Research Article

Hyers-Ulam-Rassias and Ulam-Gavruta-Rassias Stabilities of an Additive Functional Equation in Several Variables

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It is well known that the concept of Hyers-Ulam-Rassias stability was originated by Th. M. Rassias (1978) and the concept of Ulam-Gavruta-Rassias stability was originated by J. M. Rassias (1982–1989) and by P. Găvruta (1999). In this paper, we give results concerning these two stabilities.

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1. Introduction

In 1940, Ulam [13] proposed the Ulam stability problem of additive mappings. In the next year, Hyers [5] considered the case of approximately additive mappings $f : E \rightarrow E'$, where E and E' are Banach spaces and f satisfies inequality $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n \rightarrow \infty} 2^{-n} f(2^n x)$ exists for all $x \in E$ and that L is the unique additive mapping satisfying $\|f(x) - L(x)\| \leq \varepsilon$. In 1978, Rassias [14] generalized the result to an approximation involving a sum of powers of norms. In 1982–1989, Rassias [8–11] treated the Ulam-Gavruta-Rassias stability on linear and nonlinear mappings and generalized Hyers result to the following theorem.

THEOREM 1.1 (J. M. Rassias). *Let $f : E \rightarrow E'$ be a mapping, where E is a real-normed space and E' is a Banach space. Assume that there exist $\theta > 0$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \|x\|^p \|y\|^q \quad (1.1)$$

for all $x, y \in E$, where $r = p + q \neq 1$. Then there exists a unique additive mapping $L : E \rightarrow E'$

such that

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2 - 2^r|} \|x\|^r \quad (1.2)$$

for all $x \in E$.

However, the case $r = 1$ in the above inequality is singular. A counterexample has been given by Găvruta [2]. The above-mentioned stability involving a product of different powers of norms is called Ulam-Gavruta-Rassias stability by Bouikhalene and Elqorachi [1], Ravi and ArunKumar [12], and Nakmahachalasint [6]. In recent years, some other authors [3, 4, 7] have investigated the stability of additive mapping in various forms.

In this paper, we propose an n -dimensional additive functional equation and investigate its Hyers-Ulam-Rassias and Ulam-Gavruta-Rassias stabilities.

2. The functional equation and the solution

THEOREM 2.1. *Let $n > 1$ be an integer and let X, Y be real vector spaces. A mapping $f : X \rightarrow Y$ satisfies the functional equation*

$$nf\left(\sum_{i=1}^n x_i\right) = \sum_{i=1}^n f(x_i) + \sum_{1 \leq i < j \leq n} f(x_i + x_j) \quad \forall x_1, x_2, \dots, x_n \in X \quad (2.1)$$

if and only if f satisfies the Cauchy functional equation

$$f(x + y) = f(x) + f(y) \quad \forall x, y \in X. \quad (2.2)$$

Proof. We first suppose that a mapping $f : X \rightarrow Y$ satisfies (2.2). By the additivity of the Cauchy functional equation, we have

$$\begin{aligned} \sum_{i=1}^n f(x_i) + \sum_{1 \leq i < j \leq n} f(x_i + x_j) &= \sum_{i=1}^n f(x_i) + \sum_{1 \leq i < j \leq n} (f(x_i) + f(x_j)) \\ &= n \sum_{i=1}^n f(x_i) = nf\left(\sum_{i=1}^n x_i\right) \end{aligned} \quad (2.3)$$

for all $x_1, x_2, \dots, x_n \in X$. Hence, f satisfies (2.1).

Now suppose that a mapping $f : X \rightarrow Y$ satisfies (2.1). Putting $x_1 = x_2 = \dots = x_n = 0$ in (2.1), we have $nf(0) = nf(0) + \binom{n}{2}f(0)$, which leads to $f(0) = 0$. Putting $x_1 = x, x_2 = y$ and, if $n > 2$, $x_3 = x_4 = \dots = x_n = 0$ in (2.1), we get

$$nf(x + y) = f(x) + f(y) + (n - 2)f(x) + (n - 2)f(y) + f(x + y) \quad \forall x, y \in X, \quad (2.4)$$

which simplifies to $f(x + y) = f(x) + f(y)$ as desired. $\square$

3. Hyers-Ulam-Rassias stability

The following theorem treats the Hyers-Ulam-Rassias stability of (2.1).

THEOREM 3.1. *Let $n > 1$ be an integer, let X be a real vector space, and let Y be a Banach space. Given real numbers $\delta, \theta \geq 0$ and $p \in (0, 1) \cup (1, \infty)$ with $\delta = 0$ when $p > 1$. If a mapping $f : X \rightarrow Y$ satisfies the inequality*

$$\left\| nf\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n f(x_i) - \sum_{1 \leq i < j \leq n} f(x_i + x_j) \right\| \leq \delta + \theta \sum_{i=1}^n \|x_i\|^p \quad (3.1)$$

for all $x_1, x_2, \dots, x_n \in X$, then there exists a unique additive mapping $L : X \rightarrow Y$ that satisfies (2.1) and the inequality

$$\|f(x) - L(x)\| \leq \frac{2\delta}{n} + \frac{2\theta}{(n-1)|2-2^p|} \|x\|^p \quad \forall x \in X. \quad (3.2)$$

The mapping L is given by

$$L(x) = \begin{cases} \lim_{m \rightarrow \infty} 2^{-m} f(2^m x) & \text{if } 0 < p < 1 \\ \lim_{m \rightarrow \infty} 2^m f(2^{-m} x) & \text{if } p > 1 \end{cases} \quad \forall x \in X. \quad (3.3)$$

Proof. Putting $x_1 = x_2 = \dots = x_n = 0$ in (3.1), we have $\|nf(0) - nf(0) - \binom{n}{2}f(0)\| \leq \delta$. Thus, $\|f(0)\| \leq 2\delta/(n^2 - n)$. Setting $x_1 = x_2 = x$ and, if $n > 2$, $x_3 = x_4 = \dots = x_n = 0$ in (3.1), we have

$$\left\| nf(2x) - 2f(x) - (n-2)f(0) - f(2x) - 2(n-2)f(x) - \binom{n-2}{2}f(0) \right\| \leq \delta + 2\theta\|x\|^p, \quad (3.4)$$

which simplifies to

$$(n-1) \left\| f(2x) - 2f(x) - \frac{n-2}{2}f(0) \right\| \leq \delta + 2\theta\|x\|^p. \quad (3.5)$$

Therefore,

$$\|2f(x) - f(2x)\| \leq \frac{n-2}{2} \|f(0)\| + \frac{\delta + 2\theta\|x\|^p}{n-1} \leq \frac{2\delta}{n} + \frac{2\theta}{n-1} \|x\|^p. \quad (3.6)$$

We first consider the case where $0 < p < 1$. Rewrite the above inequality (3.6) as

$$\|f(x) - 2^{-1}f(2x)\| \leq \frac{\delta}{n} + \frac{\theta}{n-1} \|x\|^p. \quad (3.7)$$

For every positive integer m ,

$$\begin{aligned} \|f(x) - 2^{-m}f(2^m x)\| &= \left\| \sum_{i=0}^{m-1} (2^{-i}f(2^i x) - 2^{-(i+1)}f(2^{i+1}x)) \right\| \\ &\leq \sum_{i=0}^{m-1} \|2^{-i}f(2^i x) - 2^{-(i+1)}f(2^{i+1}x)\| \\ &= \sum_{i=0}^{m-1} 2^{-i} \|f(2^i x) - 2^{-1}f(2 \cdot 2^i x)\|. \end{aligned} \quad (3.8)$$

Substituting x with $x, 2x, 2^2x, \dots, 2^{m-1}x$ in (3.7), the above inequality becomes

$$\|f(x) - 2^{-m}f(2^m x)\| \leq \frac{\delta}{n} \sum_{i=0}^{m-1} 2^{-i} + \frac{\theta}{n-1} \|x\|^p \sum_{i=0}^{m-1} 2^{i(p-1)}. \quad (3.9)$$

Consider the sequence $\{2^{-m}f(2^m x)\}$. For all positive integers $k < l$, we have

$$\begin{aligned} \|2^{-k}f(2^k x) - 2^{-l}f(2^l x)\| &= 2^{-k} \|f(2^k x) - 2^{-(l-k)}f(2^{l-k} \cdot 2^k x)\| \\ &\leq 2^{-k} \left(\frac{\delta}{n} \sum_{i=0}^{l-k-1} 2^{-i} + \frac{\theta}{n-1} \|2^k x\|^p \sum_{i=0}^{l-k-1} 2^{i(p-1)} \right) \\ &\leq \frac{2^{-k}\delta}{n} \sum_{i=0}^{\infty} 2^{-i} + \frac{\theta}{n-1} 2^{-k(1-p)} \|x\|^p \sum_{i=0}^{\infty} 2^{i(p-1)}. \end{aligned} \quad (3.10)$$

The right-hand side of the above inequality approaches 0 as $k \rightarrow \infty$. Therefore, $L(x) = \lim_{m \rightarrow \infty} 2^{-m}f(2^m x)$ is well defined. Taking the limit of (3.9) as $m \rightarrow \infty$, we have

$$\|f(x) - L(x)\| \leq \frac{\delta}{n} \sum_{i=0}^{\infty} 2^{-i} + \frac{\theta}{n-1} \|x\|^p \sum_{i=0}^{\infty} 2^{i(p-1)} = \frac{2\delta}{n} + \frac{2\theta}{(n-1)(2-2^p)} \|x\|^p \quad \forall x \in X. \quad (3.11)$$

To show that L satisfies (2.1), replace each x_i in (3.1) with $2^m x_i$. This results in

$$\left\| nf \left(\sum_{i=1}^n 2^m x_i \right) - \sum_{i=1}^n f(2^m x_i) - \sum_{1 \leq i < j \leq n} f(2^m x_i + 2^m x_j) \right\| \leq \left(\delta + \theta \sum_{i=1}^n \|2^m x_i\|^p \right). \quad (3.12)$$

Dividing the above inequality by 2^m and taking the limit as $m \rightarrow \infty$, we obtain

$$\left\| nL \left(\sum_{i=1}^n x_i \right) - \sum_{i=1}^n L(x_i) - \sum_{1 \leq i < j \leq n} f(x_i + x_j) \right\| \leq \lim_{m \rightarrow \infty} \left(\frac{\delta}{2^m} + \frac{\theta}{2^{m(1-p)}} \sum_{i=1}^n \|x_i\|^p \right) = 0, \quad (3.13)$$

which verifies that L indeed satisfies (2.1).

To prove the uniqueness of L , suppose there is a mapping $L' : X \rightarrow Y$ such that L' satisfies (2.1) and (3.2). The additivity of L and L' is asserted by Theorem 2.1; hence,

$$\begin{aligned} \|L(x) - L'(x)\| &= 2^{-m} \|L(2^m x) - L'(2^m x)\| \\ &\leq 2^{-m} (\|L(2^m x) - f(2^m x)\| + \|L'(2^m x) - f(2^m x)\|) \\ &\leq 2^{-m} \cdot 2 \left(\frac{2\delta}{n} + \frac{2\theta}{(n-1)(2-2^p)} \|2^m x\|^p \right) \xrightarrow{m \rightarrow \infty} 0. \end{aligned} \quad (3.14)$$

Thus, $L(x) = L'(x)$ for all $x \in X$.

For the case $p > 1$, $\delta = 0$ and (3.7) must be replaced by

$$\|f(x) - 2f(2^{-1}x)\| \leq \frac{2\theta}{n-1} \|2^{-1}x\|^p. \quad (3.15)$$

The rest of the proof can be done in the same fashion as that of the case $0 < p < 1$. $\square$

4. Ulam-Gavruta-Rassias stability

The following theorem treats the Ulam-Gavruta-Rassias stability of (2.1).

THEOREM 4.1. *Let $n > 1$ be an integer, let X be a real vector space, and let Y be a Banach space. Given real numbers $\delta, \theta \geq 0$ and $p \in (0, 1) \cup (1, \infty)$ with $\delta = 0$ when $p > 1$. If a mapping $f : X \rightarrow Y$ satisfies the inequality*

$$\left\| nf\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n f(x_i) - \sum_{1 \leq i < j \leq n} f(x_i + x_j) \right\| \leq \delta + \theta \sum_{1 \leq i < j \leq n} \|x_i\|^{p/2} \|x_j\|^{p/2} \quad (4.1)$$

for all $x_1, x_2, \dots, x_n \in X$, then there exists a unique additive mapping $L : X \rightarrow Y$ that satisfies (2.1) and the inequality

$$\|f(x) - L(x)\| \leq \frac{2\delta}{n} + \frac{\theta}{(n-1)|2-2^p|} \|x\|^p \quad \forall x \in X. \quad (4.2)$$

The mapping L is given by (3.3).

Proof. We make the same substitution as in the proof of Theorem 3.1 and obtain instead of (3.5) the following inequality:

$$(n-1) \left\| f(2x) - 2f(x) - \frac{n-2}{2} f(0) \right\| \leq \delta + \theta \|x\|^p \quad \forall x \in X. \quad (4.3)$$

The rest of the proof, apart from a multiplicative factor of 2 appears before θ , can be carried over from that of Theorem 3.1. $\square$

It should be remarked that in the case where $n = 2$, functional equation (2.1) reduces to the Cauchy functional equation, and the Ulam-Gavruta-Rassias stability of this problem has been treated by J. M. Rassias, and the result has been restated in Theorem 1.1.

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