p+1)/\lambda-1} \frac{1+At}{1+Bt} dt = q(z) \prec \frac{1+Az}{1+Bz}. \quad (2.4)$$

Thus $f \in \Omega_p^\sigma(A, B, 0)$. □

As a special case to Theorem 2.1, we obtain the following.

COROLLARY 2.3. Let $f \in A(p)$. Then $(1/(p+1))[(zf'(z) + f(z))/z^p] \prec (1+Az)/(1+Bz)$, implies $f(z)/z^p \prec (1+Az)/(1+Bz)$.

THEOREM 2.4. For $f \in A(p)$ suppose that $f \in \Omega_p^\sigma(A, B, \lambda)$. If $0 \leq \lambda \leq p(p+1)$, then

$$\operatorname{Re} \left(\frac{I^\sigma f(z)}{z^p} \right) \geq \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)/\lambda-1} \frac{1-Au}{1-Bu} du. \quad (2.5)$$

The result is sharp.

Proof. Set $p(z) = I^\sigma f(z)/z^p$. Then, by Theorem 2.1, we have

$$p(z) \prec \frac{p(p+1)}{\lambda} z^{-p(p+1)/\lambda} \int_0^z t^{p(p+1)/\lambda-1} \frac{1+At}{1+Bt} dt \prec \frac{1+Az}{1+Bz}. \quad (2.6)$$

This is equivalent to

$$\frac{I^\sigma f(z)}{z^p} = \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)/\lambda-1} \frac{1+uAw(z)}{1+uBw(z)} du, \quad (2.7)$$

where $w(z)$ is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U . Therefore

$$\begin{aligned} \operatorname{Re} \left(\frac{I^\sigma f(z)}{z^p} \right) &= \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)/\lambda-1} \operatorname{Re} \left\{ \frac{1+uAw(z)}{1+uBw(z)} \right\} du \\ &\geq \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)/\lambda-1} \frac{1-Au}{1-Bu} du. \end{aligned} \quad (2.8)$$

Therefore

$$\frac{I^\sigma f(z)}{z^p} = \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)/\lambda-1} \frac{1+Au}{1+Bu} du, \quad (2.9)$$

such that for this function we have

$$\frac{\lambda}{p} \frac{I^{\sigma-1} f(z)}{z^p} + \frac{p-\lambda}{p} \frac{I^\sigma f(z)}{z^p} = \frac{1+Az}{1+Bz}. \quad (2.10)$$

Letting $z \rightarrow -1$ yields

$$\frac{I^\sigma f(z)}{z^p} \rightarrow \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)/\lambda-1} \frac{1-Au}{1-Bu} du. \quad (2.11)$$

□

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Saeid Shams: Department of Mathematics, University of Urmia, Urmia-57153, Iran
E-mail address: sa40shams@yahoo.com

S. R. Kulkarni: Department of Mathematics, Fergusson College, Pune-411004, India
E-mail address: kulkarni_ferg@yahoo.com

Jay M. Jahangiri: Department of Mathematical Sciences, Kent State University, Ohio, USA
E-mail address: jjahangiri@kent.edu